Response of Uncertain Dynamic Systems. II

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ABSTRACT

This paper deals with structural and mechanical systems that can be modeled as single-degree-of-freedom oscillators, the knowledge of whose mass, damping, and stiffness parameters is uncertain. In conformity with usual engineering practice, it is assumed that knowledge of *only* the upper and lower limits within which these uncertain parameters lie is available. Excitations generated by a) external forces such as wind loads, and b) base accelerations such as those caused by strong earthquake ground shaking are both considered. The statistics of the response of such systems are obtained for the following three types of excitations.

1) Harmonic excitations, yielding the statistics of the transfer function of the system. Here it is shown that Monte Carlo simulations require large sample sizes to obtain results close to those analytically deduced.
2) Deterministic time histories of excitation, yielding the statistics of the transient response of the system. This is done by Fourier decomposition using the transfer function results obtained above.
3) Random stationary excitations, yielding the statistics of the power spectral density of the response.

Thus the paper presents the results of the response of a "random system" subjected to harmonic excitations, deterministic transient excitations, and random stationary excitations.

I. INTRODUCTION

In paper I we obtained the density functions for the important properties of interest of a single-degree-of-freedom oscillatory system whose mass,

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stiffness, and damping parameters are uncertain. In this paper we shall deal with the determination of the expected response and the time-dependent variance of the response of such a random oscillatory system subjected to dynamic loads. The same notation will be used for the various entities as in paper I.

We begin by obtaining the expected value and the variance of the transfer function of the random system. We shall limit our derivations, for the sake of simplicity, to the situation where the parameters m, k, and c are truncated, uniformly distributed random variables. This situation is perhaps the one that occurs most commonly in engineering practice, for truncated Gaussian distributions, as shown in [1], are maximally unpresumptive when estimates of the variances of the uncertain parameters are available. These variance estimates in many practical engineering problems may be hard to obtain, and would be generally unreliable, at best. The expected value and variance of the amplitude and phase of the transfer function are obtained in closed form, together with closed-form expressions for the probability distributions of these quantities at any frequency.

The statistics of the system response to various types of loading environments are next investigated. The forced vibration response of the undamped system to deterministic (or random) initial conditions is obtained. The response statistics for transient deterministically known excitations as well as random stationary excitations are also provided. Base excitation (as occurs during an earthquake) as well as forced excitation of the system are considered in each case. Closed-form results for several of these quantities are obtained for the first time.

II. TRANSFER-FUNCTION STATISTICS

Consider the system described by

$$m\ddot{x} + \dot{c} + kx = d(t),$$

$$x(0) = \dot{x}_0, \qquad x(0) = v_0 \tag{1}$$

when the parameters m, k, and c are known to be in the ranges m_1 to m_2, k_1 to k_2, and c_1 to c_2, respectively. We note that for a base acceleration $\ddot{z}(t)$, Equation (1) represents the response of the oscillator relative to its base, and $d(t) = -m\ddot{z}(t)$. Thus even for a deterministic base motion, Equation (1) represents the response of a random system subjected to a loading time history which is scaled by a random variable. Taking Fourier transforms on both sides of Equation (1), we get

$$X(\omega) = H(\omega)\left[D(\omega) + mv_0 + x_0(c + i\omega m)\right], \tag{2}$$

where

$$\begin{aligned} X(\omega) &= \int_0^\infty e^{-i\omega t} x(t)\,dt, \qquad i = \sqrt{-1}, \\ H(\omega) &= (k - m\omega^2 + ic\omega)^{-1}. \end{aligned} \tag{3}$$

The transfer function for the two cases when $d(t)$ is an externally applied force and when $d(t) = -m\ddot{z}(t)$ are then obtained by the relations

$$\begin{aligned} X_f(\omega) &= (k - m\omega^2 + ic\omega)^{-1} D(\omega) = H_f(\omega) D(\omega), \\ X_e(\omega) &= -m(k - m\omega^2 + ic\omega)^{-1} \ddot{Z}(\omega) = H_e(\omega)\ddot{Z}(\omega), \end{aligned} \tag{4}$$

where the subscripts stand for external forcing and earthquake excitation. The transfer functions $H_e(\omega)$ and $H_f(\omega)$ are related by $H_e(\omega) = -mH_f(\omega)$.

