A New Approach to Stable Optimal Control of Complex Nonlinear Dynamical Systems

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This paper gives a simple approach to designing a controller that minimizes a user-specified control cost for a mechanical system while ensuring that the control is stable. For a user-given Lyapunov function, the method ensures that its time rate of change is negative and equals a user specified negative definite function. Thus a closed-form, optimal, nonlinear controller is obtained that minimizes a desired control cost at each instant of time and is guaranteed to be Lyapunov stable. The complete nonlinear dynamical system is handled with no approximations/linearizations, and no a priori structure is imposed on the nature of the controller. The methodology is developed here for systems modeled by second-order, nonautonomous, nonlinear, differential equations. The approach relies on some recent fundamental results in analytical dynamics and uses ideas from the theory of constrained motion. [DOI: 10.1115/1.4024874]

1 Introduction

Lyapunov's second method has today become the method of choice in determining the stability of a proposed control design for a dynamical system. Most often for complex nonlinear systems, a controller is postulated, often on heuristic grounds, and its stability is checked by searching for a suitable Lyapunov function V that ensures that its time derivative is nonpositive [1]. Though there are some standard methods that one can get guidance from in the search for a suitable Lyapunov function, when handling complex, nonlinear, high-dimensional dynamical systems, this can become a difficult and time consuming process, which may at times not be fruitful. When one is unable to find such a function, the stability of the postulated control is left uncertain.

This paper uses Lyapunov's second method as the essential vehicle to obtain sets of stable controllers that minimize a desired control cost at *each* instant of time. We consider systems modeled by second order, nonautonomous, nonlinear differential equations of the form [2,3]

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) \tag{1}$$

where $M(q, t) > 0$ is an n by n matrix, q is an n-vector, and Q is an n-vector whose components are known C^1 functions of the arguments q, $\dot{q}$, and t. The dots over the various quantities denote derivatives with respect to time. Such descriptions often arise in the modeling of complex structural and mechanical systems when employing Lagrangian and/or Newtonian mechanics.

We shall assume that Eq. (1) is defined in the domain $D \times R^+$ where $D \subset R^n \times R^n$. Our aim is to control the system by using a controller $Q^c(q,\dot{q},t) \in D \times R^+$ so that the controlled system [1,4]

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q^C(q,\dot{q},t) := f(q,\dot{q},t) \tag{2}$$

is brought to the fixed point of the system, which is assumed to be given by $f(0,0,t) = 0 \ \forall \ t$, that is at $q = \dot{q} = 0$. The usual approach in control design is to first postulate a controller Q^C, and then check its stability, most often using Lyapunov's second method.

In this paper, we show a simple method for finding a controller Q^C for nonlinear mechanical systems described by Eq. (1) so that:

(1) a user-given control cost is minimized at each instant of time, and
(2) a user-specified (candidate) Lyapunov function is required to decrease at a user-specified rate, which is prescribed by a given function of the state of the dynamical system.

No a priori structure is imposed on the controller, no approximations/linearizations are made with respect to the dynamics of the nonlinear system, and the set of nonlinear controllers is obtained in closed form.

The inspiration for the results developed here come from principles that underlie the foundations of analytical mechanics and recent developments in the theory of constrained motion. In fact, we view our problem within the context of constrained motion and take as our objective the minimization of the control cost when the system is "constrained" to move so that it satisfies the requirement imposed by item (2) above. Nature, in like manner, determines the control force Q^C to be applied to a constrained mechanical system by minimizing the Gaussian, which she takes as the control cost, subject to any given consistent set of constraints that the dynamical system is required to satisfy [2,5].

We begin by considering a Lyapunov function $V(q,\dot{q},t)$ such that [1]

$$V_1(q,\dot{q}) \le V(q,\dot{q},t) \le V_2(q,\dot{q}) \tag{3}$$

where $V_1(q,\dot{q})$ and $V_2(q,\dot{q})$ are positive definite functions on a domain D, and

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}}\ddot{q} < 0 \tag{4}$$

in D. In what follows, we shall require that the time rate of change of V along the trajectories of the dynamical system not merely be negative but decrease at a user-specified rate so that

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}}\ddot{q} = \dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial \dot{q}}\ddot{q} = -w(q,\dot{q}) \tag{5}$$

where $w(q,\dot{q})$ is a user-specified positive definite function in D.

Any controller that causes the dynamics of the controlled system to satisfy the relation in Eq. (4) for a given candidate

Manuscript received October 31, 2011; final manuscript received June 14, 2013; accepted manuscript posted June 21, 2013; published online September 18, 2013. Assoc. Editor: Alexander F. Vakakis.

