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Time delayed control of classically damped structural systems

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FIRDAUS E. UDWADIA† and RAVI KUMAR‡

The effects of time delays on collocated as well as non-collocated point control of classically damped discrete dynamic systems have been examined. Controllers of PID type have been considered. Analytical estimates of time delays to maintain/obtain stability for small gains have been given. Several new results dealing with the effect of time delays on collocated and non-collocated control designs are obtained. It is shown that undamped structural systems cannot be stabilized with pure velocity (or integral) feedback without time delays while using a controller that is not collocated with the sensor, when the mass matrix is diagonal. However, with the appropriate choice of time delays, for certain classes of commonly occurring structural systems, stable non-collocated control can be achieved. Analytical results providing the upper bound on the controller's gain necessary for stability have been presented. The theoretical results obtained are illustrated and verified with numerical examples.

1. Introduction

The development of methodologies for the active control of structural systems, which are modelled by linear matrix differential equations, is an area of considerable interest today. Such methods lend themselves to a wide range of applications in civil, aerospace and mechanical engineering. Examples such as the control of tall building structures to strong earthquake ground shaking, the vibration control of Large Space Structures and the control of robot manipulators are some applications where the proper control of structural systems to disturbances becomes essential to the continued usefulness of the systems concerned.

Many such systems are spatially distributed and are represented by multi-degree-of-freedom (MDOF) systems (Meirovitch and Baruch 1982). It has been known for some time that *direct* velocity feedback control for such systems, when using collocated sensors and actuators, results in the damping out of all modes of vibration with no spillover effects (Aubrun 1980 and Balas 1979 a). Often (*a*) direct state feedback is not possible because of the involved dynamics of the sensor and the actuator; and (*b*) the collocation of the sensor with the actuator (or controller) may pose great practical problems. In fact, in most large structural systems, collocation of the sensors and actuators is seldom possible. Balas (1979 b) has investigated the potential of direct output feedback control for such systems, where sensors and actuators need not to be collocated. Later on, Goh and Caughey (1985) and Fanson and Caughey (1990) have shown that position feedback is preferable to velocity feedback (for the collocated case), especially when actuators' dynamics are taken into account. However, their results do not indicate the effects of time delays (explicitly) and the effect of

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dislocation of the sensor and the actuator on the stability of the control system. Cannon and Rosenthal (1984) deal with the experimental studies of collocated and non-collocated control of flexible structures. Based on these studies, it has been concluded that it is very difficult to achieve robust non-collocated control of such systems. Constructive conditions to recognize *a priori* which non-collocated control systems (using no time delays) are symmetrizable are given by Piché (1990).

Effective necessary and sufficient conditions of modal controllability for time delayed control of linear stationary systems are derived by Marchenko (1989). Kwon *et al*. (1989) suggested the use of intentional delays in the state feedback control for the stabilization of ordinary systems. It has been suggested that the delayed state feedback controller may possess some useful advantages of PID actions. Gu and Lee (1989) propose a technique for the stability testing of time delay systems. Their technique, which involves the solvability of some algebraic Riccati equations in testing the stability of time delay systems, is computationally intensive and complex. In Udwadia (1991), it is shown that using finite-dimensional controllers and appropriate time delays, the control can be made stable with no spillover. However, his results, like many others who have dealt with time delayed signals in the feedback loop, are restricted to special classes of continuous systems; they are relevant to simple structures which can be modelled as continua which are non-dispersive. This leaves out large classes of structural and mechanical systems which are commonly encountered in real life. In this paper, we investigate the non-collocated feedback control of general classically damped structural and mechanical MDOF systems. Thus, the results presented here are applicable to a much wider class of systems. The effect of time delays on collocated and non-collocated control has been investigated.

Specifically, we model a structural or mechanical system as a classically damped MDOF system and direct our attention to the general non-collocated feedback control design using several sensors and one controller. The multiple sensors collect response signals at various locations in the structural system. The control is taken to be of the PID type. Both collocated and non-collocated sensor-actuator positions are considered.

It has sometimes been erroneously assumed that the instability in the closed loop system is mainly because of the phenomenon of spillover, which is a consequence of a system being continuous and therefore having an infinite number of modes of vibration (see Balas 1978 a, b, 1982). We show in this paper that the primary reason for such instability is actually the time delay in the information between the sensor and actuator location. Thus, if the time delay is large, destabilization is guaranteed for systems which are undamped (or very lightly damped), and controlled by a single actuator, which is collocated with the sensors, while using PID control.

A general formulation for non-collocated feedback control of discrete systems is presented. Results for collocated and non-collocated control of both undamped and under-damped systems are given. The use of time delay in collocated control systems' design has been shown to have adverse effects on the stability of the systems. However, these time delays, which are not desirable for collocated systems, when appropriately chosen, can cause non-collocated control to stabilize the system. Numerical examples exhibiting the validity of the theoretical results are presented.

2. System model

Consider a linear classically damped structural system whose response $x(t)$, is described by the matrix differential equation

$$M\ddot{x}(t) + \tilde{C}\dot{x}(t) + \tilde{K}x(t) = g(t);\ x(0) = \dot{x}(0) = 0 \tag{1}$$

where M is a positive definite, symmetric, $n \times n$ mass matrix, $\tilde{C}$ is the symmetric damping matrix and $\tilde{K}$ is the positive definite, symmetric, stiffness matrix. The force n-vector, $g(t)$, is considered to be the distributed force. Making the substitution $y(t) = M^{1/2}x(t)$, yields

$$\ddot{y}(t) + C\dot{y}(t) + Ky(t) = f(t);\ y(0) = \dot{y}(0) = 0 \tag{2}$$

where $C = M^{-1/2}\tilde{C}M^{-1/2}$, $K = M^{-1/2}\tilde{K}M^{-1/2}$ and $f(t) = M^{-1/2}g(t)$.

For ease of understanding, each equation in the equation set (2) can be thought of as representing the equilibrium condition related to a particular node, x_i, $i = 1, 2, \ldots, n$, of the system. Using the transformation $y(t) = Tz(t)$, we get

$$\ddot{z}(t) + \Xi\dot{z}(t) + \Lambda z(t) = T^{\mathrm{T}}f(t);\ z(0) = \dot{z}(0) = 0 \tag{3}$$

where $\Xi = \operatorname{diag}\{2\xi_1, 2\xi_2, 2\xi_3, \ldots, 2\xi_n\}$, and $\Lambda = \operatorname{diag}\{\lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots, \lambda_n^2\}$, and the matrix $T = [t_{ij}]$ is the orthogonal matrix of real eigenvectors of K. We note that the simultaneous diagonalization of C and K is implied by the fact that the system is classically damped. Taking the Laplace transform we obtain

$$\hat{x}(s) = M^{-1/2}\hat{y}(s) = M^{-1/2}T\hat{z}(s) = M^{-1/2}T\Theta T^{\mathrm{T}}M^{-1/2}\hat{g}(s) \tag{4}$$

where the hats indicate transformed quantities, and the matrix

$$\Theta = \operatorname{diag}\{(s^2 + 2s\xi_1 + \lambda_1^2)^{-1}, (s^2 + 2s\xi_2 + \lambda_2^2)^{-1}, \ldots, (s^2 + 2s\xi_n + \lambda_n^2)^{-1}\} \tag{5}$$

The open loop poles of the system are therefore given by the roots of the equations

$$s^2 + 2s\xi_q + \lambda_q^2 = (s - \gamma_{+q})(s - \gamma_{-q}) = 0, \quad q = 1, 2, \ldots, n \tag{6}$$

We have denoted the poles by $\gamma_{\pm q}$, $q = 1, 2, \ldots, n$, where the plus (minus) indicates the positive (negative) sign taken in front of the radical in solving the quadratic equations given in equation set (6). In this paper we shall always assume that no two of these equations yield the same roots, i.e. the open loop poles are *all* distinct.

3. General formulation for non-collocated feedback control

We utilize p responses $x_{s_k}(t)$, $k = 1, 2, \ldots, p$, in our feedback control design. Each response $x_{s_k}(t)$ could, in general, be time-delayed by T_{s_k} and then linearly combined with other such time delayed responses being fed to a controller which then generates the desired feedback control force. The control methodology, applied to a building structure, is shown in Fig. 1. The actuator causes a force to be applied to the system thereby affecting the jth equation in the equation set (1).

When $j \notin \{s_k: k = 1, 2, \ldots, p\}$ we obtain a situation where the sensors and the actuator are non-collocated. If $j \in \{s_k: k = 1, 2, \ldots, p\}$ the sensor and the actuator are collocated.

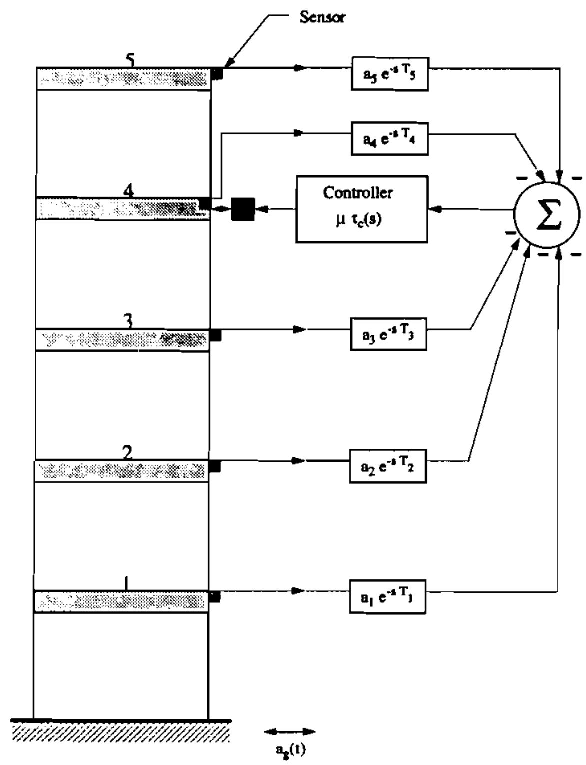

Figure 1. Shear frame building structure and control methodology.