The expected values of the transfer functions then become

$$E\left[H_{e,f}(\omega)\right] = \frac{1}{V}\int_{k_1}^{k_2}\int_{c_1}^{c_2}\int_{m_1}^{m_2} H_{e,f}(\omega)\,dc\,dm\,dk \tag{5}$$

with

$$V = (c_2 - c_1)(k_2 - k_1)(m_2 - m_1). \tag{6}$$

The integral in Equation (34) can be evaluated in closed form to give

$$E\left[H_{e,f}(\omega)\right] = \frac{1}{V}\left[\sum_{s_i=1}^{2}\sum_{s_j=1}^{2}\sum_{s_l=1}^{2} \varepsilon_{s_i s_j s_l} h_{e,f}\left(m_{s_i}, k_{s_j}, c_{s_l}\right)\right]. \tag{7}$$

Here we use the following notation:

$$\varepsilon_{pqr}=\begin{cases}(-1)^{p+q+r} & \text{when} \quad p\neq q\neq r,\\ 0 & \text{when any two indices are the same,}\end{cases} \tag{8}$$

$$h_e(m_s,k_j,c_l)=\frac{-i}{\omega^5}\left[\frac{y^2}{6}\left(2m_s\omega^2+ic_l\omega+k_j\right)\ln y\right.$$

$$\left.-\frac{y^2}{36}\left(4m\omega^2+5ic_l\omega+5k_j\right)\right],\quad \omega\neq 0, \tag{9}$$

$$h_f(m_s,k_j,c_l)=\frac{-i}{2\omega^2}\left[\frac{y^2}{2}-y^2\ln y\right],\qquad \omega\neq 0, \tag{10}$$

$$y=k_j-m_s\omega^2+i\omega c_l, \tag{11}$$

$$\ln y=\ln|y|+i\theta,\qquad 0\leqslant\theta\leqslant 2\pi. \tag{12}$$

Also,

$$E\left[H_e(0)\right]=(m_2+m_1)\ln(k_1/k_2) \tag{9a}$$

and

$$E\left[H_f(0)\right]=\frac{1}{V}\ln(k_2/k_1). \tag{10a}$$

The expected value of the amplitude of the transfer function may now be expressed for uniform density as

$$E\left[|H_f(\omega)|\right]\triangleq E\left[A_f(\omega)\right]=\frac{1}{V}\int_{c_1}^{c_2}\int_{k_1}^{k_2}\int_{m_1}^{m_2}\frac{dm\,dk\,dc}{\sqrt{(k-m\omega^2)^2+(c\omega)^2}}, \tag{13}$$

where V is defined in (6). After some algebra, this can be written as

$$E\left[A_f(\omega)\right]=\sum_{s_i=1}^{2}\sum_{s_j=1}^{2}\sum_{s_l=1}^{2}\frac{1}{V}\varepsilon_{s_is_js_k}a_f(m_{s_i},k_{s_j},c_{s_l}), \tag{14}$$

where

$$a_f(m_s, k_j, c_l) = \frac{1}{2\omega^3}\left\{q\sqrt{p^2+q^2} - p^2 \ln\left(q + \sqrt{p^2+q^2}\right)\right\}$$
$$- \frac{pq}{\omega^3}\sinh^{-1}\left(\frac{p}{q}\right), \qquad \omega \neq 0, \tag{15}$$

with

$$p = \left(k_j - m_s\omega^2\right) \quad \text{and} \quad q = c_l\omega. \tag{16}$$

For $\omega = 0$,

$$E\left[|H_f|\right] = (k_2 - k_1)^{-1}\ln(k_2/k_1). \tag{17}$$

The expression for $E[A_f^2]$ can also be obtained again after integration as

$$E\left[A_f^2\right] = \sum_{s_i=1}^{2}\sum_{s_j=1}^{2}\sum_{s_l=1}^{2}\frac{1}{V}\varepsilon_{s_is_js_l} b_f\left(m_{s_i}, k_{s_j}, c_{s_l}\right), \tag{18}$$

where V and ε_{pqr} are defined by (6) and (8), and

$$b_f(m_s, k_j, c_l) = \frac{p}{\omega^3}\tan^{-1}\left(\frac{q}{p}\right) + \frac{q}{2\omega^3}\ln\left(p^2+q^2\right)$$
$$+ \frac{p}{\omega^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\left(\frac{p}{q}\right)^{2n-1}, \qquad \omega \neq 0, \tag{19}$$

with p and q defined by Equation (16).

