Stability of Multidegree of Freedom Potential Systems to General Infinitesimal Positional Perturbation Forces

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The stability of general linear multidegree of freedom stable potential systems that are perturbed by general arbitrary positional forces, which may be neither conservative nor purely circulatory/conservative, is considered. It has been recently recognized that such perturbed potential systems with multiple frequencies of vibration are susceptible to instability, and this paper is centrally concerned with the situation when potential systems have such multiple natural frequencies. An approach based on perturbation theory that includes nonlinear terms in the expansions of the perturbed eigenvalues is developed. Explicit conditions under which the system either remains stable or becomes unstable due to flutter are provided. These results show that the stability/instability picture that emerges is far subtler and more complex than what might be intuitively inferred. The manner in which prior results related to narrower classes of perturbation matrices, like circulatory matrices, get included in the more general results obtained here is pointed out. Several numerical examples illustrate the applicability of the analytical results. An engineering application is provided demonstrating the power of the analytical results. [DOI: 10.1115/1.4055305]

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1 Introduction

Consider the potential (conservative) system described by the equation

$$\tilde{M}\ddot{q} + \tilde{K}q = 0 \tag{1}$$

where the n by n real matrix $\tilde{M}$ is a symmetric positive definite matrix and $\tilde{K}$ is a real symmetric matrix. The n-vector of generalized coordinates is denoted by q, and the dots indicate differentiation. $\tilde{M}$ is the inertia (mass) matrix and $\tilde{K}$ describes the potential (conservative) forces. It is well-known that the potential system is stable, i.e., every solution $q(t)$ of Eq. (1) is bounded for all non-negative t, if and only if the potential matrix $\tilde{K}$ is positive definite ($\tilde{K} > 0$), and in this paper we will assume that this condition is satisfied. It is an important question from both theoretical and practical points of view to know how the introduction of disturbing positional forces into system (1) affects its stability [1]. Positional perturbations arise in many real-life applications ranging from aerospace structures and aero-elasticity to brake squeal, wheel shimmy, and bipedal motion [2,3], and the investigation of the stability of such systems has an old and rich history (see monographs [2,4–6] and review papers [7–9]).

In recent years, interest in these stability problems has revived, and it has mainly been focused on purely circulatory force perturbations, both infinitesimal and finite [1,3,10–15]. However, despite the rich literature that has been developed over numerous decades of research, our knowledge, even for this rather narrow and limited class of (positional circulatory) perturbations, still seems incomplete. In fact, it was only recently observed that the presence of multiple eigenvalues in stable potential systems is a central cause in making them unstable under perturbational forces [1]. Furthermore, the "received view" from decades of past investigations that has been handed down to us is that infinitesimal positional circulatory perturbations to a stable multidegree of freedom potential system with multiple frequencies always cause the system to become unstable. It is only recently, through a more detailed perturbation analysis that considers higher order effects beyond those captured by linear perturbation theory that this long-held view has been shown to be incorrect [16]. A considerably more complex picture of stability is shown to emerge in which stability and instability can alternate depending on the nature of the circulatory perturbation and the manner of its interaction with the eigenstructure of the potential system [16]. Though pointing to a paradigm different from our long-held received view, these new stability results are, however, restricted to only the class of infinitesimal circulatory perturbation forces. In most real-life situations that arise in nature as well as in engineered civil, aerospace, and mechanical systems, the perturbing forces are usually of a more general nature and they could be circulatory as well as noncirculatory.

This paper explores the question of the stability and instability of stable multidegree of freedom systems subjected to arbitrary infinitesimal positional perturbation forces. As shown here, and as expected, widening the class of perturbations to which a general multidegree of freedom, stable potential system is subjected from those that are described by only circulatory and/or only conservative matrices to those described by general, arbitrary matrices make the determination of the stability of such perturbed systems significantly more complex from a mathematical standpoint. The results obtained though, compensate for this mathematical complexity by having far greater generality. Since perturbations that occur in real-life physical systems do not follow the pigeon-holing done by scientists/engineers/mathematicians, as being only circulatory, or only conservative, etc., the compass of applicability of the results obtained in this paper is much larger. They are therefore applicable to, and useful for, actual, real-world situations encountered in nature and in engineered systems. To the best of our knowledge, the results obtained in this paper are new and go well beyond

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those available in the literature to date as well as those obtained in the references provided.

A stable multidegree of freedom potential system (1) when subjected to a *general* perturbing positional force is described by the equation

$$\tilde{M}\ddot{q} + \tilde{K}q + \varepsilon\tilde{P}q = 0 \quad (2)$$

where $\tilde{P}$ is an *arbitrary* real matrix and $\varepsilon \geq 0$ is a dimensionless parameter which we introduce to characterize the intensity of the perturbing force.

Making the transformation $x = \tilde{M}^{1/2}q$, where the exponent ½ indicates the unique positive definite square root of the matrix $\tilde{M}$, and premultiplying Eqs. (1) and (2) by $\tilde{M}^{-1/2}$ we get the following equations that describe the potential system and the perturbed potential system

$$\ddot{x} + Kx = 0 \quad (3)$$

$$\ddot{x} + Kx + \varepsilon Px = 0 \quad (4)$$

where the symmetric matrix $K = \tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$ and the matrix $P = \tilde{M}^{-1/2}\tilde{P}\tilde{M}^{-1/2}$. Clearly, system (2) is equivalent to system (4), and we shall from here on consider this system.

We begin in Sec. 2 with the case when all natural frequencies of the potential system are distinct and give an estimate of the parameter ε below which system (4) remains stable. The case where we have multiple natural frequencies is more complex, and it is treated subsequently at some length in Sec. 3. Section 4 considers an example of practical engineering interest which demonstrates the power of the analytical results obtained herein. Section 5 gives the conclusions.

2 Case of Simple Natural Frequencies

To obtain a sufficient condition for the stability of the systems under consideration, we first state two lemmas.

Lemma 1. *For a fixed value of the real parameter ε, the system (4) is stable if and only if all eigenvalues of the matrix $K + \varepsilon P$ are positive and simple (unrepeated) or semi-simple (i.e., the number of linearly independent eigenvectors associated with a multiple eigenvalue of the matrix $K + \varepsilon P$ coincides with its multiplicity).*

Proof. See, for example, [2]. ∎

Lemma 2. *Let $A, B \in \Re^{n\times n}$, with A symmetric, and let $\|.\|_2$ be the spectral matrix norm. If A has eigenvalues $\lambda_1,\ldots, \lambda_n$, then*

(a) *all the eigenvalues of $A + B$ lie in the union of the discs*

$$D_i = \{z : |z - \lambda_i| \leq \|B\|_2\}, \quad i = 1, 2, \ldots, n \quad (5)$$

in the complex z-plane;

(b) *a set of k disks having no point in common with the remaining n—k discs contains exactly k eigenvalues of $A + B$.*

Proof. This is the well-known Bauer–Fike localization theorem specialized for the case when the unperturbed matrix A is real and symmetric (see, for example [11], Lemma 1). ∎

Recall that $\|B\|_2$ is equal to the square root of the largest eigenvalue of the matrix B^TB.

The following assertion provides us with an estimation of the parameter ε in terms of the eigenvalues of K and the spectral norm of P under which system (4) remains stable.

Result 1. Let $\lambda_1,\ldots, \lambda_n$ be the distinct eigenvalues of the positive definite potential matrix K. If

$$\varepsilon < \eta/\|P\|_2 \quad (6)$$

where

$$\eta = \min\left\{\lambda_{\min},\ \min_{1\leq i\neq j\leq n} |\lambda_i - \lambda_j|/2\right\} \quad (7)$$

then the system (4) is stable.

Proof. It follows from the part (a) of Lemma 2 that all the eigenvalues of $K + \varepsilon P$ are contained in the union of the disks

$$D_i = \{z : |z - \lambda_i| \leq \varepsilon\|P\|_2\}, \quad i = 1, 2, \ldots, n \quad (8)$$

with the same radius $\varepsilon\|P\|_2$ and centers at the points λ_i on the real axis. The condition $\varepsilon < \lambda_{\min}/\|P\|_2$ ensures that the union of disks (8) is located in the open right half plane, i.e. that every eigenvalue of $K + \varepsilon P$ has positive real part. On the other hand, if $\varepsilon\|P\|_2 < \min_{1\leq i\neq j\leq n} |\lambda_i - \lambda_j|/2$, then any disk (8) is disjoint from all other disks, and, in view of the part (b) of Lemma 2, it contains one and only one eigenvalue of $K + \varepsilon P$. In this case any eigenvalue of the matrix $K + \varepsilon P$ is real because the spectrum of a real matrix is symmetrically placed with respect to the real axis. Thus, under conditions (6) and (7), the matrix $K + \varepsilon P$ has all simple positive eigenvalues and, according to Lemma 1, the system (4) is stable. ∎

Remark 1. When the perturbing matrix P is circulatory ($P^T = -P$), then (7) can be replaced by $\eta = \min_{1\leq i\neq j\leq n}|\lambda_i - \lambda_j|/2$, as shown in Ref. [11]. ∎

Result 1 says that a stable potential system all of whose natural frequencies are distinct remains stable after the addition of sufficiently small positional forces of arbitrarily structure. In the next section, we will consider the case of multiple frequencies of the potential system.

3 Case of Multiple Natural Frequencies

In what follows, we will need the following two assertions.