Lyapunov function $V(q, \dot{q}, t)$, which satisfies the relation in Eq. (3), ensures that the fixed point $q = \dot{q} = 0$ is uniformly asymptotically stable. Furthermore, we shall require V to decrease at a user-prescribed, specified rate $w(q, \dot{q})$ described by Eq. (5). We shall call any function that only satisfies the condition in Eq. (3) a positive definite function or a candidate Lyapunov function for short.

We assume that we have a suitable candidate Lyapunov function $V(q, \dot{q}, t)$, which we would like to use as a Lyapunov function for our system described by Eq. (2). As such, the Lyapunov function is a kind of surrogate for the energy of the system, and the relation in Eq. (5) says, in a rough manner, that we require the time rate of change of energy of the system to reduce in a specific manner described by the given function $w(q, \dot{q})$; the system is thus continually losing energy at a prespecified rate w until it eventually reaches its minimum energy, which occurs at the fixed point $q = \dot{q} = 0$.

In the control literature a variant of this problem appears to have been first broached by Sontag [6][1]. Using a given so-called control Lyapunov function (CLF), V, and with the linear quadratic (LQ) problem as inspiration, closed form controls are found in Ref. [6] for a given CLF that satisfies the *inequality* relation in Eq. (4), i.e., the proposed controllers guarantee that the inequality $\dot{V} < 0$ is satisfied. There are several points of divergence between Ref. [6] and the present work. (i) Instead of framing the problem in terms of a family of linear stabilizable systems parameterized by the state (which requires considerably more mathematical machinery), herein the problem is framed as one of constrained motion of the mechanical system. It is formulated by requiring the minimization of a given cost function at each instant of time subject to the *equality* constraint given by Eq. (5). We thereby ensure a specified rate at which the Lyapunov function V decays over time along the system's dynamical trajectory. For a mechanical system, as mentioned before, the Lyapunov function is often taken to be a surrogate for the energy of the system, and this constraint then corresponds to controlling, in a definite manner, the energy decay rate in the system. (ii) Perhaps the most important difference lies in the totally different mathematical approach used as compared to the development in Ref. [6]; the central result obtained herein follows quite simply from only the use of elementary linear algebra. (iii) No cost minimization is done in Ref. [6]. Our central goal is to minimize the control cost at each instant of time under the constraint provided by Eq. (5).

Improvements and extensions of the basic landmark result in Ref. [6] have been obtained in the controls literature over the following decade (e.g., Refs. [7,8]). These improvements have culminated in ensuring the inequality given by the relation in Eq. (4) along with minimization of the integral of the control cost, albeit at the expense of considerable mathematical sophistication [7]. By comparison, the approach used here is very simple and relies on, and gets its inspiration from, some fundamental results in the analytical dynamics of constrained motion; additionally, it minimizes the control costs at each instant of time (and not as an integral over time as has been done and is common in LQ problems). Furthermore, it allows one to employ a user-specified time decay rate of a user-specified candidate Lyapunov function.

A well-developed method in the control literature for handling output feedback of a class of nonlinear systems is backstepping [9]. Here, every state of the system is essentially controlled in a recursive fashion—one could think of the state as a "virtual control"—and the Lyapunov function is successively modified. In this approach, the successive "folding in" of the control provided at each stage of the recursive process through successive changes in the Lyapunov function precludes the use of a user-specified candidate Lyapunov function for the entire system. Also, backstepping does not deal with minimizing the control costs. Thus standard backstepping does not address the central issues in this paper, namely, minimizing the total control cost at each instant of time while causing a given user-specified candidate Lyapunov function (for the entire system) to decay at a user-specified rate.

Another popular method used for the control of nonlinear systems is the state-dependent Riccati equation (SDRE) method, which gets its inspiration from linear quadratic regulator (LQR) theory. Here an autonomous nonlinear system is described through factorization of the nonlinear dynamics into a state-dependent matrix and the state vector thereby yielding for the nonlinear system a nonunique linear structure. A performance index with a quadratic-like structure is minimized by solving an algebraic Riccati equation to give the suboptimal control law at each point in state space. Thus, the SDRE approach is far more complex than the one presented herein from both analytical and computational standpoints. Since solving the necessary Riccati equations on-line is computationally quite intensive especially for systems with a large number of degrees of freedom, the method has considerable limitations, besides being applicable to only autonomous systems. For an extensive list of references on SDRE, see Ref. [10].