Denoting the transfer function of the controller by $\mu\tau_c(s)$, where μ is the non-negative control gain, the closed loop system is defined by the equation

$$\hat{A}(s)\hat{x}(s) = [Ms^2 + \tilde{C}s + \hat{K}]\hat{x}(s) = \hat{g}(s) - \mu\tau_c(s)\sum_{k=1}^{p} a_{s_k} x_{s_k}(s)\exp[-sT_{s_k}]e_j \tag{7}$$

where e_j is the unit vector with unity in its jth element and zeros elsewhere. The real numbers a_{s_k} provide a linear combination of the responses which are fed to the controller. Moving the second term on the right-hand side of (7) to the left, we obtain

$$\hat{A}_1(s)\hat{x}(s) = \hat{g}(s) \tag{8}$$

where $\hat{A}_1(s)$ is obtained by adding to the (j, s_k)th element of matrix $\hat{A}(s)$ the quantity $\mu\tau_c(s)a_{s_k}\exp[-sT_{s_k}]$ for $k = 1, 2, \ldots, p$. The closed loop poles are obtained from the relation

$$\det[\hat{A}_1(s)] = 0 \tag{9}$$

This determinant can be expressed as

$$\det[\hat{A}_1(s)] = \det[\hat{A}(s)]\left\{1 + \mu\tau_c(s)\sum_{k=1}^{p} a_{s_k}\exp[-sT_{s_k}]\hat{x}^{(\delta)}_{s_k,j}(s)\right\} \tag{10}$$

where $\hat{x}^{(\delta)}_{s_k,j}(s)$ is the Laplace transform of the open loop response $x_{s_k}(t)$ to an impulsive force applied at node j, at time $t = 0$. But the open loop response to such an excitation is given by (4) as

$$\hat{x}^{(\delta)}_{s_k,j}(s) = \sum_{i=1}^{n}\left[\frac{t^{(M)}_{s_k,i}t^{(M)}_{j,i}}{s^2 + 2s\xi_i + \lambda_i^2}\right] \tag{11}$$

where

$$t^{(M)}_{s_k,r} = \sum_{u=1}^{n} m^{-1/2}_{s_k,u} t_{u,r} \tag{12}$$

and we have denoted the (i, j)th element of the symmetric matrix $M^{-1/2}$ by $m^{-1/2}_{i,j}$. We note that the matrix $T^{(M)} = M^{-1/2}T = [t^{(M)}_{ij}]$ is not orthogonal while the matrix $T = [t_{ij}]$ is.

If the mass matrix M is a diagonal matrix then $M^{-1/2}$ is also diagonal, and (11) reduces to

$$\hat{x}^{(M)}_{s_k,j}(s) = \sum_{i=1}^{n}\frac{1}{(m_j m_{s_k})^{1/2}}\left[\frac{t_{s_k,i}t_{j,i}}{s^2 + 2s\xi_i + \lambda_i^2}\right] \tag{13}$$

where m_j denotes the (j, j)th element of the diagonal matrix M.

We now assume that the following set of conditions, whose physical meaning will be provided later, are satisfied

$$\left.\begin{array}{ll} (1)\ \tau_c(\gamma_{\pm m}) \neq 0 & \text{for } m = 1, 2, \ldots, n \\ (2)\ \sum_{k=1}^{p} a_{s_k}\exp[-\gamma_{\pm m}T_{s_k}]t^{(M)}_{s_k,m} \neq 0 & \text{for } m = 1, 2, \ldots, n, \text{ and} \\ (3)\ t^{(M)}_{j,m} \neq 0 & \text{for } m = 1, 2, \ldots, n \end{array}\right\} \tag{C1}$$

The $\gamma_{\pm i}$ are the open loop poles of the system. Then we have the following result.

Result 3.1: When the open loop system has distinct poles and condition set $C1$ is satisfied, the open loop and closed loop systems do not have any pole in common. □

Proof: We note that when $\mu = 0$, the system is open loop and $\hat{A}_1(s) = \hat{A}(s)$. Furthermore

$$\det[\hat{A}(s)] = \det[M]\det[\Theta^{-1}] = \det[M]\prod_{r=1}^{n}(s - \gamma_{+r})(s - \gamma_{-r}) \tag{14}$$

Let us assume that the closed loop system and the open loop system, for some value of $\mu > 0$, have a pole in common, say at $s = \gamma_m$, for some $m = \pm 1, \pm 2, \ldots, \pm n$. Then $\det[\hat{A}_1(\gamma_m)] = 0$. Using (10) and (11), the condition $\lim_{s\to\gamma_m}(\det[\hat{A}_1(\gamma_m)]) = 0$ requires that

$$\lim_{s\to\gamma_m}\left(\mu\tau_c(s)\sum_{k=1}^{p} a_{s_k}\exp[-sT_{s_k}]\sum_{r=1}^{n}\left\{\frac{\prod_{q=1}^{n}(s - \gamma_{+q})(s - \gamma_{-q})}{(s - \gamma_{+r})(s - \gamma_{-r})}\right\}t^{(M)}_{s_k,r}t^{(M)}_{j,r}\right) = 0 \tag{15}$$

and yields

$$\mu\tau_c(\gamma_m)\sum_{k=1}^{p} a_{s_k}\exp[-\gamma_m T_{s_k}]t_{s_k,m}^{(M)}t_{j,m}^{(M)}\prod_{\substack{q=1\\q\neq m}}^{n}(\gamma_m-\gamma_{+q})(\gamma_m-\gamma_{-q})=0 \quad (16)$$

Noting that the open loop poles are distinct, we find that for the closed loop and open loop systems to have a common pole, we require

$$\mu\tau_c(\gamma_m)\left\{\sum_{k=1}^{p} a_{s_k}\exp[-\gamma_m T_{s_k}]t_{s_k,m}^{(M)}\right\}t_{j,m}^{(M)}=0 \quad (17)$$

which is impossible as long as conditions $C1$ are met. Hence the result. □

Remark 3.1: The first condition in $C1$ requires that the zeros of the controller transfer function do not coincide with the open loop poles of the system; the second condition is a generalized observability condition, which requires that all mode shapes be observable from the summed, time-delayed, sensor measurements; and the third condition is a controllability condition, and requires that the controller cannot be located at any node of any mode of the system. □

Remark 3.2: If for a given open loop pole $s=\gamma_m$, any one of the three conditions in $C1$ is not satisfied, then the open loop and closed loop systems share a common pole at $s=\gamma_m$ for all values of the gain μ. This is true because $\lim_{s\to\gamma_m}(\det[\hat{A}(\gamma_m)])=0$. □

Result 3.2: Under conditions $C1$, the closed loop poles for $\mu>0$ are given by those values of 's' which satisfy the relation

$$1+\mu\tau_c(s)\sum_{k=1}^{p}\sum_{i=1}^{n} a_{s_k}\exp[-sT_{s_k}]\left[\frac{t_{s_k,i}^{(M)}t_{j,i}^{(M)}}{s^2+2s\xi_i+\lambda_i^2}\right]=0 \quad (18)$$

□

Proof: Noting (10) and (11) and Result 3.1, the result follows. □

4. A result on stability of the feedback control

When the controller gain, μ, equals zero, the system becomes open loop and the poles of the system are the open loop poles. Multiplying (18) by $(s^2+2s\xi_r+\lambda_r^2)$ we obtain

$$(s^2+2s\xi_r+\lambda_r^2)+\mu\tau_c(s)\sum_{k=1}^{p}\sum_{\substack{i=1\\i\neq r}}^{n} a_{s_k}\exp[-sT_{s_k}]\left[\frac{s^2+2s\xi_r+\lambda_r^2}{s^2+2s\xi_i+\lambda_i^2}\right]t_{s_k,i}^{(M)}t_{j,i}^{(M)}$$

$$=-\mu\tau_c(s)\sum_{k=1}^{p} a_{s_k}\exp[-sT_{s_k}]t_{s_k,r}^{(M)}t_{j,r}^{(M)} \quad (19)$$

Differentiating with respect to μ and letting $s\to\gamma_{\pm r}=-\xi_r\pm i(\lambda_r^2-\xi_r^2)^{1/2}$ and $\mu\to 0$, we obtain

$$\left.\frac{ds}{d\mu}\right|_{\substack{\mu\to 0\\ s\to\gamma_{\pm r}}}=-\frac{\tau_c(\gamma_{\pm r})}{\pm 2i(\lambda_r^2-\xi_r^2)^{1/2}}\left[\sum_{k=1}^{p} a_{s_k}\exp[-\gamma_{\pm r}T_{s_k}]t_{s_k,r}^{(M)}\right]t_{j,r}^{(M)} \quad (20)$$

When M is a diagonal matrix, this becomes

$$\left.\frac{ds}{d\mu}\right|_{\substack{\mu\to 0\\ s\to\gamma_{\pm r}}} = -\frac{\tau_c(\gamma_{\pm r})}{\pm 2i(\lambda_r^2 - \xi_r^2)^{1/2}}\left[\sum_{k=1}^{p} b_{s_k}\exp[-\gamma_{\pm r}T_{s_k}]t_{s_k,r}\right]t_{j,r} \tag{21}$$

where $b_{s_k} = a_{s_k}/(m_{s_k}m_j)^{1/2}$.