Lemma 3. *Suppose that the positive definite potential matrix K has one multiple eigenvalue λ_0 and that its other eigenvalues are simple. If all eigenvalues $\mu_j(\varepsilon)$ of the matrix $K + \varepsilon P$, such that $\mu_j(0) = \lambda_0$, are real and simple or semi-simple as $\varepsilon \to 0$, then the system remains stable for small enough ε; otherwise it will be unstable by flutter for arbitrarily small nonzero ε.*

Proof. This follows easily from Lemmas 1 and 2. ∎

Lemma 4. *Let A be a real $k \times k$ matrix and let $\overset{(A)}{H} = [h_{i+j-2}]_{i,j=1}^{k}$, where*

$$h_{i+j-2} = \mathrm{Tr}(A^{i+j-2}), \quad i, j = 1, 2, \ldots, k \quad (9)$$

Then, the matrix A has

(a) *all real eigenvalues if and only if the matrix $\overset{(A)}{H}$ is positive semi-definite*;

(b) *k distinct real eigenvalues if and only if $\overset{(A)}{H}$ is positive definite;*

(c) *at least one pair of complex conjugate eigenvalues if and only if $\overset{(A)}{H}$ is indefinite.*

Proof. Observe that

$$\mathrm{Tr}(A^j) = \sum_{i=1}^{k} \alpha_i^j, \quad j = 0, 1, 2, \ldots$$

where α_i are eigenvalues of A [17], i.e., $\overset{(A)}{H}$ is the Hankel matrix of the Newton sums associated with the characteristic polynomial of A. Then, in view of the Borhardt–Jacobi theorem [18, Section 13.10], if $\{\pi, \nu, \delta\}$ is the inertia of the matrix $\overset{(A)}{H}$ (i.e., the triplet of numbers of positive, negative, and zero eigenvalues of $\overset{(A)}{H}$) the matrix A has ν different pairs of complex conjugate eigenvalues and $\pi - \nu$ different real eigenvalues. From this the result easily follows. ∎

Remark 2. If A is 2×2 matrix with elements a_{ij}, then conditions (a), (b), and (c) of Lemma 4 reduce to the well-known simple conditions $\delta \geq 0$, $\delta > 0$, and $\delta < 0$, respectively, where $\delta = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$ is the discriminant of the characteristic equation of A.

In this section, we suppose that the potential matrix K has one eigenvalue λ_0 of multiplicity $m \geq 2$, and that the other eigenvalues are simple. Let $T=[T_m|T_{n-m}]$ be an orthogonal matrix, where the $n \times m$ submatrix T_m contains m eigenvectors of K corresponding to the multiple eigenvalue, and the $n\times(n-m)$ submatrix T_{n-m} contains the remainder of the eigenvectors of K. The orthogonal matrix T reduces K and P to the forms

$$\hat{\Lambda}=T^TKT=\mathrm{diag}(\lambda_0 I_m,\ \Lambda_{n-m}),\ \ \hat{P}=T^TPT=\begin{bmatrix}\hat{P}_{11} & \hat{P}_{12}\\ \hat{P}_{21} & \hat{P}_{22}\end{bmatrix}$$

where $\Lambda_{n-m}=T_{n-m}^TKT_{n-m}$ is the $(n-m)$-dimensional diagonal matrix, and

$$\hat{P}_{11}=T_m^TPT_m,\quad \hat{P}_{12}=T_m^TPT_{n-m},\quad \hat{P}_{21}=T_{n-m}^TPT_m,\quad \hat{P}_{22}=T_{n-m}^TPT_{n-m} \tag{10}$$

Let $\mu(\varepsilon)$ be an eigenvalue of the matrix $(\hat{\Lambda}+\varepsilon\hat{P})$ for which $\mu(0)=\lambda_0$ and let $w(\varepsilon)$ be corresponding eigenvector, i.e.,

$$(\hat{\Lambda}+\varepsilon\hat{P})w=\mu w \tag{11}$$

According to a classical result related to the perturbation of a multiple semi-simple eigenvalue, there is a number a and a positive integer $r\le m$ such that $\mu(\varepsilon)=\lambda_0+a\varepsilon+O(\varepsilon^{1+(1/r)})$ as $\varepsilon\to 0$ [18, δ11.7] (see also the interesting paper [19], which discusses the appearance of fractional powers in the expansions of perturbed semi-simple eigenvalues). With this in mind, we substitute in $\mu=\lambda_0+\varepsilon\beta$ and $w=\begin{bmatrix}\bar{w}^T & \varepsilon\tilde{w}^T\end{bmatrix}^T$ in (11), where $\bar{w}$ and $\tilde{w}$ are m and $(n-m)$ dimensional vectors, respectively, so that we get

$$(\hat{P}_{11}-\beta I_m)\bar{w}+\varepsilon\hat{P}_{12}\tilde{w}=0 \tag{12}$$

and

$$(\Lambda_{n-m}-\lambda_0 I_{n-m})\tilde{w}+\hat{P}_{21}\bar{w}+\varepsilon\hat{P}_{22}\tilde{w}-\varepsilon\beta\tilde{w}=0 \tag{13}$$

Also, we next write

$$\beta(\varepsilon)=\beta_0+\beta_1\varepsilon^{1/r}+\beta_2\varepsilon^{2/r}+\cdots+\beta_r\varepsilon+\cdots \tag{14}$$

and

$$\bar{w}=\bar{w}_0+\varepsilon^{1/r}\bar{w}_1+\varepsilon^{2/r}\bar{w}_2+\cdots,\qquad \tilde{w}=\tilde{w}_0+\varepsilon^{1/r}\tilde{w}_1+\varepsilon^{2/r}\tilde{w}_2+\cdots \tag{15}$$

Substituting (14) and (15) into (12) and collecting coefficients of equal powers of ε, we find

$$\{\varepsilon^0\}\quad (\hat{P}_{11}-\beta_0 I_m)\bar{w}_0=0 \tag{16}$$

$$\{\varepsilon^{k/r}\}\quad (\hat{P}_{11}-\beta_0 I_m)\bar{w}_k=\begin{cases}\sum_{i=1}^{k}\beta_i\bar{w}_{k-i}, & k=1,\ldots,r-1\\ \sum_{i=1}^{k}\beta_i\bar{w}_{k-i}-\hat{P}_{12}\tilde{w}_{k-r}, & k=r,\ r+1,\ldots\end{cases} \tag{17}$$

Also, in the same manner, from (13) we get

$$\{\varepsilon^0\}\quad \tilde{w}_0=-D\hat{P}_{21}\bar{w}_0 \tag{18}$$

$$\{\varepsilon^{k/r}\}\quad \tilde{w}_j=\begin{cases}-D\hat{P}_{21}\bar{w}_j, & j=1,\ldots,r-1\\ -D\left(\hat{P}_{21}\bar{w}_j+\hat{P}_{22}\tilde{w}_{j-r}-\sum_{i=0}^{j-r}\beta_i\tilde{w}_{j-r-i}\right), & j=r,\ r+1,\ldots\end{cases} \tag{19}$$

where

$$D=(\Lambda_{n-m}-\lambda_0 I_{n-m})^{-1} \tag{20}$$

It follows from (16) that β_0 is an eigenvalue of the m by m real matrix $\hat{P}_{11}=T_m^TPT_m$. If this matrix has all real distinct eigenvalues β_{0i}, $i=1,\ldots,m$, then when $\varepsilon\to 0$, the matrix $\hat{\Lambda}+\varepsilon\hat{P}$ (and, of course, $K+\varepsilon P$ too) has m real distinct eigenvalues of the forms

$$\mu_i(\varepsilon)=\lambda_0+\varepsilon\beta_{0i}+O(\varepsilon^{1+(1/r)}),\quad i=1,\ldots,m$$

and, according to Lemma 3, system (4) is stable for sufficiently small ε. On the other hand, if the matrix $\hat{P}_{11}$ has at least one pair of eigenvalues with nonzero imaginary part, then the matrix $\hat{\Lambda}+\varepsilon\hat{P}$ has a pair of eigenvalues of the form

$$\mu(\varepsilon)=\lambda_0+\varepsilon(\alpha\pm i\nu)+O(\varepsilon^{1+(1/r)})$$

with $\nu>0$, and again in view of Lemma 3, system (4) is unstable by flutter for arbitrarily small nonzero ε.

Recalling that the n by m matrix T_m is composed of the m orthonormal eigenvectors of K associated with the eigenvalue λ_0 and applying Lemma 4, we have proved the following result. ∎

Result 2. The addition of a general perturbation εP to the n degree of freedom potential system described by Eq. (3) that has an eigenvalue λ_0 of multiplicity m with $2\le m\le n$ will cause:

(a) the system described by Eq. (4) to remain stable for sufficiently small values of ε if the matrix $\overset{(\hat{P}_{11})}{H}$ associated with the m by m matrix $\hat{P}_{11}=T_m^TPT_m$ is positive definite;
(b) the system described by Eq. (4) becomes unstable by flutter for arbitrarily small nonzero values of ε if $\overset{(\hat{P}_{11})}{H}$ is indefinite. ∎

Corollary 1. *[16,20]. If the perturbation matrix is purely circulatory (i.e., $P^T=-P$) and $\hat{P}_{11}\ne 0$, then the system (4) is unstable by flutter for arbitrarily small nonzero values of ε.*
Proof. Obviously, if $P^T=-P$ and $\hat{P}_{11}\ne 0$, then the matrix $\overset{(\hat{P}_{11})}{H}$ is indefinite because $h_2=\mathrm{Tr}(\hat{P}_{11}^2)<0$. ∎