For $\omega = 0$,

$$E\left[A_f^2\right] = 1/k_1k_2. \tag{20}$$

For base excitation, the closed-form expansion for the expected value of the amplitude of the transfer function becomes

$$E\left[|H_e(\omega)|\right] \triangleq E\left[A_e(\omega)\right] = \sum_{s_i=1}^{2}\sum_{s_j=1}^{2}\sum_{s_l=1}^{2}\frac{1}{V}\varepsilon_{s_is_js_l} a_e\left(m_{s_i}, k_{s_j}, c_{s_l}\right), \tag{21}$$

where

$$a_e(m_s, k_j, c_l) = \left[\frac{3}{8}\frac{p}{\omega^5} + \frac{m_s}{2\omega^3}\right]\left[qr + p^2 \ln r\right]$$
$$-\frac{p^2}{\omega}\left[\frac{p^2}{2\omega^5} + \frac{m_s}{\omega^3}\right]\left[\frac{q}{p}\sinh^{-1}\left(\frac{p}{q}\right) + \ln\left(\frac{r}{|p|}\right)\right]$$
$$-\frac{p^3}{2\omega^5}\left[\frac{1}{6}\ln\left(\frac{r}{|p|}\right) - \frac{1}{3}\left(\frac{q}{p}\right)^3 \sinh^{-1}\left(\frac{q}{p}\right) - \frac{1}{6}\frac{q}{p^2}\sqrt{p^2+q^2}\right]$$
$$\omega \neq 0. \quad (22)$$

The quantities p and q are as defined before, and

$$r = q + \sqrt{p^2 + q^2}. \quad (23)$$

For $\omega = 0$,

$$E[A_e] = \frac{m_1 + m_2}{2(k_2 - k_1)} \ln\left(\frac{k_2}{k_1}\right). \quad (24)$$

Also,

$$E\left[A_e^2(\omega)\right] = \sum_{s_i=1}^{2} \sum_{s_j=1}^{2} \sum_{s_l=1}^{2} \frac{1}{\omega^7}\frac{1}{V}\varepsilon_{s_i s_j s_l} b_e(m_{s_i}, k_{s_j}, c_{s_l}), \quad (25)$$

where

$$b_e(m_s, k_j, c_l) = \left[\frac{z_l y_j^2}{2} - \frac{z_l^3}{18}\right]\ln\left[1 + \left(\frac{x_s - y_j}{z_l}\right)^2\right] + \frac{x_s^2 z_l + 7x_s y_j z_l - 8y_j^2 z_l}{18}$$
$$+ \left(\frac{x_s - y_j}{18}\right)\left(2x_s^2 - 5x_s y_j + 11y_j^2\right)\tan^{-1}\left(\frac{z_l}{x_s - y_j}\right)$$
$$-\frac{y_j z_l^2}{2}\tan^{-1}\left(\frac{x_s - y_j}{z_l}\right) - \left(\frac{x_s^3 - y_j^3}{3}\right)$$
$$\times \int^{z_l} \frac{1}{a}\tan^{-1}\left(\frac{x_s - y_j}{a}\right) da, \qquad \omega \neq 0, \quad (26)$$

where

$$x_s = m_s \omega^2, \qquad y_j = k_j, \quad \text{and} \quad z_l = c_l \omega. \tag{27}$$

For $\omega = 0$,

$$E\left[A_e^2(\omega)\right] = \frac{\frac{1}{3}\left(m_1^2 + m_1 m_2 + m_2^2\right)}{k_1 k_2}. \tag{28}$$

Let us obtain some physical insight into the results analytically obtained thus far. Consider a box containing a large number of oscillators each having its mass lying at random between m_1 and m_2, its stiffness between k_1 and k_2, and its damping between c_1 and c_2. Each oscillator with mass m_i, stiffness k_j, and damping c_l can be represented by a point in a three-dimensional parameter space with coordinates m_i, k_j, and c_l. The entire ensemble of oscillators is thus represented by the box shown in Figure 1(a). Were we to plot the amplitude of the transfer function (either A_e or A_f) corresponding to an arbitrary point (m_i, k_j, c_l) contained in this box, we would obtain the function shown by the dashed line in Figure 1(b). In fact, to each point in the box would correspond a particular oscillatory system, and a corresponding curve in Figure 1(b). The average over all such curves at a frequency ω will then give the expected value of the transfer-function amplitude at that frequency. It is this that the expressions (13) and (17) provide. The variance of the transfer-function amplitude at a frequency ω can be interpreted in a similar fashion.