2 Central Result

For a given positive definite function $V(q, \dot{q}, t)$, Eq. (5) can be expressed as

$$\frac{\partial V}{\partial \dot{q}} \ddot{q} = -w(q, \dot{q}) - \frac{\partial V}{\partial t} - \frac{\partial V}{\partial q} \dot{q} \tag{6}$$

which can be rewritten in the form [2,3]

$$A(q, \dot{q}, t) \ddot{q} = b(q, \dot{q}, t) \tag{7}$$

with

$$A(q, \dot{q}, t) = \frac{\partial V}{\partial \dot{q}}, \quad \text{and} \quad b(q, \dot{q}, t) = -w(q, \dot{q}) - \frac{\partial V}{\partial t} - \frac{\partial V}{\partial q} \dot{q} \tag{8}$$

We note that A is a 1 by n matrix, and b is a scalar. We view Eq. (7) as a consistent constraint imposed on the dynamical system described by Eq. (1).

Our aim is to find the control force Q^C that we need to apply to the system described by Eq. (1) so that the control cost is minimized at each instant of time and the ensuing dynamics causes the consistent constraint in Eq. (6) (or alternatively Eq. (7)) to be satisfied. We have the following result.

Result. Given the dynamical system

$$M(q, t) \ddot{q} = Q(q, \dot{q}, t) \tag{9}$$

with $M(q, t) > 0$ and

(i) a suitable candidate Lyapunov function $V(q, \dot{q}, t)$ that is positive definite, and
(ii) a candidate function $w(q, \dot{q})$ that is positive definite

the control force $Q^C(q, \dot{q}, t)$ that causes the controlled system

$$M(q, t) \ddot{q} = Q(q, \dot{q}, t) + Q^C(q, \dot{q}, t) := f(q, \dot{q}, t) \tag{10}$$

(1) to minimize, at each instant of time t, the control cost

$$J(t) = [Q^C]^T N(q, t) [Q^C] := \|Q^C\|_N^2 \tag{11}$$

where N is a user-prescribed positive definite matrix, and

(2) to have the asymptotically stable equilibrium point given by $f(0, 0, t) = 0$ by ensuring that the candidate Lyapunov function $V(q, \dot{q}, t)$ *is* a Lyapunov function for the controlled system through the satisfaction of the relation in Eq. (5)

[1]The author is indebted to an anonymous reviewer who brought this to his attention.

is explicitly given by

$$Q^C(q,\dot{q},t) = N^{-1/2}G^{+}(b - AM^{-1}Q) \tag{12}$$

where,

$$G = A(N^{1/2}M)^{-1}, \quad A(q,q,t) = \frac{\partial V}{\partial \dot{q}}$$
$$\text{and} \quad b = -w(q,\dot{q}) - \frac{\partial V}{\partial t} - \frac{\partial V}{\partial q}\dot{q} \tag{13}$$

In Eq. (12), we denote by G^+ the Moore–Penrose inverse of the matrix G.

Proof. We need to find the generalized control force Q^C that is such that the constraint given by Eq. (7) is satisfied for the chosen positive definite candidate functions V and w. For brevity, from here on we shall suppress the arguments of the various quantities unless needed for clarity.

Since by Eq. (10) $Q^C = M\ddot{q} - Q$, let us denote

$$z(t) = N^{1/2}Q^C = N^{1/2}(M\ddot{q} - Q) \tag{14}$$

so that from the relation in Eq. (11) we get

$$J(t) = \|z(t)\|^2 \tag{15}$$

Furthermore, Eq. (14) can be rewritten as

$$\ddot{q} = (N^{1/2}M)^{-1}(z(t) + N^{1/2}Q) \tag{16}$$

Since the controlled system must satisfy Eq. (7), namely $A\ddot{q} = b$, Eq. (16) yields

$$A(N^{1/2}M)^{-1}z = b - AM^{-1}Q := b_1 \tag{17}$$

where A and b are defined in the relations given in Eq. (13).

Setting $G = A(N^{1/2}M)^{-1}$, the vector z that satisfies Eq. (17) while simultaneously minimizing $J(t)$ shown in Eq. (15), is given by

$$z(t) = G^{+}(b - AM^{-1}Q) = G^{+}b_1 \tag{18}$$

so that the necessary control force Q^C is given, on using the first equality in Eq. (14), by

$$Q^C = N^{-1/2}G^{+}(b - AM^{-1}Q) = N^{-1/2}\frac{G^T}{\|G\|^2}(b - AM^{-1}Q) \tag{19}$$

where in the last equality we have made use of the fact that the matrix G has only one row [5]. □

Remark 1. It is important to realize that in solving Eq. (17) for z we are assuming that the right hand side of Eq. (17) is in the range space of the matrix G at each instant of time. Thus, both the candidate functions V and w need to be specified in a manner such that this would be true at each instant of time. A necessary and sufficient condition for this to be true is that $GG^{+}b_1 = b_1$. Also, the equation $A\ddot{q} = b$ must be consistent. Hence, when $A(q^*,\dot{q}^*,t) = 0$, we require $b(q^*,\dot{q}^*,t) = 0$, with A and b given in Eqs. (7) and (8).