Remark 3. The matrix $\hat{P}_{11}=T_m^TPT_m$ is, as we shall see, of pivotal importance in ascertaining the stability/instability of the perturbed multidegree of freedom system. The m orthonormal columns of T_m are the eigenvectors of K corresponding to the multiple eigenvalue λ_0 of multiplicity m, and they form a subspace. The matrix $\hat{P}_{11}$ then represents the work done by the perturbational forces for displacements in the subspace spanned by the columns of the matrix T_m. Any displacement vector in this subspace can be represented by $x_m=T_m\chi$ and the work done by the perturbation force, Px_m, is then $\chi^T\hat{P}_{11}\chi$. ∎

Remark 4. If the matrix T_m can be partitioned such that $T_m=[T_p|T_{m-p}]$, $2\le p\le m$, $T_p^TP\begin{bmatrix}T_{m-p} & T_{n-m}\end{bmatrix}=0$, and the $p\times p$ matrix $T_p^TPT_p$ has at least one pair of complex conjugate eigenvalues with nonzero imaginary parts, then the condition (b) of Result 2 is satisfied. Moreover, in this case instability follows for every ε, infinitesimal or finite, as shown in Ref. [1]. ∎

The following example illustrates the applicability of Result 2.
Example 1. Consider system (4) with

$$K=\begin{bmatrix}2 & -1 & 0 & -1\\ -1 & 2 & 0 & 1\\ 0 & 0 & 1 & 0\\ -1 & 1 & 0 & 2\end{bmatrix}\quad\text{and}$$
$$P=\begin{bmatrix}8 & a+b-4 & a-2b+4 & -a-b-3\\ a-b & 2a+2 & -b-4 & -a-b-2\\ a+2b+2 & b-2 & 3-2a & a+b-2\\ b-a-1 & b-a & a-b-4 & 2\end{bmatrix} \tag{21}$$

where a and b are real numbers.

The matrix K has the following eigenvalues and corresponding mutually orthogonal eigenvectors:

$$t_1=\frac{1}{\sqrt{3}}[1\quad 0\quad 1\quad 1]^T,\quad \lambda_{1,2,3}=\lambda_0=1,$$
$$t_2=\frac{1}{\sqrt{3}}[1\quad 1\quad -1\quad 0]^T,\quad t_3=\frac{1}{\sqrt{3}}[0\quad 1\quad 1\quad -1]^T,$$
$$\lambda_4=4,\quad t_4=\frac{1}{\sqrt{3}}[1\quad -1\quad 0\quad -1]^T$$

For this system of eigenvectors, we have

$$\hat{P}_{11}=\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}^T[P]\begin{bmatrix}t_1 & t_2 & t_3\end{bmatrix}=3\begin{bmatrix}1 & b & 0\\ -b & 1 & a\\ 0 & a & 1\end{bmatrix}\tag{22}$$

Now, it is easy to calculate that the matrix $\overset{(\hat{P}_{11})}{H}$ associated with (22) is as follows:

$$\overset{(\hat{P}_{11})}{H}=\begin{bmatrix}h_0 & h_1 & h_2\\ h_1 & h_2 & h_3\\ h_2 & h_3 & h_4\end{bmatrix}$$

with $h_0=3$, $h_1=9$, $h_2=9(3+2(a^2-b^2))$, $h_3=81(1+2(a^2-b^2))$, $h_4=81(3+12(a^2-b^2)+2(a^2-b^2)^2)$, and its leading principal minors are: $D_1=3$, $D_2=6(a^2-b^2)$ and $D_3=108(a^2-b^2)^3$. From this, in view of the Sylvester's criterion, $\overset{(\hat{P}_{11})}{H}>0$ if $a^2-b^2>0$. Conversely, if $a^2-b^2<0$, $\overset{(\hat{P}_{11})}{H}$ is indefinite, and in the case $a^2=b^2$ the matrix $\overset{(\hat{P}_{11})}{H}$ is positive semi-definite, since all its principal minors are non-negative. Thus, according to Result 2, if $|a|>|b|$ system (4), (21) remains stable for sufficiently small ε, and if $|a|<|b|$ the system becomes unstable by flutter for arbitrarily nonzero ε. ■

Result 2 leaves open the question of what happens when the matrix $\overset{(\hat{P}_{11})}{H}$ is positive semi-definite, i.e., when the matrix $\hat{P}_{11}$ with the real spectrum has a repeated eigenvalue.

Let p_i, $i=1,\ \ldots,\ m$, be real eigenvalues of the $m\times m$ matrix $\hat{P}_{11}=T_m^TPT_m$. Suppose, for convenience, that the eigenvalues p_i are ordered in such a way that $p_1=p_2=\cdots=p_s=p_0$, $s\geq 2$, and that the other eigenvalues $p_{s+1},\ldots,\ p_m$ are simple. Substituting $\beta_0=p_0$ into (16) and (17), taking into account (18) and (19), we get the chain of equations for the unknowns $\beta_1,\beta_2,\ldots$ and $\bar{w}_0,\ \bar{w}_1\ldots$

$$\begin{aligned}&(\hat{P}_{11}-p_0I_m)\bar{w}_0=0\\ &(\hat{P}_{11}-p_0I_m)\bar{w}_k=\sum_{j=1}^{k}\beta_j\bar{w}_{k-j},\quad k=1,\ldots,r-1\\ &(\hat{P}_{11}-p_0I_m)\bar{w}_r+R\bar{w}_0=\sum_{j=1}^{r}\beta_j\bar{w}_{r-j}\end{aligned}\tag{23}$$

$$\begin{aligned}&(\hat{P}_{11}-p_0I_m)\bar{w}_{r+k}+R\bar{w}_k=\sum_{j=1}^{r+k}\beta_j\bar{w}_{r+k-j},\quad k=1,\ldots,r-1\\ &(\hat{P}_{11}-p_0I_m)\bar{w}_{2r}+R\bar{w}_r+Q\bar{w}_0=\sum_{j=1}^{2r}\beta_j\bar{w}_{2r-j}\\ &\vdots\\ &\text{with}\quad R=-\hat{P}_{12}D\hat{P}_{21},\quad Q=\hat{P}_{12}D(\hat{P}_{22}-p_0I_{n-m})D\hat{P}_{21}\end{aligned}\tag{24}$$

In what follows, we consider two important cases when p_0 is either nonderogatory or semi-simple eigenvalue of $\hat{P}_{11}$. The case of a multiple eigenvalue with an arbitrary Jordan structure is possible, but it is usually very rare in real-world settings [5] and it is not considered here.

(**i**) *Nonderogatory eigenvalue* p_0

Suppose that the eigenvalue p_0 of $\hat{P}_{11}$ with algebraic multiplicity $s\geq 2$ is nonderogatory, i.e., it has geometric multiplicity 1. The Jordan chains consisting of the right and left eigenvectors $\bar{w}_0$ and $\bar{v}_0$, and associated vectors $u_1,\ldots,\ u_{s-1}$ and $v_1,\ldots,\ v_{s-1}$ satisfy the equations

$$\begin{aligned}&(\hat{P}_{11}-p_0I_m)\bar{w}_0=0, && (\hat{P}_{11}^T-p_0I_m)\bar{v}_0=0\\ &(\hat{P}_{11}-p_0I_m)u_1=\bar{w}_0, && (\hat{P}_{11}^T-p_0I_m)v_1=\bar{v}_0\\ &\vdots && \vdots\\ &(\hat{P}_{11}-p_0I_m)u_{s-1}=u_{s-2}, && (\hat{P}_{11}^T-p_0I_m)v_{s-1}=v_{s-2}\end{aligned}\tag{25}$$

and the normalization conditions

$$\bar{v}_0^Tu_{s-1}=1,\quad v_1^Tu_{s-1}=\cdots=v_{s-1}^Tu_{s-1}=0\tag{26}$$

Substituting $r=s$ into (23) and proceeding similarly to [5, Section 2.7], from the first $(s+1)$ equations of (23), taking into account (25) and (26), and assuming that

$$v_{s-1}^T\bar{w}_0=1,\quad v_{s-1}^T\bar{w}=1\tag{27}$$

we find

$$\beta_1^s=c,\quad c=\bar{v}_0^TR\bar{w}_0\tag{28}$$

If $c\neq 0$, then $\beta_1=\sqrt[s]{c}$ has s different nonzero values, and between them, when $s>2$, there is at least one complex conjugate pair with nonzero imaginary part. Consequently, in this case the matrix $\hat{\Lambda}+\varepsilon\hat{P}$ has at least one complex conjugate pair of eigenvalues of the form

$$\mu(\varepsilon)=\lambda_0+\varepsilon p_0+\varepsilon^{1+(1/s)}(\delta\pm i\nu)+o(\varepsilon^{1+(1/s)}),\quad \delta\in\mathfrak{R},\ \nu>0\tag{29}$$

If $s=2$ in (28) and $c<0$, we have

$$\mu(\varepsilon)=\lambda_0+\varepsilon p_0\pm i\sqrt{|c|}\varepsilon^{1+(1/2)}+o(\varepsilon^{1+(1/2)})\tag{30}$$

while the case $c>0$ produces two different real eigenvalues

$$\mu(\varepsilon)=\lambda_0+\varepsilon p_0\pm\sqrt{c}\varepsilon^{1+(1/2)}+o(\varepsilon^{1+(1/2)})\tag{31}$$

Note that to check the condition $c\neq 0$ we can use any right and left eigenvector corresponding to the eigenvalue p_0, not just normalized ones. This, taking into account Lemma 3, leads to our next result.

Result 3. Suppose that the m by m matrix $\hat{P}_{11}$ has a real eigenvalue p_0 of algebraic multiplicity $s\geq 2$ and geometric multiplicity 1, and the rest of the eigenvalues are real and distinct.