We thus note that while the amplitude of the transfer function of an undamped oscillator is unbounded at the natural frequency ω_0 of the system, the expected value of the corresponding amplitude of a random system is bounded. This is because of the averaging process indicated pictorially in Figure 1(b). In fact, the expected value of the response of a SDOF undamped system, whose mass and stiffness are random variables, to given initial conditions can be expressed as

$$E\left[x(t)\right] = E\left[x_0 \cos \omega t + v_0 \frac{\sin \omega t}{\omega}\right], \tag{29}$$

where x_0 and v_0 are the initial displacement and the initial velocity, respectively, and the expectation is to be taken over the distribution of ω, the undamped natural frequency. For m and k uniformly distributed, the p.d.f. of ω is given by Equation (7) in [1]. Performing the integration over the

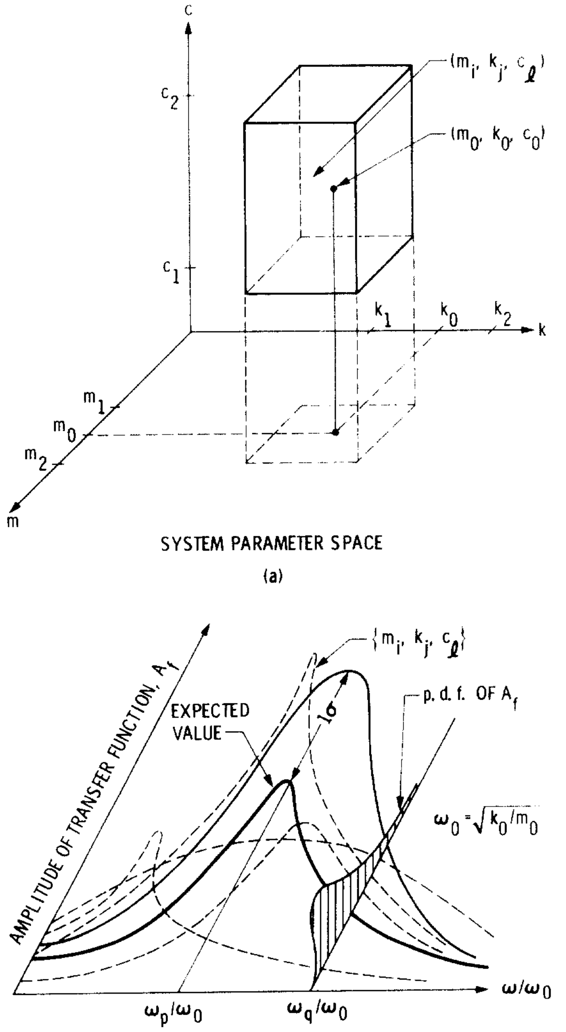
c
c_2
c_1
(m_i, k_j, c_ℓ)
(m_0, k_0, c_0)
k_1
k_0
k_2
k
m_1
m_0
m_2
m
SYSTEM PARAMETER SPACE
(a)
AMPLITUDE OF TRANSFER FUNCTION, A_f
$\{m_i, k_j, c_\ell\}$
EXPECTED VALUE
1σ
p. d. f. OF A_f
$\omega_0 = \sqrt{k_0/m_0}$
ω/ω_0
ω_p/ω_0
ω_q/ω_0
(b)

FIG. 1.

range of ω, I_Ω, over which $p(\omega) \neq 0$, closed-form expressions for $E[x(t)]$ and $E[x^2(t)]$ can be easily found (see Appendix). In fact,

$$E[x(t)] = O\left(\frac{1}{t}\right) \tag{30}$$

and

$$E[x^2(t)] = \frac{v_0^2}{2}\left[\int_{I_\Omega} \frac{p(\omega)}{\omega^2}\,d\omega\right] + \frac{x_0^2}{2} + O\left(\frac{1}{t}\right). \tag{31}$$

The generalization to the case where the initial conditions are also random can be easily made by replacing x_0^2 and v_0^2 with $E[x_0^2]$ and $E[v_0^2]$, provided that the variables x_0, v_0, m, and k are independent. Thus, the expected value of the impulse response of the random system dies down as $1/t$, while the variance tends to a constant as $t \to \infty$. Since the transfer function is the Fourier transform of the impulse response, the amplitude of the transfer function therefore remains bounded at all frequencies.

The analytical results in the Appendix are plotted in Figure 2, which shows the expected response $E[x(t)]$ for an undamped system with $m_0 = 1$, $k_0 = 100$, $\eta_1 = 0.05$, and $\eta_2 = 0.15$ for two different initial conditions [1]. The response $x^0(t)$ of the system with the mean properties (m_0, k_0) is also shown. The 1σ response band on either side of the expected response is indicated. Figure 2(a) gives the response for a case when the initial velocity $v_0 = 1$. Figure 2(b) gives the response for an initial displacement $x_0 = 1$. The analytically obtained responses are compared with Monte Carlo simulations. We note that as $t \to \infty$, $E[x(t)] \to 0$ and $\mathrm{Var}[x(t)] \to$ constant.