Remark 2. No a priori structure is imposed on the nonlinear controller and no approximations/linearizations of the nonlinear dynamical system described by Eq. (9) are done.

Remark 3. For each candidate Lyapunov function V and each positive definite function w chosen, a stable, optimal controller is obtained in closed form that minimizes a desired norm $\|Q^C\|_N^2$ of the control cost at each instant of time.

Remark 4.

(a) When the positive definite matrix N is chosen to be M^{-1}, then $G = AM^{-1/2}$ and the function $J(t) = [Q^C]^T M^{-1}[Q^C]$ becomes the well-known Gaussian used in analytical dynamics [5]. Nature uses this $J(t)$ as the control cost when a mechanical system is required to move in the presence of constraints, and it is this insight from analytical dynamics that is the inspiration for the approach proposed in this paper. The control force she provides is given explicitly by

$$Q^C = M^{1/2}\left(AM^{-1/2}\right)^{+}(b - AM^{-1}Q)$$
$$= \frac{A^T}{(AM^{-1}A^T)}(b - AM^{-1}Q) \tag{20}$$

as in the description of constrained motion in analytical dynamics [3–5]. The last equality arises because A is a row matrix.

(b) When N is chosen to be M^{-2}, then $G = A$, and the control force is explicitly given by

$$Q^C = MA^{+}(b - AM^{-1}Q) = M\frac{A^T}{(AA^T)}(b - AM^{-1}Q) \tag{21}$$

The cost function $J(t) = [Q^C]^T M^{-2}[Q^C]$ engendered by this choice of the matrix N is often used in fields like multibody dynamics and robotics.

(c) When $N = I$, then $G = AM^{-1}$, and the control force is explicitly given by

$$Q^C = (AM^{-1})^{+}(b - AM^{-1}Q)$$
$$= M^{-1}\frac{A^T}{(AM^{-2}A^T)}(b - AM^{-1}Q) \tag{22}$$

In Eqs. (20)–(22), we are assuming that the scalars in the denominator on the right hand sides of these equations are not zero, that is, that the matrix $A \neq 0$.

3 Numerical Examples

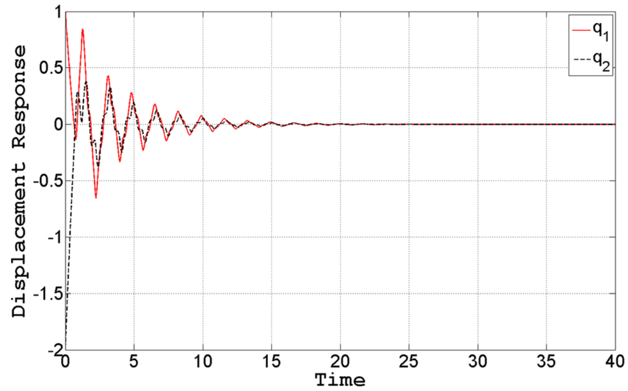
Numerical Example A. Consider the coupled nonlinear two degree-of-freedom system described by the equations

$$M\ddot{q} = -Kq + Lq^{(3)} + H := Q(q,\dot{q},t) \tag{23}$$

where $q = [q_1, q_2]^T$. The matrices in Eq. (22) are given by

$$M(t) = \begin{bmatrix} m_1\dfrac{(t+1)}{(t+2)} & 0 \\ 0 & m_2\dfrac{(t+3)}{(t+2)} \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix},$$
$$L = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix}, \quad H = q_1 q_2 \begin{bmatrix} c_1\dot{q}_1 \\ c_2\dot{q}_2 \end{bmatrix} \tag{24}$$

and the two-vector $q^{(3)} = [q_1^3 \quad q_2^3]^T$. The $m_i's > 0$, $k_i's$, $l_i's$, and $c_i's$ are constants. We note that the system has a Duffing-type nonlinearity and coupled nonlinear damping.