(a) If $s>2$ and $\bar{v}_0^TR\bar{w}_0\neq 0$, where the m by m matrix R is given in Eq. (24) and $\bar{w}_0$ ($\bar{v}_0$) is a right (left) eigenvector of $\hat{P}_{11}$ corresponding to p_0, then system (4) becomes unstable by flutter for arbitrarily small nonzero values of ε.

(b) Let $s=2$ and let $\bar{w}_0$, u_1 ($\bar{v}_0$, v_1) be the right (left) Jordan chain corresponding to p_0 normalized as $v_1^T\bar{w}_0=\bar{v}_0^Tu_1=1$, then
  (b-1) If $\bar{v}_0^TR\bar{w}_0<0$, the system becomes unstable by flutter for arbitrarily small nonzero values of ε;
  (b-2) If $\bar{v}_0^TR\bar{w}_0>0$, the system remains stable for sufficiently small values of ε.

■

Example 2. Consider Example 1 when $|b|=|a|\neq 0$.

We first consider the case when $b=a\neq 0$. For this case the matrix (22) becomes

$$\hat{P}_{11}=3\begin{bmatrix}1 & a & 0\\ -a & 1 & a\\ 0 & a & 1\end{bmatrix}\tag{32}$$

which has the triple nonderogatory eigenvalue $p_0=3$ with right and left eigenvectors $\bar{w}_0=[1\quad 0\quad 1]^T$ and $\bar{v}_0=[1\quad 0\quad -1]^T$, respectively. On the other hand, we find that

$D=\frac{1}{3}I_1$, $\hat{P}_{12}=[\,t_1\quad t_2\quad t_3\,]^T[P][t_4]=3[\,2\quad 1\quad 1\,]^T$, $\hat{P}_{21}=[t_4]^T[P][\,t_1\quad t_2\quad t_3\,]=3[\,2\quad -1\quad 1\,]$, and

$$R=-\hat{P}_{12}D\hat{P}_{21}=-3\begin{bmatrix}4&-2&2\\2&-1&1\\2&-1&1\end{bmatrix}$$

so that $\bar{v}_0^TR\bar{w}_0=-9\neq 0$. Observe that $\hat{P}_{11}(a,b=-a)=[\hat{P}_{11}(a,b=a)]^T$ and, consequently, the vectors $\bar{v}_0=[\,1\quad 0\quad -1\,]^T$ and $\bar{w}_0=[\,1\quad 0\quad 1\,]^T$ are the right and left eigenvectors of the matrix $\hat{P}_{11}(a,b=-a)$, and we find again that $\bar{w}_0^TR\bar{v}_0=-9\neq 0$. Hence, according to Result 3-a, if $|b|=|a|\neq 0$, then system (4), (21) is unstable by flutter for arbitrarily small nonzero values of ε. ∎

Example 3. Let

$$\hat{\Lambda}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}\quad\text{and}\quad\hat{P}=\begin{bmatrix}1&1&1\\0&1&1\\-1&0&2\end{bmatrix}\tag{33}$$

For this example $n=3$, $\lambda_0=1$ and $m=2$. The matrix

$$\hat{P}_{11}=\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

has the double nonderogatory eigenvalue $p_0=1$ with corresponding right and left Jordan chains

$$\bar{w}_0=\begin{bmatrix}1\\0\end{bmatrix},\quad u_1=\begin{bmatrix}0\\1\end{bmatrix},\quad\text{and}\quad\bar{v}_0=\begin{bmatrix}0\\1\end{bmatrix},\quad v_1=\begin{bmatrix}1\\0\end{bmatrix}$$

which satisfy the normalization conditions $\bar{v}_0^Tu_1=v_1^T\bar{w}_0=1$. Moreover, $D=I_1$, $\hat{P}_{12}=[\,1\quad 1\,]^T$, $\hat{P}_{21}=[\,-1\quad 0\,]$ and

$$R=\begin{bmatrix}1&0\\1&0\end{bmatrix}$$

so that $\bar{v}_0^TR\bar{w}_0=1>0$. Thus, in view of Result 3-b-2, system (4), (33) is stable for sufficiently small values of ε. ∎

(ii) *Semi-simple eigenvalue* p_0

Suppose that the eigenvalue p_0 of $\hat{P}_{11}$ with algebraic multiplicity $s\geq 2$ is semi-simple, i.e., its geometric multiplicity is also s. Define m by m matrices

$$U=[\,U_s\quad U_{m-s}\,],\quad V=[\,V_s\quad V_{m-s}\,]$$

whose columns are the m right and left eigenvectors of $\hat{P}_{11}$, respectively, determined so that $V^TU=I_m$ and

$$V^T\hat{P}_{11}U=\begin{bmatrix}V_s^T\hat{P}_{11}U_s&V_s^T\hat{P}_{11}U_{m-s}\\V_{m-s}^T\hat{P}_{11}U_s&V_{m-s}^T\hat{P}_{11}U_{m-s}\end{bmatrix}=\begin{bmatrix}p_0I_s&0\\0&\bar{\Lambda}_{m-s}\end{bmatrix}$$

where $\bar{\Lambda}_{m-s}=\mathrm{diag}(p_{s+1},\ldots,p_m)$. Setting $\bar{w}_j=U[\,\bar{\bar{w}}_j^T\quad\tilde{\bar{w}}_j^T\,]^T$, where $\bar{\bar{w}}_j$ and $\tilde{\bar{w}}_j$ are s and $(m-s)$ dimensional vectors, respectively, in Eq. (23) and premultiplying these by V^T from the left, from the first r vector Eq. (23) we obtain

$$\tilde{\bar{w}}_0=\tilde{\bar{w}}_1=\cdots=\tilde{\bar{w}}_{r-1}=0\tag{34}$$

and

$$\beta_1=\beta_2=\cdots=\beta_{r-1}=0\tag{35}$$

Taking into account (34) and (35), the $(r+1)$th equation of (23) becomes

$$\begin{bmatrix}0&0\\0&\bar{\Lambda}_{m-s}-p_0I_{m-s}\end{bmatrix}\begin{bmatrix}\bar{\bar{w}}_r\\\tilde{\bar{w}}_r\end{bmatrix}+\begin{bmatrix}\hat{R}_{11}&\hat{R}_{12}\\\hat{R}_{21}&\hat{R}_{22}\end{bmatrix}\begin{bmatrix}\bar{\bar{w}}_0\\0\end{bmatrix}=\beta_r\begin{bmatrix}\bar{\bar{w}}_0\\0\end{bmatrix}\tag{36}$$

where

$$\hat{R}_{11}=V_s^TRU_s,\quad\hat{R}_{12}=V_s^TRU_{m-s},\quad\hat{R}_{21}=V_{m-s}^TRU_s,\quad\hat{R}_{22}=V_{m-s}^TRU_{m-s}\tag{37}$$

while the next r equations, again taking into account (34) and (35), have the forms

$$\begin{bmatrix}0&0\\0&\bar{\Lambda}_{m-s}-p_0I_{m-s}\end{bmatrix}\begin{bmatrix}\bar{\bar{w}}_{r+k}\\\tilde{\bar{w}}_{r+k}\end{bmatrix}+\begin{bmatrix}\hat{R}_{11}&\hat{R}_{12}\\\hat{R}_{21}&\hat{R}_{22}\end{bmatrix}\begin{bmatrix}\bar{\bar{w}}_k\\0\end{bmatrix}=\sum_{j=0}^{k}\beta_{r+j}\begin{bmatrix}\bar{\bar{w}}_{k-j}\\0\end{bmatrix},\quad k=1,\ldots,r-1\tag{38}$$

and

$$\begin{bmatrix}0&0\\0&\bar{\Lambda}_{m-s}-p_0I_{m-s}\end{bmatrix}\begin{bmatrix}\bar{\bar{w}}_{2r}\\\tilde{\bar{w}}_{2r}\end{bmatrix}+\begin{bmatrix}\hat{R}_{11}-\beta_rI_s&\hat{R}_{12}\\\hat{R}_{21}&\hat{R}_{22}-\beta_rI_{m-s}\end{bmatrix}\begin{bmatrix}\bar{\bar{w}}_r\\\tilde{\bar{w}}_r\end{bmatrix}+\begin{bmatrix}\hat{Q}_{11}&\hat{Q}_{12}\\\hat{Q}_{21}&\hat{Q}_{22}\end{bmatrix}\begin{bmatrix}\bar{\bar{w}}_0\\0\end{bmatrix}=\sum_{j=1}^{r}\beta_{r+j}\begin{bmatrix}\bar{\bar{w}}_{r-j}\\0\end{bmatrix}\tag{39}$$

with

$$\hat{Q}_{11}=V_s^TQU_s,\quad\hat{Q}_{12}=V_s^TQU_{m-s},\quad\hat{Q}_{21}=V_{m-s}^TQU_s,\quad\hat{Q}_{22}=V_{m-s}^TQU_{m-s}\tag{40}$$

It follows from (36) that

$$(\hat{R}_{11}-\beta_rI_s)\bar{\bar{w}}_0=0,\quad\tilde{\bar{w}}_r=-\bar{D}\hat{R}_{21}\bar{\bar{w}}_0\tag{41}$$

where

$$\bar{D}=(\bar{\Lambda}_{m-s}-p_0I_{m-s})^{-1}\tag{42}$$

and, consequently, β_r is an eigenvalue of the s by s matrix $\hat{R}_{11}=V_s^TRU_s$. If this matrix has all real distinct eigenvalues ρ_i, $i=1,\ldots,s$, then when $\varepsilon\to 0$, the matrix $\hat{\Lambda}+\varepsilon\hat{P}$ (and, of course, $K+\varepsilon P$) has m real distinct eigenvalues of the forms

$$\mu_i(\varepsilon)=\lambda_0+\varepsilon p_0+\varepsilon^2\rho_i+o(\varepsilon^2),\quad i=1,\ldots,s\tag{43}$$

and

$$\mu_i(\varepsilon)=\lambda_0+\varepsilon p_i+o(\varepsilon),\quad i=s+1,\ldots,m\tag{44}$$

If the matrix $\hat{R}_{11}$ has at least one pair of eigenvalues of the form $\delta\pm i\nu$, with $\nu>0$, then the matrix $\hat{\Lambda}+\varepsilon\hat{P}$ has at least one complex conjugate pair of eigenvalues of the form

$$\mu(\varepsilon)=\lambda_0+p_0\varepsilon+(\delta\pm i\nu)\varepsilon^2+o(\varepsilon^2)\tag{45}$$

Thus, according to Lemmas 3 and 4, we get the following result.