The expressions (13)–(28) used here are in terms of the parameters m_1, m_2, k_1, k_2, c_1, and c_2 for ease of application to practical engineering problems. However, Equation (5) can be expressed in nondimensional form as

$$k_0 E[A_f] = \frac{1}{8\eta_1\eta_2\eta_3}\int_{1-\eta_3}^{1+\eta_3}\int_{1-\eta_2}^{1+\eta_2}\int_{1-\eta_1}^{1+\eta_1} \frac{d\alpha\, d\beta\, d\gamma}{\left[(\alpha - \beta\Omega^2)^2 + (2\xi_0\gamma\Omega)^2\right]^{1/2}}, \tag{32}$$

where η_1, η_2, η_3 are the scatters in our knowledge of m, k, and c as defined by Equations (5) and (21) of Reference [1], and $\Omega = \omega/\omega_0$ with ω_0 being the natural undamped frequency of the system with the mean properties (see Equations (5) and (6) of Reference [1]). The expected value of the nondimen-

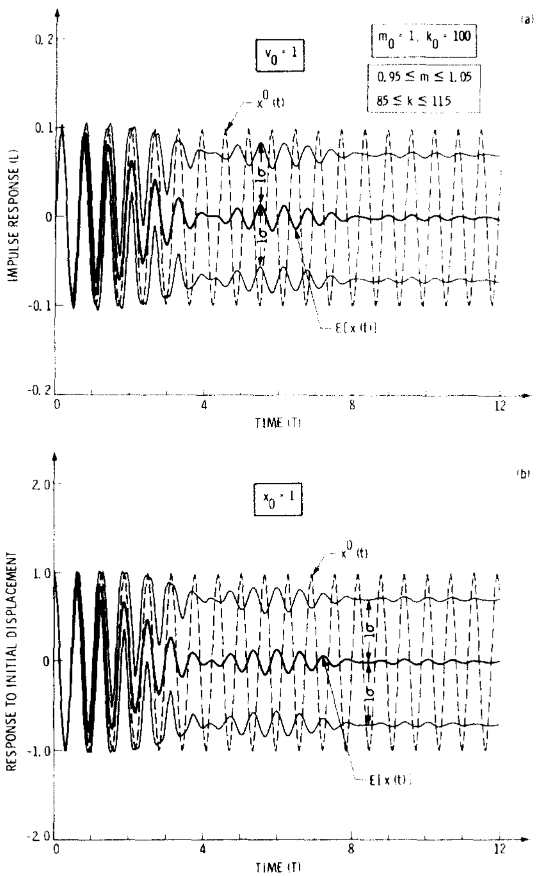

FIG. 2.

sional transfer-function amplitude is therefore only dependent on ξ_0, the percentage of critical damping corresponding to the system with the mean properties, and on the dimensionless scatters η_1, η_2, and η_3. A similar argument can be made for base excitation, where the corresponding quantity would be $\omega_0^2 E[A_e]$.

Figures 3(a) and (b) show plots of the dimensionless transfer-function amplitude for an oscillator subjected to a harmonic excitation. The uncertainty levels are consistent with what one generally experiences in the analysis of tall buildings. The dashed line indicates the transfer-function amplitude $k_0 A_f^0$ corresponding to the system with the mean parameter values m_0, c_0, k_0, and the thick solid line indicates $E[k_0 A_f]$, for which the

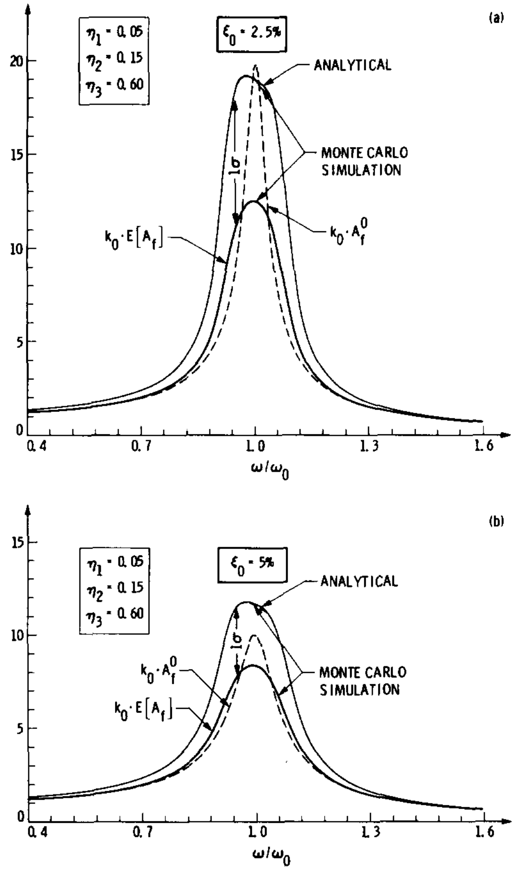

FIG. 3.

results from Equations (14)–(16) are used. Using the expression for $E[A_f^2]$ from Equations (18)–(20), the variance at each frequency ω/ω_0 is analytically obtained. The figure shows the 1σ band thus obtained, plotted above the mean at each frequency. The results are verified by Monte Carlo simulation using a sample size of 20,000. Good agreement with the analytical results is observed. Figure 3(b) shows the effect of ξ_0 on the expected value and the variance of the amplitude of the transfer function, while keeping the scatters η_1, η_2, and η_3 the same.