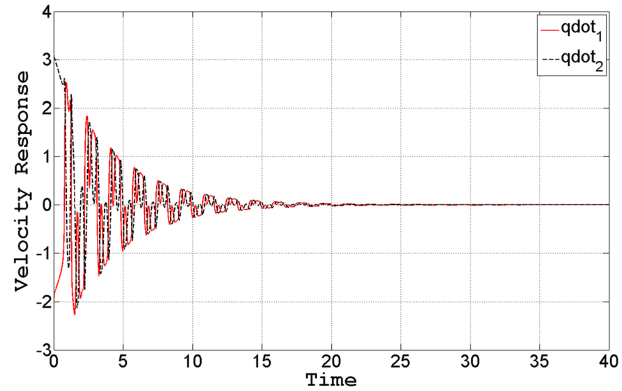


Fig. 1 Displacement and velocity response of the controlled nonlinear system

Let us assume that we would like to use the positive definite candidate Lyapunov function

$$V(q,\dot{q}) = \frac{1}{2}a_1 q^T q + \frac{1}{2}a_2 \dot{q}^T \dot{q} + a_{12}\dot{q}^T q \tag{25}$$

with the constants $a_1, a_2 > 0, a_1 a_2 > a_{12}^2$, and the candidate function

$$w(q,\dot{q}) = \alpha V(q,\dot{q}), \quad \alpha > 0 \tag{26}$$

In view of the relation in Eq. (5), our object is to design a controller whose dynamics will be such as to satisfy the equation

$$\frac{dV(q,\dot{q})}{dt} := \frac{\partial V}{\partial \dot{q}}\ddot{q} + \frac{\partial V}{\partial q}\dot{q} = -\alpha V \tag{27}$$

thereby ensuring that the function V automatically becomes an appropriate Lyapunov function for the ensuing dynamics and, therefore, that the fixed point $q = \dot{q} = 0$ is asymptotically stable. Placing Eq. (27) in the form $A\ddot{q} = b$, we find the matrix

$$A(q,\dot{q}) = a_2\dot{q}^T + a_{12}q^T = [a_2\dot{q}_1 + a_{12}q_1, a_2\dot{q}_2 + a_{12}q_2] \tag{28}$$

and the scalar

$$b(q,\dot{q}) = -\alpha\left[\frac{1}{2}a_1 q^T q + \frac{1}{2}a_2\dot{q}^T\dot{q} + a_{12}\dot{q}^T q\right] - a_1 q^T \dot{q} - a_{12}\dot{q}^T\dot{q} \tag{29}$$

Furthermore, we shall ensure stability while requiring that the control effort Q^C minimizes the weighted norm $J(t) = [Q^C]^T N(q,t)[Q^C]$ at each instant of time. Taking the matrix $N(t) = M^{-1}(t)$, we can write the controlled system as

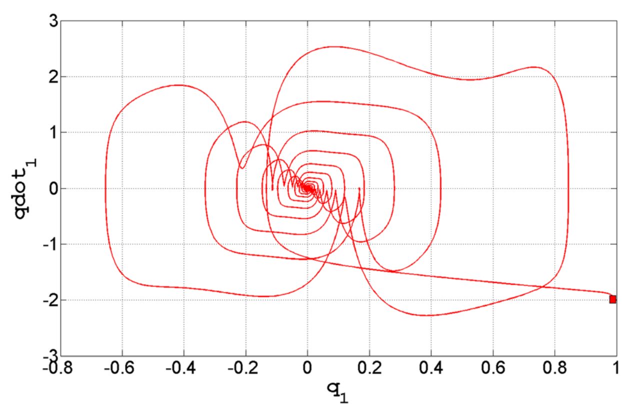
$$M\ddot{q} = -Kq + Lq^{(3)} + H + Q^C(q,\dot{q},t) \tag{30}$$

where the control force Q^C is now explicitly given in Eq. (20) with A, b, Q, and M defined in Eqs. (28), (29), (23), and (24).

Our choice of constants a_1, a_2, a_{12}, and α will be made subject to the condition that when $A = 0$ then $b = 0$ (see *Remark 1*). From the first equality in Eq. (28), we see that $A(q,\dot{q}) = 0$ when $\dot{q} = -(a_{12}/a_2)q$. In order then for $b(q, -(a_{12}/a_2)q)$ given in Eq. (29) to equal zero, it can be easily shown that we require $\alpha = 2(a_{12}/a_2)$.

Using the numerical values $m_1 = 1, m_2 = 2, \quad k_1 = 100$, $k_2 = 100$, $l_1 = l_2 = 4$, and $c_1 = c_2 = 1$ in the relations shown in Eqs. (24) and the initial conditions $q(0) = [1, -2]^T$ and $\dot{q}(0) = [-2, 3]^T$, the uncontrolled system given by Eq. (23) for these parameter values is *unstable*. We shall use the candidate Lyapunov function V given in Eq. (25) with $a_1 = 1, a_2 = 4$, and $a_{12} = 1$ so that $\alpha = 1/2$.

We obtain a simulation in the MATLAB environment of the controlled system given by Eq. (30) (using Q^C explicitly obtained from Eq. (20)). Throughout this paper, numerical integration of the ode's has been done using ode15s using a relative error tolerance of 10^{-8} and an absolute error tolerance of 10^{-12}. Figure 1 shows the displacement and velocity response as a function of time of the controlled system showing the asymptotic convergence of the controlled system to the fixed point $q = \dot{q} = 0$.