Result 4. Suppose that the m by m matrix $\hat{P}_{11}$ has a real eigenvalue p_0 of algebraic multiplicity $s\geq 2$ and geometric multiplicity s, the rest of the eigenvalues being real and distinct. Consider the matrix $\hat{R}_{11}=V_s^TRU_s$ given in Eq. (37) in which the columns of the m by s matrix U_s (V_s) are right (left) eigenvectors of $\hat{P}_{11}$ corresponding to the multiple eigenvalue p_0. Then

(a) the system described by Eq. (4) remains stable for sufficiently small values of ε if the matrix $\overset{(\hat{R}_{11})}{H}$ associated with $\hat{R}_{11} = V_s^T R U_s$ is positive definite;

(b) the system described by Eq. (4) becomes unstable by flutter for arbitrarily small nonzero values of ε if the matrix $\overset{(\hat{R}_{11})}{H}$ associated with $\hat{R}_{11} = V_s^T R U_s$ is indefinite. ■

COROLLARY 2. *[16]. If $P^T = -P$ and $\hat{P}_{11} = 0$, then the system (4) remains stable for sufficiently small values of ε when all eigenvalues of the symmetric matrix $R = \hat{P}_{12} D \hat{P}_{12}^T$ are distinct.*

Proof. Clearly, in this case $p_0 = 0$ is an m-fold semi-simple eigenvalue of $\hat{P}_{11}$, $\hat{P}_{21} = -\hat{P}_{12}^T$, and consequently $\hat{R}_{11} = R = \hat{P}_{12} D \hat{P}_{12}^T$. ■

Example 4. Let

$$\hat{\Lambda} = \mathrm{diag}(1,\ 1,\ 1,\ 1,\ 2,\ 3) \quad \text{and}$$

$$\hat{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & -4 & 0 & 1 & 0 & -3 \\ 2 & 2 & 1 & -1 & 3 & 2 \end{bmatrix} \tag{46}$$

For this example $n = 6$, $\lambda_0 = 1$, $m = 4$, $\hat{P}_{11} = \mathrm{diag}(1,\ 1,\ 1,\ 2)$, i.e., $p_0 = 1$, $s = 3$, and

$$\hat{P}_{12}^T = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{bmatrix}, \quad \hat{P}_{21} = \begin{bmatrix} 0 & -4 & 0 & 1 \\ 2 & 2 & 1 & -1 \end{bmatrix}$$

Now, taking into account that $D = \mathrm{diag}(1,\ 1/2)$, we have

$$R = -\hat{P}_{12} D \hat{P}_{21} = -\frac{1}{2} \begin{bmatrix} 4 & -20 & 2 & 4 \\ 4 & 4 & 2 & -2 \\ 4 & -4 & 2 & 0 \\ 2 & 2 & 1 & -1 \end{bmatrix}$$

from which follows

$$\hat{R}_{11} = -\begin{bmatrix} 2 & -10 & 1 \\ 2 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

The Hankel matrix $\overset{(\hat{R}_{11})}{H}$ associated with $\hat{R}_{11}$ is indefinite because its diagonal element $h_2 = \mathrm{Tr}(\hat{R}_{11}^2) = -31$ is negative, and, in view of Result 4-b, instability of system (4), (46) follows for arbitrarily small nonzero values of ε. ■

We next consider the case when the s by s matrix $\hat{R}_{11}$ has a real eigenvalue ρ_0 of algebraic multiplicity l ($\rho_1 = \cdots = \rho_l = \rho_0$) that is either nonderogatory or semi-simple, with its other eigenvalues $\rho_{l+1}, \ldots, \rho_s$ being real and distinct.

Putting $\beta_r = \rho_0$ in Eqs. (36), (38), and (39), and assuming that the integer $r \le l$, we find the following equations with the unknowns $\beta_{r+1}, \beta_{r+2}, \ldots$ and $\bar{\bar{w}}_0, \bar{\bar{w}}_1 \ldots$

$$(\hat{R}_{11} - \rho_0 I_s)\bar{\bar{w}}_0 = 0 \tag{47}$$

$$(\hat{R}_{11} - \rho_0 I_s)\bar{\bar{w}}_k = \sum_{j=1}^{k} \beta_{r+j} \bar{\bar{w}}_{k-j}, \quad k = 1, \ldots, r-1 \tag{48}$$

and, taking into account the second equation of (41),

$$(\hat{R}_{11} - \rho_0 I_s)\bar{\bar{w}}_r + L\bar{\bar{w}}_0 = \sum_{j=1}^{r} \beta_{r+j} \bar{\bar{w}}_{r-j} \tag{49}$$

where the $s \times s$ matrix L is determined as

$$L = \hat{Q}_{11} - \hat{R}_{12} \bar{D} \hat{R}_{21} \tag{50}$$

The matrices $\hat{R}_{12}$, $\hat{R}_{21}$, $\hat{Q}_{11}$, and $\bar{D}$ are given in Eqs. (37), (40), and (42).

(ii-1) *Nonderogatory eigenvalue ρ_0*

Let $\bar{\bar{w}}_0$ and $\bar{\bar{v}}_0$ be right and left eigenvectors corresponding to the l-fold nonderogatory eigenvalue ρ_0 of the matrix $\hat{R}_{11}$. Substituting $r = l$ into (48) and (49), and working through in the same manner as in subsection (i), for the case $l > 2$ when $\bar{\bar{v}}_0^T L \bar{\bar{w}}_0 \neq 0$, we obtain that the matrix $\hat{\Lambda} + \varepsilon \hat{P}$ has at least one complex conjugate pair of eigenvalues of the form

$$\mu(\varepsilon) = \lambda_0 + \varepsilon p_0 + \varepsilon^2 \rho_0 + \varepsilon^{2+(1/l)}(\delta \pm i\nu) + o(\varepsilon^{2+(1/l)}), \quad \delta \in \Re,\ \nu > 0 \tag{51}$$

On the other hand, in the case $l = 2$ assuming that the right and left Jordan chains ($\bar{\bar{w}}_0$, u_1) and ($\bar{\bar{v}}_0$, v_1) corresponding to ρ_0 satisfy the normalization conditions $v_1^T \bar{\bar{w}}_0 = \bar{\bar{v}}_0^T u_1 = 1$, we see that if $c_1 = \bar{\bar{v}}_0^T L \bar{\bar{w}}_0 < 0$, then the matrix $\hat{\Lambda} + \varepsilon \hat{P}$ has one complex conjugate pair of eigenvalues of the form

$$\mu(\varepsilon) = \lambda_0 + \varepsilon p_0 + \varepsilon^2 \rho_0 \pm i\sqrt{|c_1|}\varepsilon^{2+(1/2)} + o(\varepsilon^{2+(1/2)}) \tag{52}$$

while if $c_1 > 0$, then

$$\mu(\varepsilon) = \lambda_0 + \varepsilon p_0 + \varepsilon^2 \rho_0 \pm \sqrt{c_1}\varepsilon^{2+(1/2)} + o(\varepsilon^{2+(1/2)}) \tag{53}$$

This leads to the following result.

Result 5. Suppose that the s by s matrix $\hat{R}_{11}$ has a real eigenvalue ρ_0 of algebraic multiplicity $l \ge 2$ and geometric multiplicity 1, the rest of the eigenvalues being real and distinct.

(a) If $l > 2$ and $\bar{\bar{v}}_0^T L \bar{\bar{w}}_0 \neq 0$, where the matrix L is given in Eq. (50), and $\bar{\bar{w}}_0$ ($\bar{\bar{v}}_0$) is a right (left) eigenvector of $\hat{R}_{11}$ corresponding to ρ_0, then system (4) becomes unstable by flutter for arbitrarily small nonzero values of ε.

(b) If $l = 2$ and $\bar{\bar{w}}_0, u_1$ ($\bar{\bar{v}}_0, v_1$) be the right (left) Jordan chain corresponding to ρ_0 normalized as $v_1^T \bar{\bar{w}}_0 = \bar{\bar{v}}_0^T u_1 = 1$, then

(b-1) if $\bar{\bar{v}}_0^T L \bar{\bar{w}}_0 < 0$, the system becomes unstable by flutter for arbitrarily small nonzero values of ε, and

(b-2) if $\bar{\bar{v}}_0^T L \bar{\bar{w}}_0 > 0$, the system remains stable for sufficiently small values of ε.