Figures 4(a), (b), and (c) show the separate effects of uncertainties in m, k, and c on the transfer-function amplitude. We note that the variability in the mass and stiffness parameters [Figures 4(a) and (b)] causes the curve for

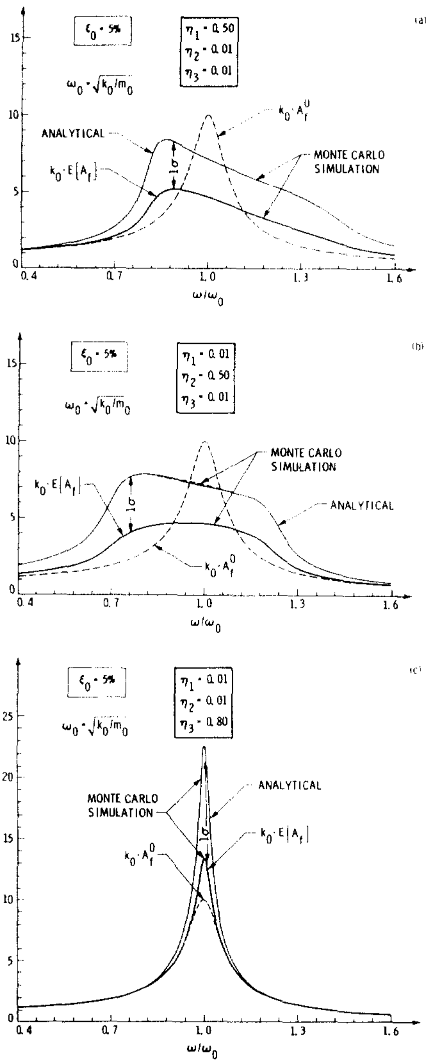

FIG. 4.

$k_0E[A_f]$ to be in general flatter and broader than that for A_f^0. Around $\omega = \omega_0$, the expected-value curve is lower than A_f^0, while at frequencies away from ω_0 it is higher than A_f^0, as could have been anticipated in the light of the interpretation in Figure 1(b). Again, the analytically obtained results for mean and variance agree well with Monte Carlo simulations. The influence of variability in the damping parameter c is indicated in Figure 4(c). We note that the $k_0 E[A_f]$ curve is not as skewed towards the lower frequencies as in Figures 4(a) and (b) and that it is higher than the A_f^0 curve.

Figures 5 and 6 show the results for the transfer function corresponding to earthquake loading. For the parameters chosen, most of the curves show generally the same qualitative behavior as those of Figures 3 and 4; the curves in Figure 4(c), however, are quite different from those of 6(a). This illustrates the effect of the variability of the inertia forces, a consequence of uncertainty in the system's mass as it enters on the right-hand side of Equation (1).

The analytically obtained 1σ bands provide only an intuitive impression of the scatter of the transfer-function amplitudes at any particular frequency. They do not indicate the probability enclosed by the 1σ bands at any frequency. It is therefore more informative to obtain the p.d.f. of the transfer-function amplitude at any frequency. For the case of an externally applied loading, the p.d.f. of A_f can be obtained, after considerable algebra, as

$$p_{A_f/A_f^0}(x) = \frac{\gamma^2}{2\xi_0 x^2 \Omega^3} \times \int_{\beta_L}^{\beta_u} \int_{\alpha_L}^{\alpha_u} \frac{f_{k/k_0}(\alpha) f_{m/m_0}\left(\dfrac{\alpha - \beta}{\Omega^2}\right) f_{c/c_0}\left(\dfrac{\sqrt{\gamma^2 - x^2\beta^2}}{2x\xi_0\Omega}\right) d\alpha\, d\beta}{\sqrt{\gamma^2 - x^2\beta^2}}, \quad (33)$$

where $\gamma^2 = (1-\Omega^2)^2 + 4\xi_0^2\Omega^2$, $\Omega = \omega/\omega_0$,

$$\alpha_L = \max\left[(1-\eta_2), (1-\eta_1)\Omega^2 + \beta\right], \quad \alpha_u = \min\left[(1+\eta_2), (1+\eta_1)\Omega^2 + \beta\right],$$
$$\beta_L = -\left[\frac{\gamma}{x}\right], \qquad \beta_u = \left[\frac{\gamma}{x}\right]. \quad (34)$$