Result 4.1: A sufficient condition for the closed loop system to remain stable for infinitesimal gains is that

$$\mathrm{Re}\left\{\left.\frac{ds}{d\mu}\right|_{\substack{\mu\to 0\\ s\to\gamma_{\pm r}}}\right\} < 0, \quad r = 1, 2, \ldots, n \tag{22}$$

□

Proof: This condition requires the root loci of the closed loop poles to move towards the left half s-plane and hence stability is ensured. □

Result 4.2: For undamped systems (i.e. $C = 0$), condition (22) is a necessary and sufficient condition for small gain stability. □

Proof: Since the open loop poles now lie on the imaginary axis in the s-plane, the result follows. □

Remark 4.1: If the actuator is located at a node of the rth mode then the position of the rth open loop pole will not be affected by the feedback control because the system is not controllable. □

Proof: When the actuator is located at a node of the rth mode, $t_{j,r}^{(M)} = 0$, and so by (20)

$$\left.\frac{ds}{d\mu}\right|_{\substack{\mu\to 0\\ s\to\gamma_{\pm r}}} = 0$$ □

Remark 4.2: If the sensors are located such that

$$\sum_{k=1}^{p} a_{s_k}\exp[-\gamma_{\pm r}T_{s_k}]t_{s_k,r}^{(M)} = 0 \tag{23}$$

for any particular r, then the rth open loop pole is not affected by the control because the system is not observable. □

Proof: For small gains, the result follows from (20). In particular, when $p = 1$, the placement of a sensor at a node of the rth mode will cause condition (20) to be equal to zero. Again, for large gains the result follows from Remark 3.2. □

We note that when using multiple sensors (i.e. $p > 1$) even when the sensors are *not* located at any of the nodes of the rth mode, the sensor outputs could be so combined that (23) is satisfied. This will leave the rth mode unobservable and thus the rth open loop pole unaffected by the feedback control. By Result 3.1, Remarks 4.1 and 4.2 are valid for all $\mu > 0$.

5. PID feedback control

We now particularize the controller's transfer function to be

$$\tau_c(s) = K_0 + K_1 s + \frac{K_2}{s}; \; K_0, \; K_1, \; K_2 \geqslant 0 \tag{24}$$

The first term on the right refers to proportional control, the second to velocity control and the third to integral control. PID controllers are commonly used in control systems and we will next investigate their efficacy.

5.1. *Results for undamped system*

For an undamped system, $C = 0$ and $\gamma_{\pm r} = \pm i\lambda_r$. Using relations (20) and (22), we would then require for stability, when the gain is small, that

$$\mathrm{Re}\left\{\left(\frac{K_0}{\pm i\lambda r} + K_1 - \frac{K_2}{\lambda_r^2}\right)\left(\sum_{k=1}^{p} a_{s_k} \exp[\mp i\lambda_r T_{s_k}] t_{s_k,r}^{(M)} t_{j,r}^{(M)}\right)\right\} > 0, \quad \text{for } r = 1, 2, \ldots, n \tag{25}$$

which yields

$$-\frac{K_0}{\lambda_r}\left[\sum_{k=1}^{p} a_{s_k} \sin(\lambda_r T_{s_k}) t_{s_k,r}^{(M)} t_{j,r}^{(M)}\right] + \left(K_1 - \frac{K_2}{\lambda_r^2}\right)\left[\sum_{k=1}^{p} a_{s_k} \cos(\lambda_r T_{s_k}) t_{s_k,r}^{(M)} t_{j,r}^{(M)}\right] > 0 \quad \text{for } r = 1, 2, \ldots, n \tag{26}$$

Now we are ready to present some results on the collocated control of the systems.

Result 5.1(*a*): When using one sensor, collocation of the sensor with an actuator will cause PID feedback control to be stable (for small gains) for an undamped system if and only if

$$a_j\left\{-\frac{K_0}{\lambda_r}\sin(\lambda_r T_j) + \left(K_1 - \frac{K_2}{\lambda_r^2}\right)\cos(\lambda_r T_j)\right\} > 0, \quad \text{for } r = 1, 2, \ldots, n \tag{27}$$

□

Proof: Here $p = 1$ and $s_1 = j$. The result follows from relation (26). □

Now we give stability results for some special cases for vanishingly small gains. Later we present results for large gains.

Result 5.1(*b*): When using one sensor, collocation of the sensor with an actuator will cause velocity feedback control (i.e. $K_0 = K_2 = 0$) to be stable (for small gains) for an undamped system as long as the time delay is such that $T_j < \pi/2\lambda_{\max}$, where $\lambda_{\max}$ is the highest undamped natural frequency of the system, i.e. $T_j < T_{\min}/4$, where $T_{\min}$ is the smallest period of vibration of the system. □

Proof: Noting that $K_1 > 0$, for $p = 1$ and $s_k = j$, the sensor and actuator are collocated, and condition (27) is satisfied for any $a_j > 0$. Hence the result. □

Result 5.1(*c*): When using one sensor, collocation of the sensor with an actuator will cause integral feedback control (i.e. $K_0 = K_1 = 0$) to be stable (for small gains) for an undamped system as long as the delay is such that $T_j < \pi/2\lambda_{\max}$. □

Proof: The proof is the same as above, with $a_j < 0$. □

Result 5.1(*d*): When using one sensor with $K_0 = 0$, collocation of the sensor with an actuator with time delay $T_j < \pi/2\lambda_{\max}$ will cause the undamped system to be stabilized for small gains when $K_1 > K_2/\lambda_{min}^2$ and $a_j > 0$, or $K_1 < K_2/\lambda_{max}^2$ and $a_j < 0$. □

Proof: The result is obvious from relation (27). □

Result 5.1(*e*): When using one sensor with $K_1 = K_2 = 0$, collocation of the sensor with an actuator with time delay $0 < T_j < \pi/\lambda_{max}$ will cause the undamped system to be stabilized for small gains. □

Proof: For $a_j < 0$, the result follows from relation (27). □

Remark 5.1(*a*): When using one sensor with negative proportional feedback, collocation of the sensor with an actuator with time delay $0 < T_j < \pi/\lambda_{max}$ will cause the undamped system to be *destabilized* for small gains. □

Proof: Here $a_j > 0$. Under these conditions, the left-hand side of (27) will be negative for all r. This indicates that all the open loop poles will start moving in the right half s-plane as the controller's gain increases from 0 to 0+. Hence the result. □

Remark 5.1(*b*): In Result 5.1(e), if time delay $T_j = 0$, then all the closed loop poles of the proportional feedback collocated control system will move along the imaginary axis, as the gain μ increases from 0. □

Proof: If $T_j = 0$, then the closed loop poles are the roots of the equation

$$\det[s^2 \ M + \bar{K}] = 0 \tag{28}$$

where matrix $\bar{K}$ is symmetric and is obtained by adding the quantity $\mu K_0 a_j$ to the (j, j)th element of the stiffness matrix $\widetilde{K}$. If $\bar{K}$ is positive definite or positive semidefinite (depending on the values of μ and coefficient a_j), then the zeros of $\det[s^2 \ M + \bar{K}]$ will lie on the imaginary axis of the s-plane. Hence the result. □

Remark 5.2: It should be noted that stability is *not* ensured when using a number of sensors, *one* among which is collocated with the actuator, even when using no time delays. □

We now move to large gains and investigate stability when $\mu > 0$.

Result 5.2: When the system is undamped, and

(1) conditions $C1$ are satisfied,

(2) one sensor is used and it is collocated with the actuator, and,

(3) no time delay is used,

then the PID control, if stable for $\mu \to 0+$, is stable for all $\mu > 0$, provided

$$\det\left[\hat{A}\left(-\frac{K_2}{K_1}\right)\right] + \mu a_j K_0 \det\left[\hat{A}_2\left(-\frac{K_2}{K_1}\right)\right] \neq 0, \text{ for any positive } \mu \tag{29}$$

where the matrices $\hat{A}$ and $\hat{A}_2$ are as defined below. □

Proof: Under these provisions, the closed loop poles must occur at the roots of

$$\det[\hat{A}(s^2)] + \mu a_j \tau_c(s) \det[\hat{A}_2(s^2)] = 0 \tag{30}$$

where $\hat{A}$ is defined in (7) and the matrix $\hat{A}_2$ is obtained by deleting the jth row and the jth column of $\hat{A}$. When the system is stable for $\mu \to 0+$, the root loci begin moving towards the left half s-plane. For the system to become unstable at least one of the loci must turn around and cross the imaginary axis before it moves into the right half s-plane. Assume that this cross-over occurs at $s = i\eta$. We note that the determinants of $\hat{A}$ and $\hat{A}_2$ are real at $s = i\eta$ (because the system is undamped). Collecting the real and imaginary parts of (30) we get

$$\left.\begin{aligned} &\det[\hat{A}(-\eta^2)] + \mu a_j K_0 \det[\hat{A}_2(-\eta^2)] = 0 \quad \text{and} \\ &\mu \det[\hat{A}_2(-\eta^2)]\left(K_1\eta - \frac{K_2}{\eta}\right) = 0 \end{aligned}\right\} \tag{31}$$

Note that both $\det[\hat{A}]$ and $\det[\hat{A}_2]$ cannot have a common zero since conditions $C1$ are satisfied (Result 3.1). To satisfy the second equation of set (31) we must therefore have $\eta^2 = K_2/K_1$. The result now follows. □

Remark 5.3: When using pure velocity (or integral) feedback, condition (29) is always satisfied and hence stability is guaranteed for all $\mu \geqslant 0$ (a well-known result). □

Proof: When $K_0 = K_2 = 0$, relation (29) is satisfied for $\mu \geqslant 0$ because $\det[\hat{A}(0)] = \det[\tilde{K}] > 0$, since $\tilde{K}$ is a positive definite matrix. We note that $\det[\hat{A}_2(0)]$ is bounded. □

When $K_0 = 0$ and $K_1 \to 0$, and, $\eta \to \infty$, a similar argument follows because matrix M is positive definite. (This can also be proved by positivity theory.)

Remark 5.4: If the system described in Result 5.2 becomes unstable, it does so at $s = \pm i(K_2/K_1)^{1/2}$. The upper bound on the gain for stability is then obtained as

$$\mu < \frac{-\det\left[\hat{A}\left(-\dfrac{K_2}{K_1}\right)\right]}{a_j K_0 \det\left[\hat{A}_2\left(-\dfrac{K_2}{K_1}\right)\right]} \tag{32}$$

provided the right-hand side in the above inequality is positive; if not, the system is stable for all $\mu > 0$, if it is stable for $\mu \to 0+$. □

Proof: The proof follows from the proof of Result 5.2. □

In Result 5.1(b) we have shown that for vanishingly small gains when using one sensor, collocation of the sensor with an actuator causes velocity feedback control to be stable (for small gains) for an undamped system as long as $T_j < \pi/2\lambda_{\max}$. Now our aim is to obtain an upper bound on gain μ, which ensures stability.