(ii-2) *Semi-simple eigenvalue ρ_0*

Let ρ_0 be a semi-simple real eigenvalue of multiplicity $l \ge 2$ of the s by s matrix $\hat{R}_{11}$ and let its other eigenvalues be real and simple. Define s by s matrices

$$\tilde{U} = [\,\tilde{U}_l \quad \tilde{U}_{s-l}\,], \quad \tilde{V} = [\,\tilde{V}_l \quad \tilde{V}_{s-l}\,]$$

whose columns are the s right and left eigenvectors of $\hat{R}_{11}$ respectively, such that $\tilde{V}^T \tilde{U} = I_s$ and

$$\tilde{V}^T \hat{R}_{11} \tilde{U} = \begin{bmatrix} \tilde{V}_l^T \hat{R}_{11} \tilde{U}_l & \tilde{V}_l^T \hat{R}_{11} \tilde{U}_{s-l} \\ \tilde{V}_{s-l}^T \hat{R}_{11} \tilde{U}_l & \tilde{V}_{s-l}^T \hat{R}_{11} \tilde{U}_{s-l} \end{bmatrix} = \begin{bmatrix} \rho_0 I_l & 0 \\ 0 & \bar{\bar{\Lambda}}_{s-l} \end{bmatrix}$$

where $\bar{\bar{\Lambda}}_{s-l} = \mathrm{diag}(\rho_{l+1}, \ldots, \rho_s)$. Setting $\bar{\bar{w}}_j = \tilde{U}[\,\bar{\bar{\bar{w}}}_j^T \quad \tilde{\bar{\bar{w}}}_j^T\,]^T$, where $\bar{\bar{\bar{w}}}_j$ and $\tilde{\bar{\bar{w}}}_j$ are l and $(s - l)$ dimensional vectors, respectively, in Eqs. (47), (48) and (49), and premultiplyng these by $\tilde{V}^T$ from the left, from (47) and (48) we see that

$$\tilde{\bar{\bar{w}}}_0 = \tilde{\bar{\bar{w}}}_1 = \cdots = \tilde{\bar{\bar{w}}}_{r-1} = 0 \tag{54}$$

and

$$\beta_{r+1} = \beta_{r+2} = \cdots = \beta_{2r-1} = 0 \tag{55}$$

while Eq. (49), taking into account (54) and (55), gives

$$(\hat{L}_{11} - \beta_{2r} I_l)\bar{\bar{w}}_0 = 0, \quad \bar{\bar{w}}_r = -\bar{\bar{D}}\hat{L}_{21}\bar{\bar{w}}_0 \tag{56}$$

where

$$\hat{L}_{11} = \tilde{V}_l^T L\tilde{U}_l, \quad \hat{L}_{21} = \tilde{V}_{s-l}^T L\tilde{U}_l, \quad \bar{\bar{D}} = (\bar{\bar{\Lambda}}_{s-l} - \rho_0 I_{s-l})^{-1} \tag{57}$$

It follows from the first equation of (56) that β_{2r} is an eigenvalue of the l by l matrix $\hat{L}_{11} = \tilde{V}_l^T L\tilde{U}_l$. If this matrix has all real distinct eigenvalues σ_i, $i = 1, \ldots, l$, then when $\varepsilon \to 0$, the matrix $\hat{\Lambda} + \varepsilon\hat{P}$ has m real distinct eigenvalues of the forms

$$\mu_i(\varepsilon) = \lambda_0 + \varepsilon p_0 + \varepsilon^2\rho_0 + \varepsilon^3\sigma_i + o(\varepsilon^3), \quad i = 1, \ldots, l$$

$$\mu_i(\varepsilon) = \lambda_0 + \varepsilon p_0 + \varepsilon^2\rho_i + o(\varepsilon^2), \quad i = l+1, \ldots, s$$

$$\mu_i(\varepsilon) = \lambda_0 + \varepsilon p_i + o(\varepsilon), \quad i = s+1, \ldots, m$$

If the matrix $\hat{L}_{11}$ has at least one pair of eigenvalues of the form $\delta \pm i\nu$, with $\nu > 0$, then the matrix $\hat{\Lambda} + \varepsilon\hat{P}$ has at least one complex conjugate pair of eigenvalues of the form

$$\mu(\varepsilon) = \lambda_0 + p_0\varepsilon + \rho_0\varepsilon^2 + (\delta \pm i\nu)\varepsilon^3 + o(\varepsilon^3)$$

Thus, according to Lemmas 3 and 4, we have the following result.

Result 6. Suppose that the s by s matrix $\hat{R}_{11}$ has a real semi-simple eigenvalue ρ_0 of multiplicity $l \geq 2$, the rest of the eigenvalues being real and distinct. Consider the l by l matrix $\hat{L}_{11} = \tilde{V}_l^T L\tilde{U}_l$ given in Eq. (57) in which the columns of the s by l matrix $\tilde{U}_l(\tilde{V}_l)$ are right (left) eigenvectors of $\hat{R}_{11}$ corresponding to the multiple eigenvalue ρ_0. Then,

(a) the system described by Eq. (4) remains stable for sufficiently small values of ε if the matrix $\overset{(\hat{L}_{11})}{H}$ associated with $\hat{L}_{11}$ is positive definite;
(b) the system described by Eq. (4) becomes unstable by flutter for arbitrarily small nonzero values of ε if the matrix $\overset{(\hat{L}_{11})}{H}$ associated with $\hat{L}_{11}$ is indefinite. ■

COROLLARY 3. *[16]. Suppose that $P^T = -P$ and $\hat{P}_{11} = 0$. If the symmetric* m × m *matrix $R = \hat{P}_{12}D\hat{P}_{12}^T$ has an eigenvalue ρ_0 of multiplicity $l \geq 2$ and $\tilde{T}_l^T\hat{P}_{12}D\hat{P}_{22}D\hat{P}_{12}^T\tilde{T}_l \neq 0$, where the columns of the m×l matrix $\tilde{T}_l$ are orthonormal eigenvectors of R corresponding to the multiple eigenvalue ρ_0, then system (4) becomes unstable by flutter for arbitrarily small nonzero values of ε.*

Proof. Obviously, in this case $\hat{R}_{11} = R$ and $\hat{L}_{11} = -\tilde{T}_l^T\hat{P}_{12}D\hat{P}_{22}D\hat{P}_{12}^T\tilde{T}_l$. Then instability follows from Result 6-b because $\hat{L}_{11}$ is an l by l nonzero skew-symmetric matrix. ■

If the matrix $\hat{L}_{11}$ has a real eigenvalue of multiplicity $g \geq 2$ and the other eigenvalues are real and simple, the procedure can be continued in the same manner as above. Particularly, we state the following two results without proof.

Result 7. Consider the matrices $\hat{P}_{12}$, $\hat{P}_{21}$, and D given in Eqs. (10) and (20), and suppose that the m by m matrices $\hat{P}_{11}$ and $R = -\hat{P}_{12}D\hat{P}_{21}$, have real semi-simple eigenvalues p_0 and ρ_0 of multiplicity $m \geq 2$ and $l \geq 2$, respectively, with the rest of the eigenvalues of R, $\rho_{l+1}, \ldots, \rho_m$, being real and distinct. Let $\tilde{U} = [\,\tilde{U}_l \quad \tilde{U}_{m-l}\,]$, $\tilde{V} = [\,\tilde{V}_l \quad \tilde{V}_{m-l}\,]$ be the matrices of right and left eigenvectors of R, respectively, such that $\tilde{V}^T\tilde{U} = I_m$ and

$$\tilde{V}^T R\tilde{U} = \begin{bmatrix} \rho_0 I_l & 0 \\ 0 & \bar{\bar{\Lambda}}_{m-l} \end{bmatrix}$$

where $\bar{\bar{\Lambda}}_{m-l} = \mathrm{diag}(\rho_{l+1} \ldots, \rho_m)$. Define the $m \times m$ matrices

$$Q = \hat{P}_{12}D(\hat{P}_{22} - p_0 I_{n-m})D\hat{P}_{21},$$
$$B = \hat{P}_{12}[D(\hat{P}_{22} - p_0 I_{n-m})]^2 D\hat{P}_{21} + \rho_0\hat{P}_{12}D^2\hat{P}_{21} \tag{58}$$

where $\hat{P}_{22}$ is determined in Eq. (10), as well as the l by l matrices

$$\hat{Q}_{11} = \tilde{V}_l^T Q\tilde{U}_l, \quad \Omega = \hat{Q}_{12}\bar{\bar{D}}\hat{Q}_{21} + \hat{B}_{11} \tag{59}$$

with

$$\hat{Q}_{12} = \tilde{V}_l^T Q\tilde{U}_{m-l}, \quad \hat{Q}_{21} = \tilde{V}_{m-l}^T Q\tilde{U}_l, \quad \bar{\bar{D}} = (\bar{\bar{\Lambda}}_{m-l} - \rho_0 I_{m-l})^{-1},$$
$$\hat{B}_{11} = \tilde{V}_l^T B\tilde{U}_l \tag{60}$$