The p.d.f. of A_e can be shown, after some manipulations, to be

$$p_{A_e/A_e^0}(x) = \int_{\beta_L}^{\beta_u} \int_{\alpha_L}^{\alpha_u} \frac{\beta^2 f_{m/m_0}(\beta) f_{k/k_0}(\alpha + \Omega^2\beta)}{2\xi_0 \Omega^2 x^2 \sqrt{\gamma^2\beta^2 - x^2\alpha^2}} \times f_{c/c_0}\left(\frac{\sqrt{\gamma^2\beta^2 - x^2\alpha^2}}{2\xi_0 x \Omega}\right) d\alpha\, d\beta \quad (35)$$

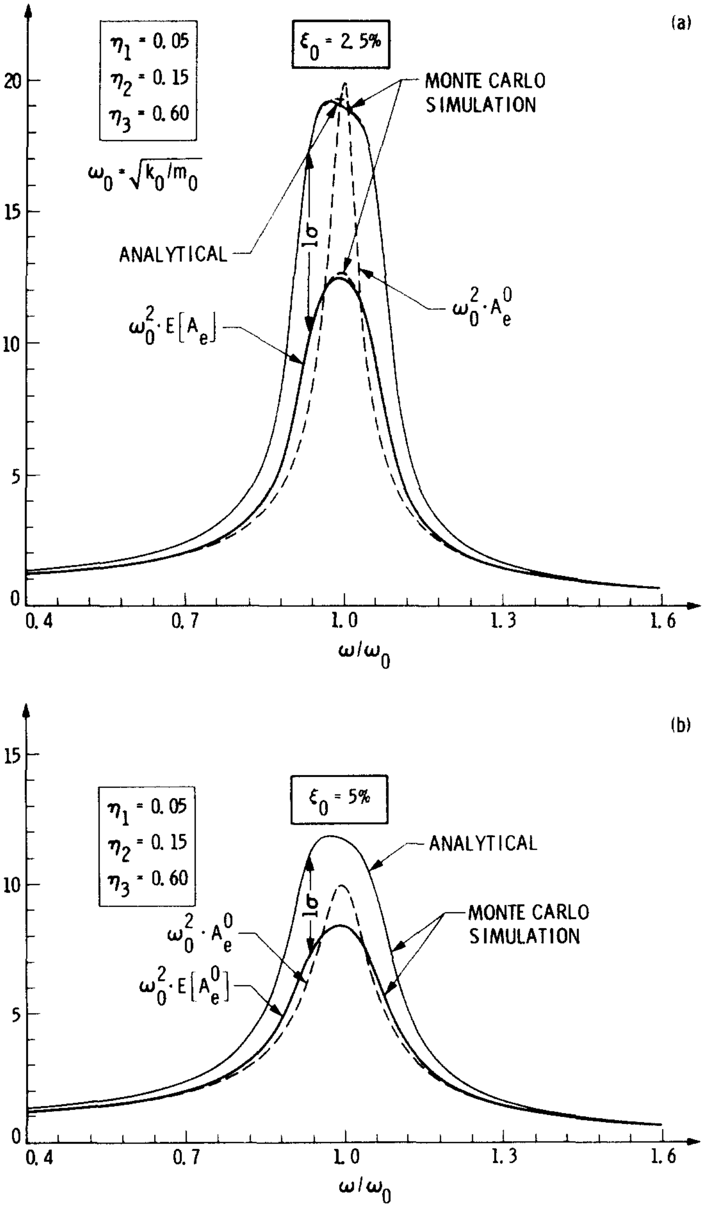
(a)
η1 = 0.05
η2 = 0.15
η3 = 0.60
ξ0 = 2.5%
ω0 = √k0/m0
MONTE CARLO SIMULATION
ANALYTICAL
1σ
ω0² · E[Ae]
ω0² · Ae0
20
15
10
5
0
0.4
0.7
1.0
1.3
1.6
ω/ω0
(b)
η1 = 0.05
η2 = 0.15
η3 = 0.60
ξ0 = 5%
ANALYTICAL
MONTE CARLO SIMULATION
1σ
ω0² · Ae0
ω0² · E[Ae0]
15
10
5
0
0.4
0.7
1.0
1.3
1.6
ω/ω0

FIG. 5.