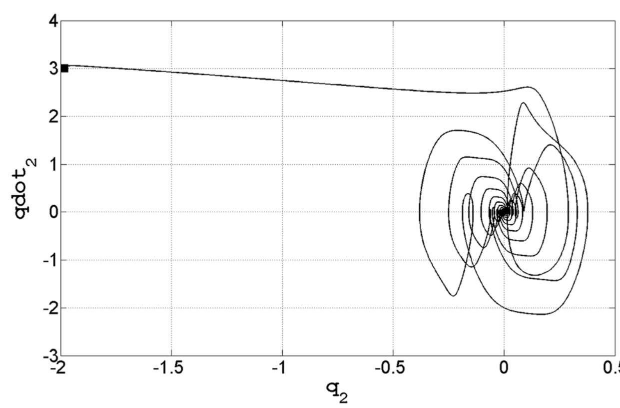


Fig. 2 Projections of the phase portrait on the $q_1 - \dot{q}_1$ and the $q_2 - \dot{q}_2$ planes. The squares show the initial values.

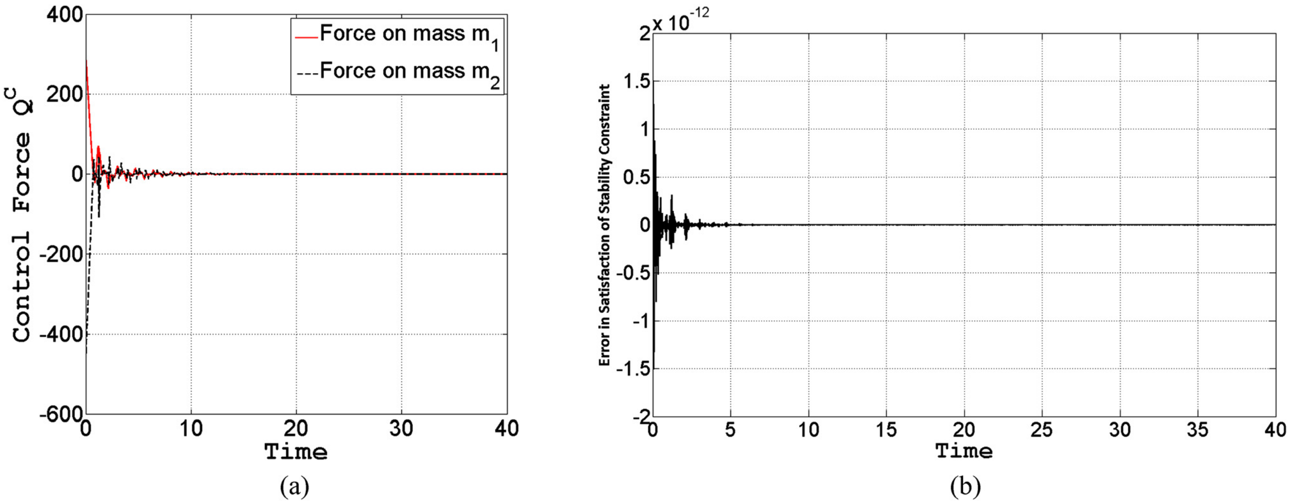


Fig. 3 (*a*) Time history of the control force Q^C. (*b*) Error $e(t)$ in satisfaction of stability requirement showing the extent to which the relation in Eq. (26) is satisfied.