(a) Suppose that the matrix $\hat{Q}_{11}$ has a real eigenvalue q_0 of algebraic multiplicity $g \geq 2$ and geometric multiplicity 1, and its other eigenvalues are real and distinct. Then
 (a-1) If $g > 2$ and $\eta_0^T\Omega\xi_0 \neq 0$, where ξ_0 (η_0) is a right (left) eigenvector of $\hat{Q}_{11}$ corresponding to q_0, then system (4) becomes unstable by flutter for arbitrarily small nonzero values of ε.
 (a-2) If $g = 2$ and ξ_0, ξ_1 (η_0, η_1) be the right (left) Jordan chain corresponding to q_0 normalized as $\eta_1^T\xi_0 = \eta_0^T\xi_1 = 1$, then
 (a-2–1) if $\eta_0^T\Omega\xi_0 > 0$, the system becomes unstable by flutter for arbitrarily small nonzero values of ε, and
 (a-2-2) if $\eta_0^T\Omega\xi_0 < 0$, the system remains stable for sufficiently small values of ε.
(b) Suppose that the matrix $\hat{Q}_{11}$ has a real semi-simple eigenvalue q_0 of multiplicity $g \geq 2$, and its other eigenvalues are real and distinct. Then,
 (b-1) the system described by Eq. (4) remains stable for sufficiently small values of ε if the matrix $\overset{(\hat{\Omega}_{11})}{H}$ associated with the $g \times g$ matrix $\hat{\Omega}_{11} = \tilde{\tilde{V}}_g^T\Omega\tilde{\tilde{U}}_g$, where the columns of the l by g matrix $\tilde{\tilde{U}}_g$ ($\tilde{\tilde{V}}_g$) are right (left) eigenvectors of $\hat{Q}_{11}$ corresponding to the multiple eigenvalue q_0, is positive definite;
 (b-2) the system described by Eq. (4) becomes unstable by flutter for arbitrarily small nonzero values of ε if $\overset{(\hat{\Omega}_{11})}{H}$ is indefinite. ■

Result 8. If $m = n - 1$ and the matrix $\hat{P}_{11}$ has the real semi-simple eigenvalue p_0 of multiplicity m, then system (4) remains stable for sufficiently small values of ε. ■

Remark 5. In the case when $P = -P^T$, the Results 7 and 8 reduce to Result 2.11 and Corollary 2.16 of [16], respectively.

Let us go back to Example 1 and consider the unsolved case when $a = b = 0$. In this case matrix (22) becomes $\hat{P}_{11} = 3I_3$, and according to Result 8, the system of this example remains stable for sufficiently small ε. Indeed, it is easy to confirm that in this case the matrix

$$\hat{\Lambda} + \varepsilon\hat{P} = \begin{bmatrix} 1+3\varepsilon & 0 & 0 & 6\varepsilon \\ 0 & 1+3\varepsilon & 0 & 3\varepsilon \\ 0 & 0 & 1+3\varepsilon & 3\varepsilon \\ 3\varepsilon & -3\varepsilon & 3\varepsilon & 4+6\varepsilon \end{bmatrix}$$

has eigenvalues

$$\mu_1 = \mu_2 = 1 + 3\varepsilon, \quad \mu_{3,4} = (5 + 9\varepsilon \pm 3\sqrt{1 + 2\varepsilon + 9\varepsilon^2})/2$$

which all are real and positive for $\varepsilon < 16/9$. On the other hand, the double eigenvalue $(1 + 3\varepsilon)$ is semi-simple, since it has the following two linearly independent eigenvectors $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T$, and, consequently, the system is stable. ■

Example 5. Let

$$\hat{\Lambda} = \mathrm{diag}(1, 1, 2, 3) \quad \text{and} \quad \hat{P} = \begin{bmatrix} 1 & 0 & 2 & a \\ 0 & 1 & 0 & 1 \\ 1 & -a & 0 & 1 \\ 0 & 4 & b & 0 \end{bmatrix} \tag{61}$$

where a and b are real numbers.

For this example $\hat{P}_{11} = \mathrm{diag}(1, 1)$ and

$$\hat{P}_{12} = \begin{bmatrix} 2 & a \\ 0 & 1 \end{bmatrix}, \quad \hat{P}_{21} = \begin{bmatrix} 1 & -a \\ 0 & 4 \end{bmatrix}, \quad \hat{P}_{22} = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix}, \quad D = \mathrm{diag}(1, 1/2) \tag{62}$$

Obviously, $p_0 = 1$ is a double semi-simple eigenvalue of the 2 by 2 matrix $\hat{P}_{11}$ and, consequently, $\hat{R}_{11} = R = -\hat{P}_{12}D\hat{P}_{21} = -2I_2$, which has only one double semi-simple eigenvalue $\rho_0 = -2$. Now, it is clear that $\hat{L}_{11} = L = \hat{Q}_{11} = Q = \hat{P}_{12}D(\hat{P}_{22} - p_0 I_2)D\hat{P}_{21}$ so, taking into account (59), we obtain

$$\hat{L}_{11} = \frac{1}{2}\begin{bmatrix} ab - 4 & 8 + 2a - a^2 b \\ b & -2 - ab \end{bmatrix}$$

The discriminant of the characteristic equation of the above matrix is $\delta = 4(1 + 8b)$, and, according to Result 6 (see also Remark 2), we conclude that: (1) if $b > -1/8$, then system (4), (61) remains stable for sufficiently small ε, and (2) if $b < -1/8$, then system (4), (61) becomes unstable by flutter for arbitrarily small nonzero values of ε. When $b = -1/8$, the matrix $\hat{L}_{11}$ becomes

$$\hat{L}_{11} = \hat{Q}_{11} = Q = \frac{1}{16}\begin{bmatrix} -a - 32 & (a + 8)^2 \\ -1 & a - 16 \end{bmatrix}$$

and it has the double nonderogatory eigenvalue $q_0 = -3/2$ with corresponding right and left Jordan chains

$$\xi_0 = \begin{bmatrix} a + 8 \\ 1 \end{bmatrix}, \quad \xi_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \eta_0 = \begin{bmatrix} -1 \\ a + 8 \end{bmatrix}, \quad \eta_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which satisfy the normalization conditions $\eta_1^T \xi_0 = \eta_0^T \xi_1 = 1$. Moreover, the matrix Ω defined in Eq. (59) for this case has the form

$$\Omega = \frac{1}{32}\begin{bmatrix} 188 + 3a & -192 - 112a - 3a^2 \\ 3 & 76 - 3a \end{bmatrix}$$

and $\eta_0^T \Omega \xi_0 = -16 < 0$. Hence, according to Result 7-a-2-2, in the case $b = -1/8$ system (4), (61) remains stable for sufficiently small ε. ■

4 Application to an Engineering Problem

While the theory developed here is quite broad and can be applied to numerous areas of application in civil, aerospace, and mechanical engineering, in this section we present a simple four degree-of-freedom linear system shown schematically in Fig. 1. The directions of the coordinates describing the motion of the two equal masses m are as shown. The stiffness of the linear springs in the x-direction is k_1 and the stiffness of the linear springs in the y-direction is k_2. The belts on the left, right, and bottom, move on rollers and have a constant velocity, v. The coefficient of dry friction between each of the belts and the snubbers that rub against them is assumed to be ε.

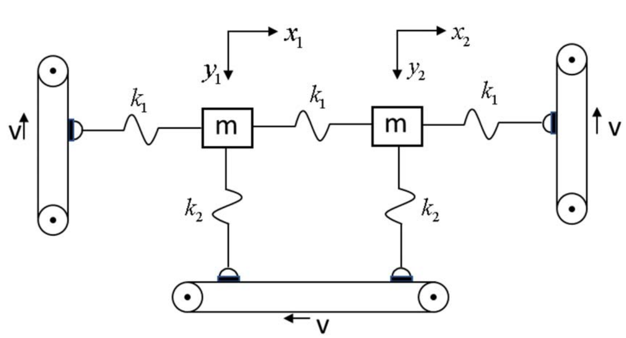

Fig. 1 A four degree-of-freedom system. The coefficient of friction between the belts and the snubbers is ε.

Denoting the four-vector $w := [x_1, x_2, y_1, y_2]^T$, the equation of motion of the system can be written as

$$\ddot{w} + Kw + \varepsilon P w = 0 \tag{63}$$

where

$$K = \begin{bmatrix} 2a & -a & 0 & 0 \\ -a & 2a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -b \\ a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \end{bmatrix} \tag{64}$$

and $a = k_1/m$ and $b = k_2/m$. We shall assume that the ratio $b/a = k_2/k_1$ is neither 1 nor 3.

The four eigenvalues of K are $\lambda_{1,2} = b$, $\lambda_3 = a$, and $\lambda_4 = 3a$. We thus see that b is a multiple eigenvalue of K, and the other eigenvalues are distinct and, by assumption, different from b. The unperturbed system ($\varepsilon = 0$) is stable because $a, b > 0$.

In the presence of the perturbing matrix εP (see Eq. (63)), the multiple eigenvalue b of K splits. We now proceed to find out the manner in which this happens, and whether this splitting would lead to an instability. One can intuit that the changes in the eigenvalues caused by the perturbing matrix could leave the perturbed system still stable, or possibly make it unstable.