Result 5.3: When the system is undamped, and

(1) conditions $C1$ are satisfied,

(2) one sensor is used and it is collocated with the actuator, and,

(3) time delay $T_j < \pi/2\lambda_{\max}$,

then velocity feedback control will be stable as long as

$$\mu < \frac{1}{a_j K_1 \eta_0 \sum_{i=1}^{n} \frac{[t_{j,i}^{(M)}]^2}{\eta_0^2 - \lambda_i^2}} \tag{33}$$

where $\eta_0 = \pi/2T_j$. □

Proof: Under conditions $C1$, the closed loop poles of the velocity feedback system, for $\mu > 0$, are given by those values of 's' which satisfy the relation (from (18))

$$1 + \mu K_1 s a_j \exp[-sT_j]\left(\sum_{i=1}^{n} \frac{[t_{j,i}^{(M)}]^2}{s^2 + \lambda_i^2}\right) = 0 \tag{34}$$

Now let us assume that any one closed loop pole crosses the imaginary axis at $s = \pm i\eta$, where η is a positive real number. So (34) becomes (with $s = \pm i\eta$)

$$1 + \mu K_1(\pm i\eta)a_j[\cos \eta T_j) \mp i \sin(\eta T_j)]\left(\sum_{i=1}^{n} \frac{[t_{j,i}^{(M)}]^2}{\lambda_i^2 - \eta^2}\right) = 0 \tag{35}$$

Separating real and imaginary parts of the above equation, we have

$$1 + \mu K_1 \eta a_j \sin(\eta T_j)\left(\sum_{i=1}^{n} \frac{[t_{j,i}^{(M)}]^2}{\lambda_i^2 - \eta^2}\right) = 0 \tag{36}$$

and

$$\pm\ \mu K_1 \eta a_j \cos(\eta T_j)\left(\sum_{i=1}^{n} \frac{[t_{j,i}^{(M)}]^2}{\lambda_i^2 - \eta^2}\right) = 0 \tag{37}$$

Now arranging the terms in (36) and (37), we get

$$\mu K_1 \eta a_j\left[\left(\sum_{i=1}^{n} \frac{[t_{j,i}^{(M)}]^2}{\lambda_i^2 - \eta^2}\right)\right] \sin(\eta T_j) = -1 \tag{38}$$

and

$$\mu K_1 \eta a_j\left[\left(\sum_{i=1}^{n} \frac{[t_{j,i}^{(M)}]^2}{\lambda_i^2 - \eta^2}\right)\right] \cos(\eta T_j) = 0 \tag{39}$$

Thus, a closed loop pole will cross the imaginary axis at $s = \pm i\eta$ if η satisfies the above two equations. Note that η cannot be zero because then (38) is not satisfied. Similarly, we can see that $\sum_{i=1}^{n}[t_{j,i}^{(M)}]^2/(\lambda_i^2 - \eta^2) \neq 0$. Therefore, to satisfy (39) we should have

$$\eta T_j = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\right\} \tag{40}$$

From the above, we note that $\eta > \lambda_{\max}$. Hence, the quantities in the brackets in (38) and (39) are negative. Knowing that a_j (from Result 5.1(b)), μ, K_1 and η all are positive quantities, it is obvious that

$$\eta T_j = \left\{\frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \ldots\right\}$$

cannot satisfy both (38) and (39) simultaneously. Therefore, for a closed loop pole to cross the imaginary axis, we should have

$$\eta_m T_j = \left(\frac{4m+1}{2}\right)\pi; \quad m = 0, 1, 2, \ldots \tag{41}$$

and for a closed loop pole not to cross the imaginary axis, the following inequality should be satisfied

$$\mu < \frac{1}{a_j K_1 \eta_m \left(\sum_{i=1}^{n} \frac{[t_{j,i}^{(M)}]^2}{\eta_m^2 - \lambda_i^2}\right)}; \quad \text{for } m = 0, 1, 2, \ldots \tag{42}$$

Furthermore, for $m = 1, 2, 3, \ldots$

$$\left[\frac{\eta_m^2 - \lambda_i^2}{\eta_m} - \frac{\eta_0^2 - \lambda_i^2}{\eta_0}\right] = \frac{4m[(4m+1)\eta_0^2 + \lambda_i^2]}{(4m+1)\eta_0} \tag{43}$$

which is positive quantity. Therefore, η_0, which is also the cross-over frequency, gives the lowest upper bound on the gain μ. Hence, the result. □

Remark 5.5: If any of the closed loop poles of the system as described in Result 5.3 cross the imaginary axis at $s = \pm i\eta$, then $\eta > \lambda_{\max}$, where $\lambda_{\max}$ is the highest natural frequency of vibration. □

Proof: From (40) and the fact that $T_j < (\pi/2\lambda_{\max})$, the result follows. □

So far, we have given results on the controllability of collocated systems. In the following, we consider the control of non-collocated systems.

Result 5.4: If both the mass matrix M and stiffness matrix $\widetilde{K}$ are positive definite, then

$$[T^{(M)}][T^{(M)}]^{\mathrm{T}} = M^{-1} \tag{44}$$

or

$$\sum_{r=1}^{n} t_{s_k,r}^{(M)} t_{j,r}^{(M)} = m_{s_k,j}^{-1} \tag{45}$$

where $m_{s_k,j}^{-1}$ denotes the (s_k, j)th element of the matrix M^{-1}. We note that $M^{-1/2}$ is symmetric. □

Proof: Since $T^{(M)} = M^{-1/2}T$, where T is orthogonal, we get

$$[T^{(M)}][T^{(M)}]^{\mathrm{T}} = M^{-\frac{1}{2}}TT^{\mathrm{T}}M^{-\frac{1}{2}} \tag{46}$$

and hence

$$[T^{(\mathrm{M})}][T^{(\mathrm{M})}]^{\mathrm{T}} = M^{-1} \tag{47}$$

From this, relation (45) follows directly.

Result 5.5: If $t_{j,r}^{(M)} \neq 0$, for $r = 1, 2, \ldots, n$ and all the (j, s_k)th elements of M^{-1} are zero, where $k = 1, 2, \ldots, p$, and $j \neq s_k$, then the inequality

$$\sum_{k=1}^{p} a_{s_k} t_{s_k,r}^{(M)} t_{j,r}^{(M)} > 0, \qquad \text{for } r = 1, 2, \ldots, n \tag{48}$$

cannot be satisfied for *any* real numbers a_{s_k}, $k = 1, 2, \ldots, p$. □

Proof: Relation (48) can be written as

$$\begin{bmatrix} t_{s_1,1}^{(M)}t_{j,1}^{(M)} & t_{s_2,1}^{(M)}t_{j,1}^{(M)} & \cdots & t_{s_p,1}^{(M)}t_{j,1}^{(M)} \\ t_{s_1,2}^{(M)}t_{j,2}^{(M)} & t_{s_2,2}^{(M)}t_{j,2}^{(M)} & \cdots & t_{s_p,2}^{(M)}t_{j,2}^{(M)} \\ \vdots & \vdots & \ddots & \vdots \\ t_{s_1,n}^{(M)}t_{j,n}^{(M)} & t_{s_2,n}^{(M)}t_{j,n}^{(M)} & \cdots & t_{s_p,n}^{(M)}t_{j,n}^{(M)} \end{bmatrix} \begin{Bmatrix} a_{s1} \\ a_{s2} \\ \vdots \\ a_{s_p} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{Bmatrix} \tag{49}$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ all are positive quantities and can be defined as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{Bmatrix} = \begin{Bmatrix} b_1\{t_{j,1}^{(M)}\}^2 \\ b_2\{t_{j,2}^{(M)}\}^2 \\ \vdots \\ b_n\{t_{j,n}^{(M)}\}^2 \end{Bmatrix} \tag{50}$$

where, $b_i > 0$, for $i = 1, 2, \ldots, n$.

Now, substituting (50) into (49) and factoring our $t_{j,r}^{(M)}$, $r = 1, 2, \ldots, n$, from the rth row, we get

$$\begin{bmatrix} t_{s_1,1}^{(M)} & t_{s_2,1}^{(M)} & \cdots & t_{s_p,1}^{(M)} \\ t_{s_1,2}^{(M)} & t_{s_2,2}^{(M)} & \cdots & t_{s_p,2}^{(M)} \\ \vdots & \vdots & \ddots & \vdots \\ t_{s_1,n}^{(M)} & t_{s_2,n}^{(M)} & \cdots & t_{s_p,n}^{(M)} \end{bmatrix} \begin{Bmatrix} a_{s_1} \\ a_{s_2} \\ \vdots \\ a_{s_p} \end{Bmatrix} = \begin{Bmatrix} b_1 t_{j,1}^{(M)} \\ b_2 t_{j,2}^{(M)} \\ \vdots \\ b_n t_{j,n}^{(M)} \end{Bmatrix} \tag{51}$$

Let us assume that there exists a set of $b_i > 0$, for $i = 1, 2, \ldots, n$ and a set of a_{s_k}, for $k = 1, 2, \ldots, p$ such that (51) is valid.

We now premultiply both sides of (51) by the row vector

$$c_j^{\mathrm{T}} = [t_{j,1}^{(M)} \quad t_{j,2}^{(M)} \quad t_{j,3}^{(M)} \quad \cdots \quad t_{j,n}^{(M)}] \tag{52}$$

giving

$$a_{s_1}\sum_{l=1}^{n} t_{s_1,l}^{(M)}t_{j,l}^{(M)} + a_{s_2}\sum_{l=1}^{n} t_{s_2,l}^{(M)}t_{j,l}^{(M)} + \cdots + a_{s_p}\sum_{l=1}^{n} t_{s_p,l}^{(M)}t_{j,l}^{(M)} = \sum_{i=1}^{n} b_i\{t_{j,i}^{(M)}\}^2 \tag{53}$$

But, $\sum_{l=1}^{n} t_{s_k,l}^{(M)}t_{j,l}^{(M)} = m_{s_k j}^{-1}$, for $k = 1, 2, \ldots, p$. Since $m_{s_k j}^{-1} = 0$, $k = 1, 2, \ldots, p$; $j \neq s_k$, the left-hand side of (53) is zero. This requires that all b_i are not all greater than zero. Hence, there do not exist $b_i > 0$ such that the equation set (51) is satisfied for some a_{s_k}, $k = 1, 2, \ldots, p$. □

Remark 5.6: The 'greater than' in relation (48) in Result 5.5 can be replaced by 'less than'. □

Proof: The proof is along the same lines, with $b_i < 0$ for $i = 1, 2, \ldots, n$. □

Remark 5.7: If the mass matrix M is diagonal then both Result 5.5 and Remark 5.6 hold good. □

Proof: Because M^{-1} is diagonal, all (j, s_k)th elements of M^{-1} are zero, for $j \neq s_k$ and $k = 1, 2, \ldots, p$.