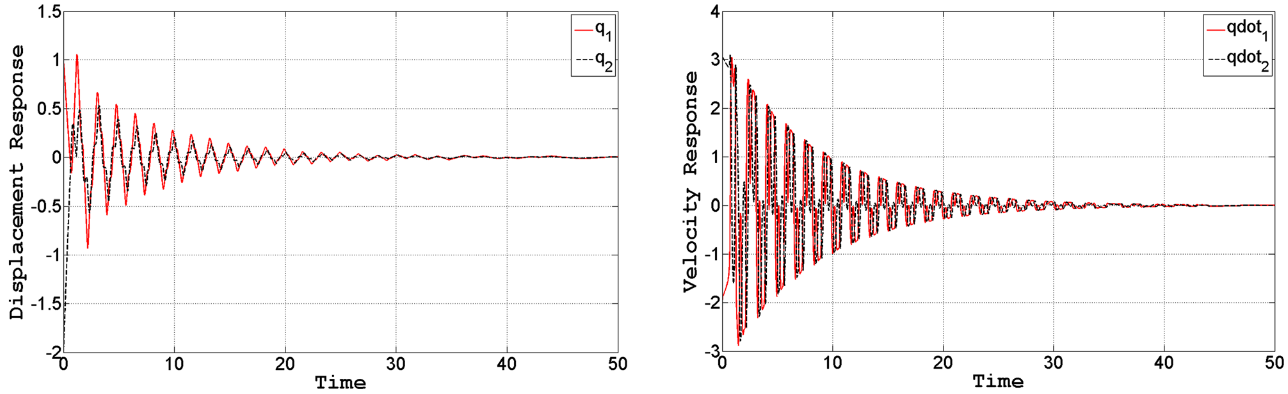


Fig. 4 Displacement and velocity response of the controlled nonlinear system

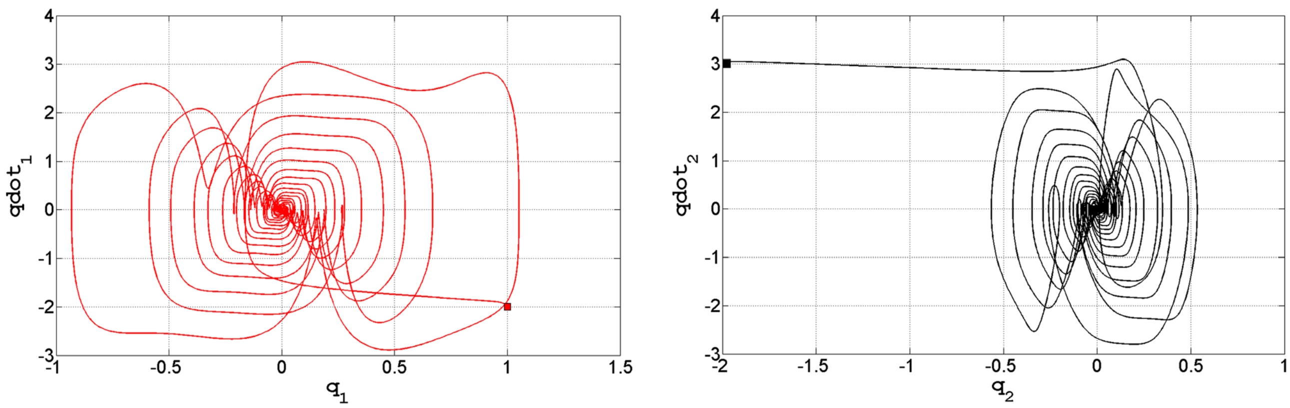
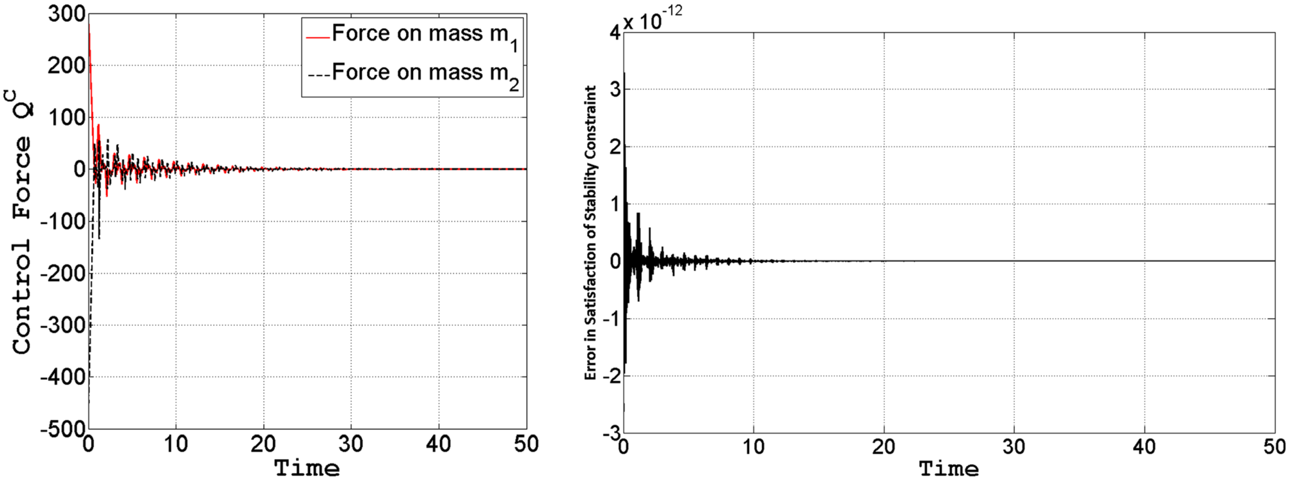


Fig. 5 Projections of the phase portrait on the $q_1 - \dot{q}_1$ and the $q_2 - \dot{q}_2$ planes. The squares show the initial values.