The orthogonal matrix containing the eigenvectors corresponding to the eigenvalues of K is

$$T = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

thereby making

$$\hat{\Lambda} = T^T K T = \mathrm{diag}(b, b, a, 3a) \quad \text{and} \quad \hat{P} = T^T P T = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} \tag{65}$$

where

$$\hat{P}_{11} = \hat{P}_{22} = 0, \quad \hat{P}_{12} = \frac{1}{\sqrt{2}}\begin{bmatrix} a & a \\ -a & a \end{bmatrix}, \quad \text{and} \quad \hat{P}_{21} = -\frac{1}{\sqrt{2}}\begin{bmatrix} b & b \\ b & -b \end{bmatrix} \tag{66}$$

From Eq. (20), we obtain $D = \mathrm{diag}((a-b)^{-1}, (3a-b)^{-1})$. Setting $w = Tu$, Eq. (63) can now be rewritten as

$$\ddot{u} + (\hat{\Lambda} + \varepsilon \hat{P})u = 0 \tag{67}$$

The eigenvalues of $\hat{P}_{11}$ are semi-simple, and we can directly apply Result 4 to this dynamical system. As required by this result, we therefore first evaluate

$$\hat{R}_{11} = R = -\hat{P}_{12}D\hat{P}_{21} = \frac{ab}{(b-a)(b-3a)}\begin{bmatrix} (2a - b) & a \\ -a & -(2a - b) \end{bmatrix} \tag{68}$$

and then evaluate the matrix $\overset{(\hat{R}_{11})}{H}$ associated with $\hat{R}_{11}$. Noting that $h_0 := \mathrm{Tr}(\hat{R}_{11}^0) = 2$, $h_1 := \mathrm{Tr}(\hat{R}_{11}) = 0$, and $h_2 := \mathrm{Tr}(\hat{R}_{11}^2) = 2a^2b^2/(b-a)(b-3a)$, we obtain

$$\overset{(\hat{R}_{11})}{H} = \begin{bmatrix} h_0 & h_1 \\ h_1 & h_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \dfrac{2a^2b^2}{(b-a)(b-3a)} \end{bmatrix} \tag{69}$$

Thus, using Result 4, the system described by Eq. (63) is stable for small enough values of the coefficient of friction ε when $(b-a)(b-3a) > 0$. We note that (i) the left-hand side of this inequality can be written as a quadratic in b/a whose zeros are at 1 and 3 and (ii) $b/a = k_2/k_1$. Hence, the condition for stability of the perturbed system, for sufficiently small values of ε, is $k_2/k_1 > 3$, or $0 < k_2/k_1 < 1$. The ratio k_2/k_1 then serves as our bifurcation parameter.

Furthermore, Result 4 states that the system described by Eq. (63) is guaranteed to be unstable by flutter for *arbitrarily small* nonzero values of ε when $(b-a)(b-3a) < 0$, and this occurs when $1 < k_2/k_1 < 3$. Using the analytical results provided, we have thus obtained the conditions on k_2/k_1 that guarantee either stability or flutter instability of the perturbed system described by Eq. (63).

The above result regarding instability states, for example, that when $b/a = k_2/k_1 = 2$, the system is guaranteed to be unstable by flutter for *arbitrarily small* values of the coefficient of friction ε. That is, the matrix (see Eq. (67))

$$\hat{\Lambda} + \varepsilon\hat{P}|_{b=2a} = \begin{bmatrix} 2a & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 3a \end{bmatrix} + \frac{\varepsilon}{\sqrt{2}} \begin{bmatrix} 0 & 0 & a & -a \\ 0 & 0 & -a & a \\ -2a & -2a & 0 & 0 \\ -2a & 2a & 0 & 0 \end{bmatrix} \tag{70}$$

has a pair of complex eigenvalues.

In fact, from Eq. (68) we see that when $b = 2a$, that is, $k_2/k_1 = 2$,

$$\hat{R}_{11} = \begin{bmatrix} 0 & -2a \\ 2a & 0 \end{bmatrix}$$

The eigenvalues of this matrix are $\pm 2ai$. In the presence of the perturbation matrix $\varepsilon\hat{P}$, the multiple eigenvalue $b(=2a)$ of $\hat{K}$ splits. Equation (45) provides an explicit estimate of the complex pair of eigenvalues of the matrix into which this multiple eigenvalue splits as

$$\mu(\varepsilon) = 2a + 0\varepsilon \pm 2ai\varepsilon^2 + o(\varepsilon^2) = 2\frac{k_1}{m} \pm 2\frac{k_1}{m} i\varepsilon^2 + o(\varepsilon^2) \tag{71}$$

Taking $\varepsilon = \sqrt{2} \times 10^{-2}$, and $k_1/m = 1$ for illustration, Eq. (71) yields $\mu_{1,2}(\varepsilon) \approx 2 \pm 4 \times 10^{-4} i$. Indeed, the determination of the eigenvalues of the matrix in (70) for these values of a and ε (using Maple) yields the complex pair $2 \pm 3.99999968 \times 10^{-4} i$, demonstrating the power of the analytical results obtained.

In a similar manner when $b = (1/2)a$, that is, $k_2/k_1 = 1/2$, Result 4 shows that the system remains stable. In this case,

$$\hat{R}_{11} = \frac{a}{5}\begin{bmatrix} 3 & 2 \\ -2 & -3 \end{bmatrix}$$

and the eigenvalues of this matrix are $\pm\sqrt{5}a/5$. Equation (43) then gives an explicit estimate of the split in the multiple eigenvalue $b = a/2 = k_1/2m$ of $\hat{\Lambda}$ that is caused by the additional perturbation $\varepsilon\hat{P}|_{b=(1/2)a}$ as

$$\mu(\varepsilon) = \frac{1}{2}\frac{k_1}{m} \pm \frac{\sqrt{5}}{5}\frac{k_1}{m}\varepsilon^2 + o(\varepsilon^2) \tag{72}$$

Again taking $\varepsilon = \sqrt{2} \times 10^{-2}$, and $k_1/m = 1$ for illustration, Eq. (72) yields $\mu_1(\varepsilon) \approx 0.5000894427$ and $\mu_2(\varepsilon) \approx 0.4999105572$. The corresponding answers obtained for the eigenvalues of the matrix $\hat{\Lambda} + \varepsilon\hat{P}|_{b=(1/2)a}$ (using Maple) are 0.5000894523 and 0.4999105669, showing the quality of the approximation given by Eq. (72).

5 Conclusions

The stability of dynamical systems is at the crossroads of interest to engineers, physicists, and mathematician. Engineers are interested in generating safe and stable designs for structures, machines, and mechanisms; physicists are concerned with the stability of natural phenomena such as ocean currents and atmospheric flows, and mathematicians aim at providing detailed mathematical conditions when stability or instability ensues in the models generated by physicists and engineers. Because of its practical applications, the study of stability of linear systems to infinitesimal perturbing forces has been a topic of interest for at least the last 150 years.

These investigations have mainly concentrated on seeing the effect of narrow classes of perturbing forces, such as positional circulatory forces, positional conservative/nonconservative forces, or combinations of these, on the stability of stable potential systems with a small number of degrees of freedom (usually 2–4). These narrow classes of perturbing forces are often used by scientists/engineers, in part, out of convenience and/or serendipity so that tractable analytical results can be obtained.

This paper deals with the stability of a general stable potential *multidegree of freedom linear* system subjected to *arbitrary* infinitesimal positional perturbation forces. Such general perturbing forces arise commonly in both naturally occurring as well as in engineered systems deployed by aerospace, civil, and mechanical engineers. As seen, the stability results pertinent to general arbitrary perturbations obtained here are considerably more complex, but they have the advantage of being applicable to real-life situations, since Nature does not necessarily limit itself to perturbing forces that belong to our man-made, narrow categorizations, which are often made up to meet mathematical convenience. Throughout the paper the manner in which the results obtained for general perturbing forces include the results for restricted (narrower) classes of forces, like circulatory forces, is discussed. To the best of the authors' knowledge, the results obtained in this paper are new and have not appeared in the literature to date.

For potential systems that have positive definite stiffness matrices whose eigenvalues are all distinct it is shown that for sufficiently 'small' general perturbation matrices (whose 'size' is precisely quantified) the perturbed system remains stable (Result 1).

When dealing with positive definite MDOF potential systems that have one eigenvalue of multiplicity greater than 1 (with the other eigenvalues being simple) the Hankel matrix of Newton sums associated with various matrices is shown to be very useful in studying stability. Sufficient conditions for stability and flutter instability are obtained that expose the interaction between the eigenstructure of the potential matrix, the eigenspace that pertains to the multiple eigenvalue, and to the general perturbation matrix.

It is shown that the subspace spanned by the orthonormal eigenvectors that correspond to the multiple eigenvalue is of primary importance, and the matrix $\hat{P}_{11}$ that represents the work done by the perturbation force in this subspace of displacements plays a pivotal role in the assessment of the stability of the system. Result 2 shows that if the Hankel matrix associated with $\hat{P}_{11}$ is positive definite (indefinite), then the potential system remains stable (becomes flutter unstable) under infinitesimal perturbing forces. When $\hat{P}_{11}$ has a real spectrum with repeated eigenvalues (and the Hankel matrix is positive semi-definite) the detailed considerations carried out show that the question of stability becomes very subtle, and a rather complex stability picture emerges. Results 3–8 in Sec. 3 also consider situations when the multiple eigenvalue of $\hat{P}_{11}$ is either semi-simple or nonderogatory. Several numerical examples are provided throughout this section.

The power of the analytical results obtained in ascertaining stability/instability of perturbed dynamical systems is illustrated by an application to an engineering problem. Use of the analytical results delineate the manner in which multiple eigenvalues of the potential system split in the presence of perturbative forces to engender flutter instability or stability. This yields, in a straight forward manner, the proper bifurcation parameter whose value determines whether the system is stable or unstable. Intervals of the bifurcation parameter over which stability/instability occur are also easily obtained.

This paper deals with the stability of linear structural and mechanical systems. However, it should be noted that while the study of such linear systems is important in and of itself, it also has considerable relevance when dealing with nonlinear systems. This is because the stability of nonlinear systems is often established by considering linearizations about their hyperbolic equilibrium points. Hence, the results obtained here have a significant bearing on investigations of the stability of nonlinear structural and mechanical systems as well.

Conflict of Interest

There are no conflicts of interest.

Data Availability Statement

No data, models, or code were generated or used for this paper.

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