Result 5.6: When using a PID controller, where

(1) the sensors and actuator are not collocated,

(2) the time delays, T_{s_k}, $k = 1, 2, \ldots, p$, are all zero,

(3) the matrix M is diagonal, and,

(4) $K_1 > (K_2/\lambda^2_{\min})$, or $K_1 < (K_2/\lambda^2_{\max})$,

it is impossible to stabilize an undamped (open loop) system for small gains. Such feedback control is guaranteed to destabilize the system. □

Proof: When no time delayed signals are used, and the mass matrix M is diagonal, condition (26) for stability, for $K_1 > (K_2/\lambda^2_{\min})$, becomes (see (20))

$$\left(\sum_{k=1}^{p} a_{s_k} t^{(M)}_{s_k,r} t^{(M)}_{j,r}\right) > 0, \quad \text{for } r = 1, 2, \ldots, n \tag{54}$$

But, by Remark 5.7 this condition cannot be satisfied for any given number of sensors, p, for some real numbers a_{s_k}, and for any sensor locations x_{s_k}, $k = 1, 2, \ldots, p$, as long as $j \notin \{s_k: k = 1, 2, \ldots, p\}$.

A similar argument is valid when $K_1 < (K_2/\lambda^2_{\max})$. We have therefore shown that such non-collocated control will *always* destabilize at least one mode of the system for (vanishingly) small gains. □

Remark 5.8: When using PD feedback, the undamped system described in Result 5.6 is guaranteed to be destabilized for small gains.

Proof: This is a special case of Result 5.6 when $K_2 = 0$ and $K_1 > 0$. □

Remark 5.9: When using PI feedback, the undamped system described in Result 5.6 is guaranteed to be destabilized for small gains. □

Proof: This is a special case of Result 5.6 when $K_2 > 0$ and $K_1 = 0$.

Result 5.7: A necessary condition for

$$\sum_{k=1}^{p} a_{s_k} t^{(M)}_{s_k,r} t^{(M)}_{j,r} > 0, \qquad \text{for } r = 1, 2, \ldots, n \tag{55}$$

is that

$$\sum_{k=1}^{p} a_{s_k} m^{-1}_{s_k,j} > 0 \tag{56}$$

where $m^{-1}_{s_k,j}$ is the (s_k, j)th element of the matrix M^{-1}. □

Proof: If the inequality in relation (55) holds, then this would require

$$\sum_{r=1}^{n}\left(\sum_{k=1}^{p} a_{s_k} t^{(M)}_{s_k,r} t^{(M)}_{j,r}\right) > 0 \tag{57}$$

Interchanging the order of summation, the result follows directly from Result 5.4. □

Remark 5.10: The 'greater than' in relations (55) and (56) in Result 5.7 can be replaced by 'less than'. The proof is along the same lines. □

Result 5.8: When using a PID controller, where

(1) the sensors and actuator are not collocated,

(2) the time delays, T_{s_k}, $k = 1, 2, \ldots, p$, are all zero, and,

(3) the matrix M is non-diagonal,

a necessary condition for the undamped system to be stabilized for small gains is

(*a*) $\sum_{k=1}^{p} a_{s_k} m_{s_k,j}^{-1} > 0$, when $K_1 > (K_2/\lambda_{\min}^2)$, and,
(*b*) $\sum_{k=1}^{p} a_{s_k} m_{s_k,j}^{-1} < 0$, when $K_1 < (K_2/\lambda_{\max}^2)$. □

Proof: If condition (22) is to be satisfied, we require that relation (26) be satisfied for all r. When, $K_1 > (K_2/\lambda_{\min}^2)$, we would require for stability that $(\sum_{k=1}^{p} a_{s_k} t_{s_k,r}^{(M)} t_{j,r}^{(M)}) > 0$, for $r = 1, 2, \ldots, n$. This would necessitate, by Result 5.7 that $\sum_{k=1}^{p} a_{s_k} m_{s_k,j}^{-1} > 0$. Thus, when the system is stable, relation (55) is always satisfied. A similar proof for $K_1 < (K_2/\lambda_{\max}^2)$ is possible. □

Remark 5.11: Result 5.8 can be 'particularized' to PD, PI, D, and I controllers, similar to Remarks 5.8 and 5.9. □

Result 5.9: If M and $\widetilde{K}$ are positive definite, M is diagonal and $\widetilde{K}$ is tridiagonal, having negative subdiagonal elements, it is possible to find a location j (for the actuator) and a location s_1 (for the sensor), $j \neq s_1$, so that sequence $\{t_{s_1,r}^{(M)} t_{j,r}^{(M)}\}_{r=1}^{n}$ will have only one sign change. □

Proof: The mass matrix M is diagonal and stiffness matrix $\widetilde{K}$ is an unreduced symmetric tridiagonal matrix. Under these circumstances, the first eigenvector will have no sign change, the second eigenvector will have one sign change, the third eigenvector will have two sign changes, and similarly the nth eigenvector will have $(n - 1)$ sign changes (Parlett 1980). It can also be proved that the first row of matrix $T^{(M)}$ will have no sign change, the second row will have one sign change, the third row will have two sign changes, and the nth row will have $(n - 1)$ sign changes (Golub and Van Loan 1989 and Parlett 1980). Therefore, for such a system, it is possible to choose two locations j and s_1, $j \neq s_1$, so that the sequence $\{t_{s_1,r}^{(M)} t_{j,r}^{(M)}\}_{r=1}^{n}$ will have ony one sign change. □

Remark 5.12: For the system defined in Result 5.9, if $j = 1$ (actuator's location), then the third condition of set $C1$ is satisfied. Also, if $s_1 = 2$ (sensor's location), the sequence $\{t_{2,r}^{(M)} t_{1,r}^{(M)}\}_{r=1}^{n}$ will have only one sign change. □

Proof: For this system, the eigenvalue problem can be written as $\widetilde{K}y = \lambda My$, where λ is any eigenvalue and y the corresponding eigenvector. Using the transformation $x = M^{1/2}y$ we should have, $\hat{K}x = \lambda x$, where tridiagonal matrix $\hat{K} = M^{-1/2}\widetilde{K}M^{-1/2}$. Now, if the first element of vector x is zero then it turns out that $x = 0$. Hence the result. Again, because the first row of modal matrix has no sign change and the second row has one sign change, the sequence $\{t_{2,r}^{(M)} t_{1,r}^{(M)}\}_{r=1}^{n}$ will have only one sign change. □

Result 5.10: When using an ID controller, for a system as defined in Result 5.9, where

(1) conditions $C1$ are satisfied,

(2) one sensor is used and it is not collocated with the actuator,

(3) the sign change in sequence $\{t_{s_1,r}^{(M)} t_{j,r}^{(M)}\}_{r=1}^{n}$ occurs when $r = m$,

(4) time delay $T_{s_1} = ((\pi/2\lambda_{m-1}) - \varepsilon)$, where ε is a small positive quantity and $(T_{s_1}\lambda_m) > (\pi/2)$,

(5) $(\lambda_{max}/\lambda_{m-1}) \leq 3$, and

(6) $K_1 > (K_2/\lambda_{min}^2)$, or $K_1 < (K_2/\lambda_{max}^2)$,

it is possible to stabilize an undamped (open loop) system for small gains. □

Proof: When using only one sensor which is non-collocated with the actuator, for an ID controller with $K_1 > K_2/\lambda_{min}^2$, the stability condition for small gains requires (see (26))

$$a_{s_1}\cos(\lambda_r T_{s_1}) t_{s_1,r}^{(M)} t_{j,r}^{(M)} > 0 \qquad \text{for } r = 1, 2, \ldots, n \tag{58}$$

Noting that $T_{s_1} = ((\pi/2\lambda_{m-1}) - \varepsilon)$, $(T_{s_1}\lambda_m) > (\pi/2)$, $(\lambda_{max}/\lambda_{m-1}) \leq 3$, and the fact that the first column of matrix $T^{(M)}$ has no sign change, relation (58) will be satisfied for $a_{s_1} > 0$. Hence the result. A similar argument can be given when $K_1 < (K_2/\lambda_{max}^2)$. □