Figure 2 shows the projection of the phase portrait of the controlled dynamical system on the $q_1 - \dot{q}_1$ and the $q_2 - \dot{q}_2$ planes. Figure 3(*a*) shows the control force Q^C given by Eq. (20) needed to enforce the stability condition (Eq. (27)) to bring about convergence of the controlled system to the fixed point $q = \dot{q} = 0$. We note that this control force minimizes the cost function $J(t) = \|Q^C\|^2_{M(t)^{-1}}$ at *each* instant of time. The error $e(t) := \dot{V} + w(q, \dot{q})$ gives us the extent to which the stability condition (Eq. (27)) is not satisfied by the ensuing dynamics. As seen in Fig. 3(*b*), this requirement is satisfied to the same order of accuracy as the absolute error tolerance used in the numerical integration of the equations describing the controlled system.

Numercal Example B. As a second example, we consider the same system given in Eq. (23) except that we include a forcing



(a) (b)

Fig. 6 (*a*) Time history of the control force Q^C. (*b*) Error $e(t)$ showing the extent to which relation in Eq. (23) is satisfied.

input given by the vector $g(t) = [0.3\cos(t),\ -0.4\sin(t)]^T$ so that the uncontrolled dynamical system's description is now given by

$$M\ddot{q} = -Kq + Lq^{(3)} + H + g(t) := Q(q, \dot{q}, t) \tag{31}$$

instead of by Eq. (23). The matrices M, K, L, and H are as given in Eq. (24). We use the same parameter values (for the elements of these matrices) and initial conditions as before, and the same positive definite functions $V(q,\dot{q})$ and $w(q,\dot{q})$ given in Eqs. (25) and (26), respectively.

Using the weighting matrix $N = M^{-1}$ again, with $a_1 = 1$, $a_2 = 8$, and $a_{12} = 1$ so that $\alpha = 1/4$, the additive control force Q^C to be applied to the uncontrolled system (on the right hand side of Eq. (31)) is again obtained by using the explicit relation in Eq. (20).

Figure 4 shows the time histories of the displacement and velocity response of the controlled system, and, as before, Fig. 5 shows the projections of the phase portrait on the $q_1 - \dot{q}_1$ and $q_2 - \dot{q}_2$ planes. The figures again show asymptotic convergence to $q = \dot{q} = 0$.

Figure 6(*a*) shows the control force Q^C that simultaneously causes the dynamics: (i) to be Lyapunov stable, since the function V *is* now a Lyapunov function, by virtue of satisfying Eq. (27), and (ii) to minimize the cost $\left\|Q^C\right\|^2_{M(t)^{-1}}$ at each instant of time. Figure 6(*b*) shows the error $e(t) := \dot{V} + w$, with V given in Eq. (25) and w given in Eq. (26). As before, it is satisfied to the same order of magnitude as the absolute error tolerance used in the numerical integration.

It should be pointed out that the magnitude and nature of the control force Q^C are dependent on the choice of the candidate functions V and w in Eq. (5) and on the user-desired weighting matrix N. Furthermore, from a computational standpoint, the choice of these functions may need to be adjusted so that $\|G\|$ is not too small, since it appears in the denominator in Eq. (19) (see *Remark 1*).

4 Conclusions

A simple approach is developed to minimize, at each instant of time, a user-specified control cost for a mechanical system while causing a user-specified Lyapunov function to decay in time at a user-specified rate. The latter ensures stability of the system. The method is based on insights from analytical dynamics that deal with the manner in which Nature executes the constrained motion of mechanical systems. The approach employs only elementary linear algebra and relies on the consistency of the constraint imposed by the Lyapunov stability condition. It is important that this consistency requirement be satisfied.

A set of nonlinear controllers are found that minimize a control cost at each instant of time while ensuring that a candidate Lyapunov function decays at a specific rate given by the function w. The latter makes this candidate function a Lyapunov function for the controlled system. For the specific functions V and w used, and a choice of the weighting matrix N, which describes the user-preferred control cost, one obtains an optimal controller. The approach allows the complete nonlinear dynamical system to be handled with no approximations/linearizations; also, no a priori structure is imposed on the nature of the nonlinear controller. The resulting set of controllers is obtained in closed form and can be easily implemented in real time. Examples showing the efficacy of the control design methodology for a highly nonlinear, nonautonomous, unstable mechanical system demonstrate the central idea behind the approach.

The results obtained here can be easily extended to systems described by a set of nonlinear, nonautonomous, first order differential equations, a topic which will be addressed more fully along with further refinements in a future communication.

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