Result 5.11: For the undamped system described in Result 5.10, velocity feedback control will be stable as long as $\mu < G$, where G is the minimum of all positive B_l, for $l = 0, 1, 2, \ldots$, where

$$B_l = \frac{-1}{a_{s_1} K_1 \eta_l \sin(\eta_l T_{s_1}) \sum_{i=1}^{n} \frac{t_{s_1,i}^{(M)} t_{j,i}^{(M)}}{\lambda_i^2 - \eta_l^2}} \tag{59}$$

and

$$\eta_l = \frac{(2l+1)\pi}{2T_{s_1}}$$

□

Proof: The proof is very similar to that of Result 5.3. □

5.2. *Results for underdamped system*

Result 5.12: When using PID control for underdamped systems, $\xi_i < \lambda_i$, $i = 1, 2, \ldots, n$, a sufficient condition for the closed loop system to be stable for small gains is

$$-\frac{1}{(\lambda_r^2 - \xi_r^2)^{1/2}}\left[K_0 - \left(K_1 + \frac{K_2}{\lambda_r^2}\right)\xi_r\right] \times \left(\sum_{k=1}^{p} a_{s_k}\exp[\xi_r T_{s_k}]\sin((\lambda_r^2 - \xi_r^2)^{1/2} T_{s_k}) t_{s_k,r}^{(M)} t_{j,r}^{(M)}\right) + \left[K_1 - \frac{K_2}{\lambda_r^2}\right]\left(\sum_{k=1}^{p} a_{s_k}\exp[\xi_r T_{s_k}]\cos((\lambda_r^2 - \xi_r^2)^{1/2} T_{s_k}) t_{s_k,r}^{(M)} t_{j,r}^{(M)}\right) > 0$$

$$\text{for } r = 1, 2, \ldots, n \tag{60}$$

□

Proof: Since by (20), for PID control

$$\operatorname{Re}\left\{\frac{ds}{d\mu}\bigg|_{\substack{\mu\to 0\\ s\to\gamma_{\pm r}}}\right\} = -\frac{1}{2}\left(K_1 - \frac{K_2}{\lambda_r^2}\right)\left(\sum_{k=1}^{p} a_{s_k}\exp[\xi_r T_{s_k}]\cos((\lambda_r^2-\xi_r^2)^{1/2}T_{s_k})t_{s_k,r}^{(M)}t_{j,r}^{(M)}\right)$$
$$+\frac{1}{2(\lambda_r^2-\xi_r^2)^{1/2}}\left(K_0-\left(K_1+\frac{K_2}{\lambda_r^2}\right)\xi_r\right)$$
$$\times\left(\sum_{k=1}^{p} a_{s_k}\exp[\xi_r T_{s_k}]\sin((\lambda_r^2-\xi_r^2)^{1/2}T_{s_k})t_{s_k,r}^{(M)}t_{j,r}^{(M)}\right) \quad (61)$$

the result follows from Result 4.1. □

Result 5.13: When the sensor and actuator are collocated and only one sensor is used, for PID control, if $(K_1-(K_2/\lambda_r^2))\neq 0$, for all r, a sufficient condition for small gains stability is

$$a_j\left(K_1-\frac{K_2}{\lambda_r^2}\right)\cos((\lambda_r^2-\xi_r^2)^{1/2}T_{s_k}+\phi)>0, \quad \text{for } r=1,2,\ldots,n \quad (62)$$

where

$$\phi=\tan^{-1}\left[\frac{K_0-\left(K_1+\frac{K_2}{\lambda_r^2}\right)\xi_r}{\left(K_1-\frac{K_2}{\lambda_r^2}\right)(\lambda_r^2-\xi_r^2)^{1/2}}\right] \quad (63)$$

Proof: Here $p=1$ and $j=s_k$. Noting that $\exp[\xi_r T_j]\{t_{j,r}^{(M)}\}^2$ is a positive quantity, we get from inequality (60)

$$\left\{\cos((\lambda_r^2-\xi_r^2)^{1/2}T_{s_k})-\left[\frac{K_0-\left(K_1+\frac{K_2}{\lambda_r^2}\right)\xi_r}{\left(K_1-\frac{K_2}{\lambda_r^2}\right)(\lambda_r^2-\xi_r^2)^{1/2}}\right]\sin((\lambda_r^2-\xi_r^2)^{1/2}T_{s_k})\right\}$$
$$\times a_j\left(K_1-\frac{K_2}{\lambda_r^2}\right)>0, \quad \text{for } r=1,2,\ldots,n \quad (64)$$

Hence the result.

Result 5.14(*a*): When using one sensor, collocation of the sensor with an actuator will cause the closed loop poles for velocity feedback to move to the left in the s-plane as long as the time delay is

$$T_j<\min_{\forall r}\left[\frac{\frac{\pi}{2}+\phi}{(\lambda_r^2-\xi_r^2)^{1/2}}\right] \quad (65)$$

where

$$\phi=\tan^{-1}\left[\frac{\xi_r}{(\lambda_r^2-\xi_r^2)^{1/2}}\right] \quad (66)$$

□

Proof: Noting that $K_0 = K_2 = 0$ and $K_1 > 0$, the condition (62) is satisfied for any $a_j > 0$. Hence the result. □

Result 5.14(*b*): When using one sensor, collocation of the sensor with an actuator will cause the closed loop poles of the integral feedback control system to move to the left in the s-plane as long as the time delay is

$$T_j < \min_{\forall r} \left[\frac{\frac{\pi}{2} - \phi}{(\lambda_r^2 - \xi_r^2)^{1/2}} \right] \tag{67}$$

where ϕ is as defined in (66). □

Proof: Here $K_0 = K_1 = 0$ and $K_2 > 0$, the condition (62) is satisfied for any $a_j < 0$. Hence the result. □

Result 5.14(*c*): When using one sensor, collocation of the sensor with an actuator will cause the closed loop poles of a PID feedback control system to move to the left in the s-plane as long as the time delay is

$$T_j < \min_{\forall r} \left[\frac{\frac{\pi}{2} - \phi}{(\lambda_r^2 - \xi_r^2)^{1/2}} \right] \tag{68}$$

where ϕ is as defined in (63), when $K_1 > (K_2/\lambda_{\min}^2)$ and $a_j > 0$, or $K_1 < (K_2/\lambda_{\max}^2)$ and $a_j < 0$. □

Proof: The result is obvious from relation (62). □

Result 5.14(*d*): When using one sensor, collocation of the sensor with an actuator will cause the closed loop poles of a pure proportional feedback control system to move to the left in the s-plane as long as the time delay is

$$0 < T_j < \min_{\forall r} \left[\frac{\pi}{(\lambda_r^2 - \xi_r^2)^{1/2}} \right] \tag{69}$$

□

Proof: When $K_1 = K_2 = 0$, from relation (60) for stability we should have

$$a_j \left[- \frac{K_0}{(\lambda_r^2 - \xi_r^2)^{1/2}} \sin((\lambda_r^2 - \xi_r^2)^{1/2} T_{s_k}) \right] > 0, \quad \text{for } r = 1, 2, \ldots, n \tag{70}$$

Noting that K_0 and $(\lambda_r^2 - \xi_r^2)^{1/2}$ both are positive, the result follows for any $a_j < 0$. □

Result 5.15: For lightly damped systems ($\xi_r \ll 1$) whose mass matrix is diagonal, non-collocated PID control with no time delays will most likely destabilize the system when either $K_1 > (K_2/\lambda_{\min}^2)$, or $K_1 < (K_2/\lambda_{\max}^2)$. An approximate bound on the gain to ensure stability can be found. □

Proof: Under these provisions, a sufficient condition for the closed loop system to be stable for small gains is (from relation (60))

$$\left[K_1 - \frac{K_2}{\lambda_r^2} \right] \left(\sum_{k=1}^{p} a_{s_k} t_{s_k,r}^{(M)} t_{j,r}^{(M)} \right) > 0, \quad \text{for } r = 1, 2, \ldots, n \tag{71}$$

By Remark 5.7, relation (71) cannot be satisfied if M is diagonal. For very lightly damped systems ($\xi_r \ll 1$), and $K_1 > (K_2/\lambda_{\min}^2)$ we thus have, for some $r \in (1, n)$

$$\left(\sum_{k=1}^{p} b_{s_k} t_{s_k,r} t_{j,r}\right) = -\delta_r < 0 \tag{72}$$

and then non-collocation will most likely lead to instability as the root locus will move towards the right half s-plane. Then, an approximate bound on μ_r to ensure stability of the rth pole, is obtained by using (61) as $\mathrm{Re}\{\Delta s\} \approx \mu_r \delta_r (K_1 - K_2/\lambda_r^2)/2$. To make this less than ξ_r, so that the pole remains in the left half s-plane, requires that

$$\mu_r < \frac{2\xi_r}{\delta_r\left(K_1 - \dfrac{K_2}{\lambda_r^2}\right)} \tag{73}$$

The value of μ for stability would then be the minimum of μ_r taken over all such rs for which $\delta_r > 0$. A similar argument can be made for $K_1 < (K_2/\lambda_{\max}^2)$. □

6. Numerical results and discussion

Consider an undamped shear frame building structure shown in Fig. 1. The mass and stiffness of each storey are 1 and 1600 (taken in SI units). The system may also be thought of as a finite dimensional representation of a bar undergoing axial vibrations. The mass matrix is the identity matrix. With these system parameters, the undamped natural frequencies are calculated as given in the Table.

Various examples of the structural response are numerically computed in this section, serving as verification of our theoretical results. For integration, we use the fourth-order Runge–Kutta scheme. The time step for integration, Δt, has been so taken that Δt (= 0·004 s) < $T_{\min}/20$, where $T_{\min}$ is the minimum period of vibration of the structure. For response results, a very small amount of damping has been introduced in each mode of vibration of the structure so that smooth integration can be carried out. The percentages of critical damping introduced in the various modes of vibration of the structure, are also given in the Table. For all the root loci plots in this section, the controller's gain μ has been varied from 0 to 100 units. Response time history plots are shown only for the first 10 s.

Mode Nos	Natural frequency (rad s^{-1})	Time period (s)	Damping ratio (per cent critical)
1	11·3852	0·552	0·18
2	33·2332	0·189	0·52
3	52·3889	0·120	0·82
4	67·3003	0·093	1·05
5	76·7594	0·082	1·20

Natural frequencies and the modal damping ratios.

Example 6.1: In this example, we have studied the collocated control of the structure with and without appropriate time delays. The controller and the sensor both are located at mass 4. The controller's transfer function has been taken as $\tau_c(s) = s$ (i.e. velocity feedback control). Coefficient a_4 has been taken as unity. Figure 2 illustrates that when the time delay $T_4 = 0{\cdot}025$ s (greater than $(\pi/2\lambda_{max})$), collocated control is unstable for all $\mu > 0$. Appropriately taking the value of this time delay, $T_4 = 0{\cdot}018$ s, we show in Fig. 3 that root loci of the closed loop poles of the collocated control system remain in the left half s-plane as long as gain μ is less than 37·9 units. We observe that the fifth pole crosses the imaginary axis of the s-plane at $\eta = 87{\cdot}27\ \text{rad s}^{-1}$. These numerically obtained values of the gain μ and the cross-over frequency η are exactly those obtained from theoretical results given in (33) and (41).

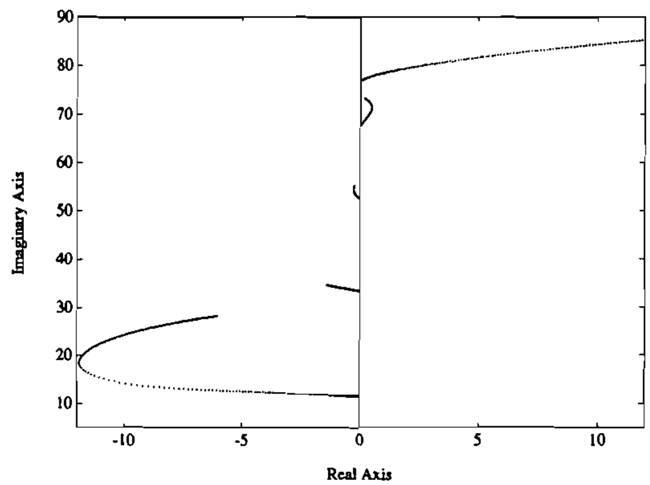

Figure 2. Root loci of closed loop poles of the velocity feedback collocated control system ($j = 4$, $s_1 = 4$, $a_4 = 1$, $T_4 = 0{\cdot}025$ s).

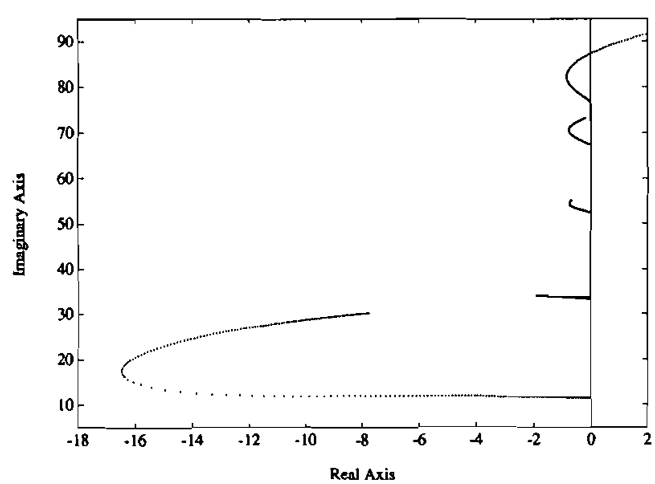

Figure 3. Root loci of closed loop poles of the velocity feedback collocated control system ($j = 4$, $s_1 = 4$, $a_4 = 1$, $T_4 = 0{\cdot}018$ s).

Figure 4 shows the displacement time history of mass 5 relative to the base, when this structure is subjected to the S00E component of the Imperial Valley Earthquake 1940 ground motion. The response has been shown for $\mu = 0$ and $\mu = 20{\cdot}0$ units. Figure 5 contains the time histories of the incoming force per storey (i.e. negative of storey mass times ground acceleration) and the control force required when the controller's gain $\mu = 20{\cdot}0$ units. When the gain is 50·0 units, which is more than the upper bound on μ needed for stability for the undamped system, as predicted by (33), the system becomes unstable. Figure 6 illustrates this result. It is observed that the upper bound on the gain for the undamped system gives a good approximation for the bound for the lightly damped system.

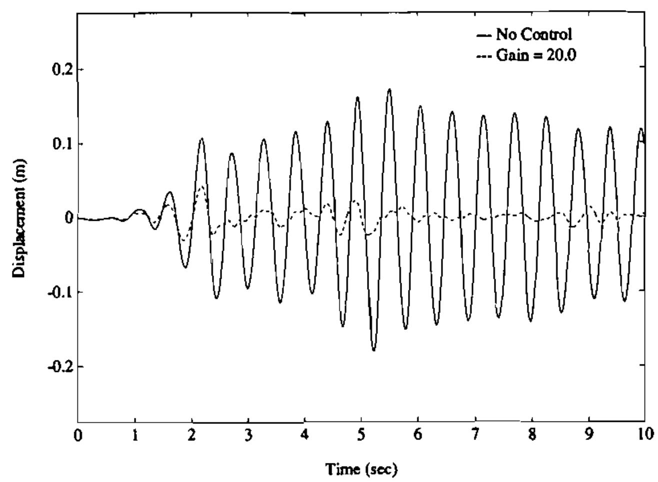

Figure 4. Relative displacement response of mass 5 ($j = 4$, $s_1 = 4$, $a_4 = 1$, $T_4 = 0{\cdot}018$ s) for collocated stable control.

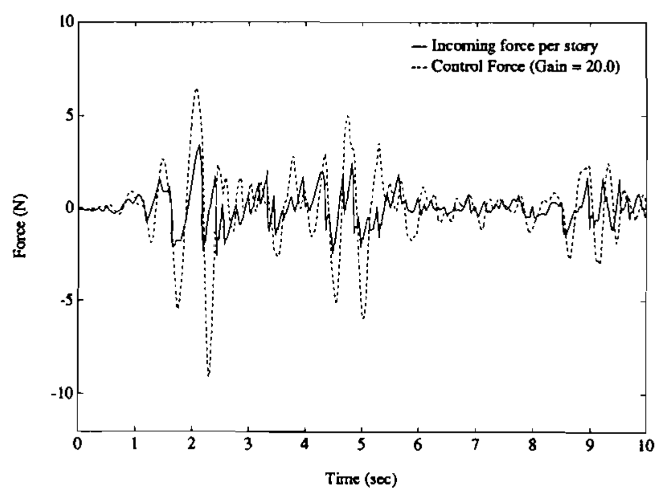

Figure 5. Incoming force per storey and control force time histories ($j = 4$, $s_1 = 4$, $a_4 = 1$, $T_4 = 0{\cdot}018$ s) for collocated stable control.

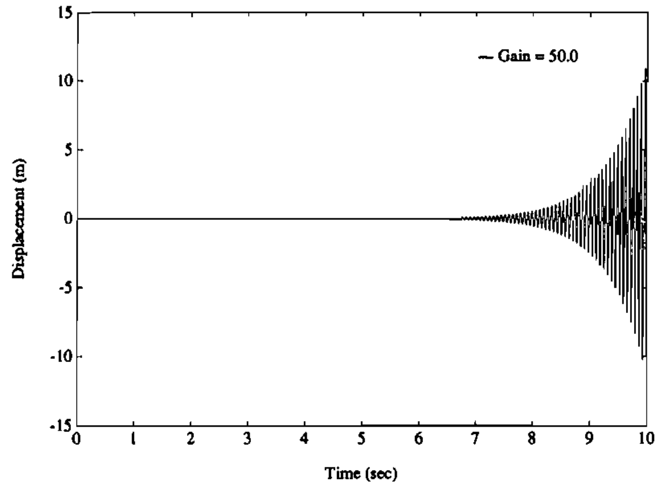

Figure 6. Relative displacement response of mass 5 ($j = 4$, $s_1 = 4$, $a_4 = 1$, $T_4 = 0{\cdot}018$ s) for collocated unstable control with $\mu = 50{\cdot}0$ units.

Example 6.2: Figure 7 illustrates Remark 5.2 and shows that when two sensors are used (at locations 4 and 2) one of which is collocated with the actuator, the control may not be stable. Here, the controller is located at location 4 and is fed the velocity signal from mass 4 and half of the velocity signal from mass 2, the other as being zero. Appropriately changing the values of the as may be thought of as a method of changing the effective damping for each closed loop pole. The effects of the dislocation of the actuator and a single sensor are shown in Fig. 8 (time delay $T_5 = 0$). Here we see ($a_5 = 1$, all other as $= 0$) that the velocity feedback non-collocated control system is unstable as guaranteed by Result 5.6. This system is made stable through the use of an appropriate time delay, i.e. $T_5 = 0{\cdot}04$ s, which is less than $\pi/2\lambda_2$ (Result 5.10). Figure 9 shows the root loci

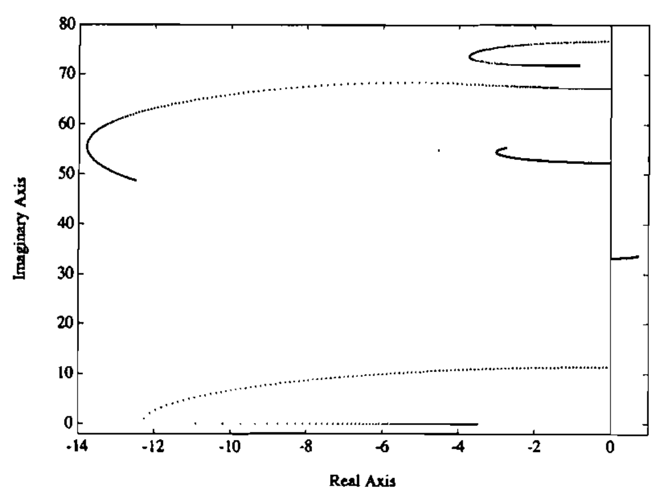

Figure 7. Root loci of closed loop poles of the velocity feedback control system ($j = 4$, $s_1 = 4$, $s_2 = 2$, $a_2 = 0{\cdot}5$, $a_4 = 1$, $T_2 = T_4 = 0$ s).

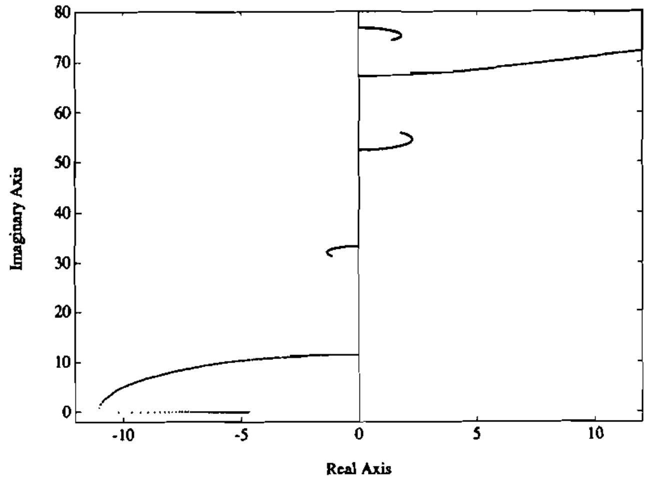

Figure 8. Root loci of closed loop poles of the velocity feedback non-collocated control system ($j = 4$, $s_1 = 5$, $a_5 = 1$, $T_5 = 0$ s).

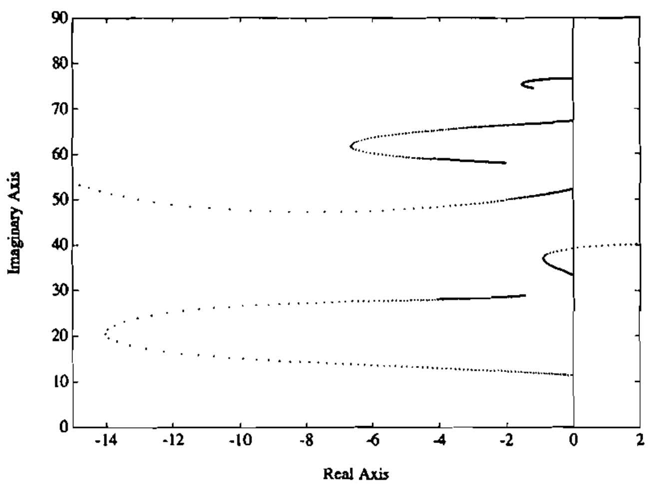

Figure 9. Root loci of closed loop poles of the velocity feedback non-collocated control system ($j = 4$, $s_1 = 5$, $a_5 = 1$, $T_5 = 0{\cdot}04$ s).

of the closed loop poles under this situation. We note that the second pole crosses the imaginary axis as $\eta = 39{\cdot}24\,\mathrm{rad\,s^{-1}}$ and that the value of the gain corresponding to this cross-over is $\mu = 39{\cdot}0$ units. This numerically obtained upper bound on μ is in exact agreement with Result 5.11.

The displacement response of mass 5 relative to the base for $\mu = 0$ and $\mu = 10{\cdot}0$ units, when the system is subjected to the same ground motion as used in Example 6.1, has been shown in Fig. 10. Figure 11 contains the required control force for gain $\mu = 10{\cdot}0$ units. When we increase the value of the gain (i.e. $\mu = 42{\cdot}0$ units) to more than the minimum upper bound needed for stability for the undamped system (as computed from Result 5.11, i.e. $\mu = 39{\cdot}0$ units),

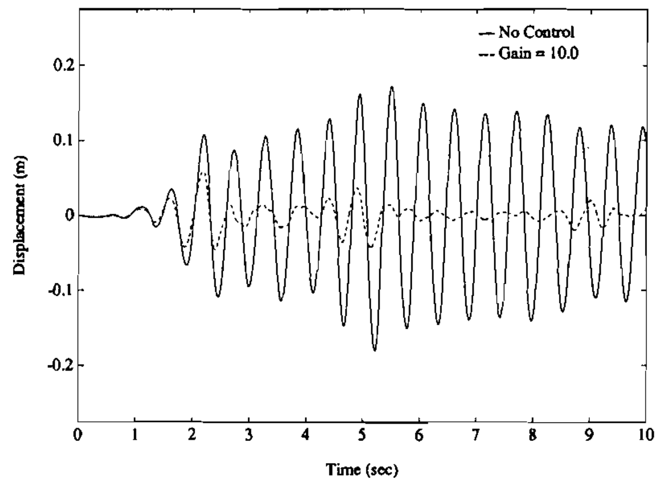

Figure 10. Relative displacement response of mass 5 ($j = 4$, $s_1 = 5$, $T_5 = 0{\cdot}04$ s) for non-collocated stable control.

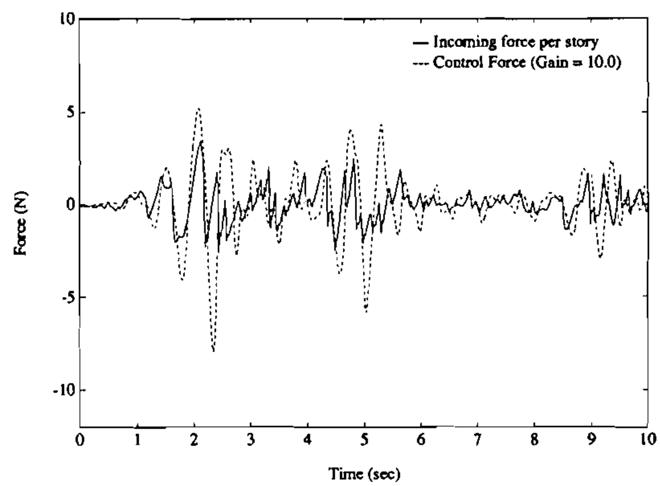

Figure 11. Incoming force per storey and control force time histories ($j = 4$, $s_1 = 5$, $T_5 = 0{\cdot}04$ s) for non-collocated stable control.

the system becomes unstable. Figure 12 depicts this result. For this case also, we numerically found that the upper bound on μ for the undamped system approximates well to that for the lightly damped system.

Figure 13 shows the root loci of the closed loop poles for velocity feedback non-collocated time delayed control of the system considered, when using three sensors. Here, the controller is located at mass 3 and sensors are put at mass 2, mass 4 and mass 5, respectively. Coefficients a_2, a_4 and a_5 are taken to be $1{\cdot}0$, $-1{\cdot}0$ and $1{\cdot}0$ and corresponding time delays are $T_2 = 0{\cdot}025$ s, $T_4 = 0{\cdot}0835$ s and $T_5 = 0{\cdot}055$ s. From this figure, it is obvious that when using more than one sensor, non-collocated control can be made stable if appropriately delayed response signals are used in the feedback loop.

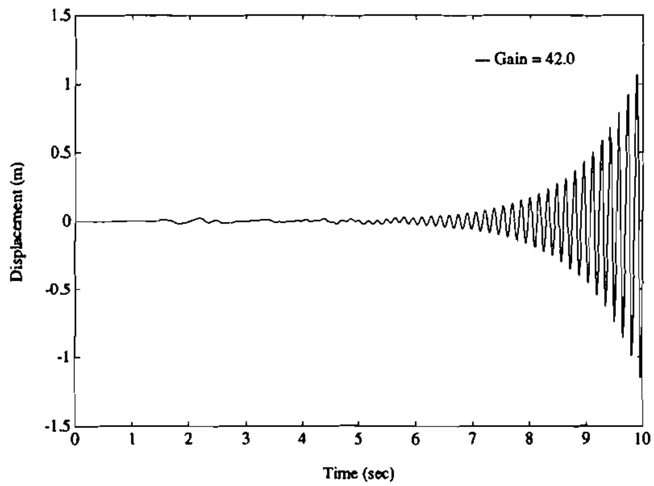

Figure 12. Relative displacement response of mass 5 ($j = 4$, $s_1 = 5$, $T_5 = 0{\cdot}04$ s) for non-collocated unstable control with $\mu = 42{\cdot}0$ units.

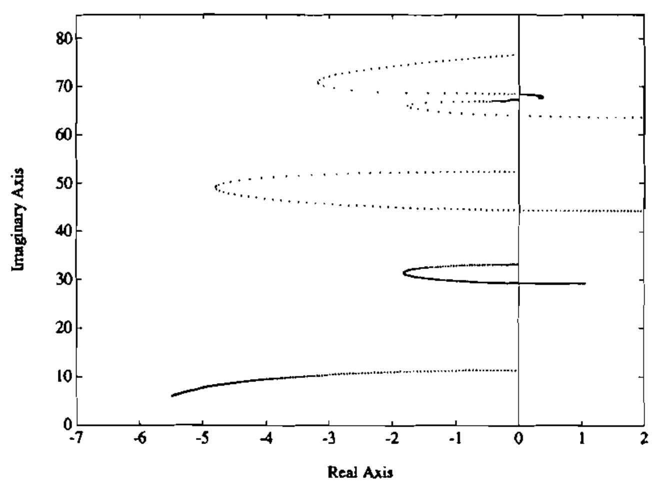

Figure 13. Root loci of closed loop poles of the velocity feedback non-collocated control system ($j = 3$, $s_1 = 5$, $s_2 = 4$, $s_3 = 2$).

7. Conclusions

Several results related to both the collocated and non-collocated time delayed control of undamped as well as underdamped multi-degree-of-freedom systems have been presented. While the results are specifically related to PID controllers, the general approach provided in this paper can be used for all finite-dimensional controllers. It is shown that time delays, which make collocated control systems unstable, can help stabilize non-collocated control systems. Some of the results of this study are summarized as follows.

(1) When using one sensor in collocation with the actuator, pure velocity (or integral) feedback control of the undamped system is stable for vanishingly small

gain as long as the time delay is less than $\pi/2\lambda_{max}$, where λ_{max} is the highest natural frequency of the undamped system. The upper bound on the gain μ (for velocity feedback) to prevent instability is explicitly given in (33).

(2) Stability of a feedback control system is not ensured when responses obtained from a number of sensors are used, one among which is collocated with the actuator, even when using no time delays.

(3) Undamped systems with diagonal mass matrices are guaranteed to become unstable under direct (no time delays) velocity (or integral) feedback control when the sensors and the actuator are dislocated, no matter how many such sensors are used.

(4) It has been shown that for special classes of undamped systems, when just one sensor is used, dislocation accompanied by a suitable time delay will ensure stability of the control system. An explicit method for determining this time delay is provided.

(5) Collocation of a sensor with the actuator causes the pure velocity (or integral) feedback control of underdamped systems to be stable as long as the time delay in the information between the sensor and the actuator is less than some prescribed value, which depends on the open loop system's parameters.

(6) For very lightly damped systems, in the presence of small time delay, collocation of a sensor with the actuator will most likely cause *negative* proportional feedback control to destabilize the closed loop system.

(7) For very lightly damped systems, non-collocation with zero time delays will most likely lead to instability. An approximate bound on the gain μ to prevent this instability is provided.

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