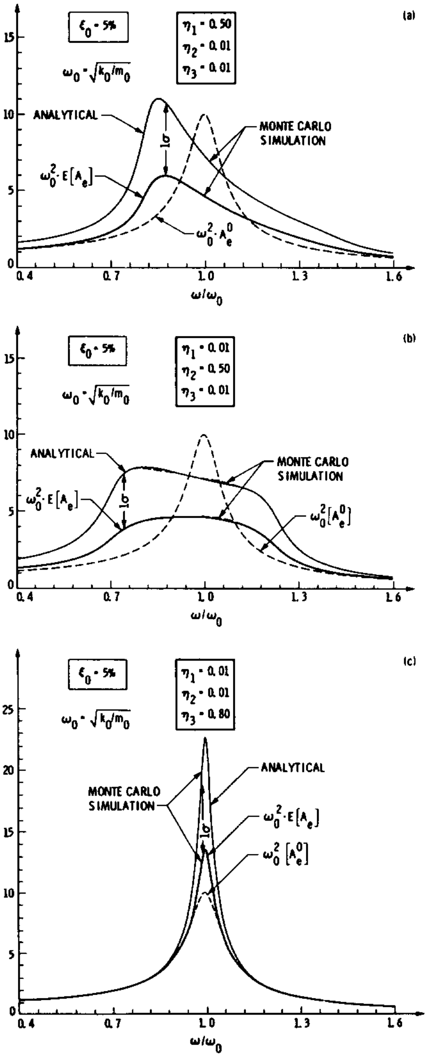
(a)
ε0 = 5%
ω0 = √(k0/m0)
η1 = 0.50
η2 = 0.01
η3 = 0.01
ANALYTICAL
MONTE CARLO SIMULATION
1σ
ω0²·E[Ae]
ω0²·Ae0
0
5
10
15
0.4
0.7
1.0
1.3
1.6
ω/ω0
(b)
ε0 = 5%
ω0 = √(k0/m0)
η1 = 0.01
η2 = 0.50
η3 = 0.01
ANALYTICAL
MONTE CARLO SIMULATION
1σ
ω0²·E[Ae]
ω0²[Ae0]
0
5
10
15
0.4
0.7
1.0
1.3
1.6
ω/ω0
(c)
ε0 = 5%
ω0 = √(k0/m0)
η1 = 0.01
η2 = 0.01
η3 = 0.80
ANALYTICAL
MONTE CARLO SIMULATION
1σ
ω0²·E[Ae]
ω0² [Ae0]
0
5
10
15
20
25
0.4
0.7
1.0
1.3
1.6
ω/ω0

FIG. 6.

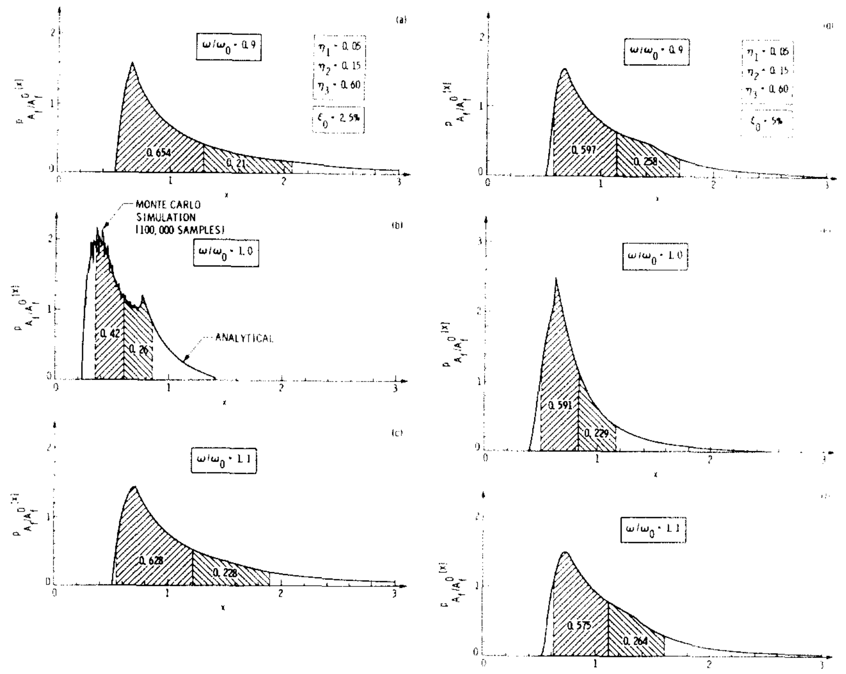

FIG. 7.

with

$$\alpha_L = \max\left[1 - \eta_2 - \beta\Omega^2, \; -\frac{1}{x}\right],$$

$$\alpha_M = \min\left[1 + \eta_2 - \beta\Omega^2, \; \frac{1}{x}\right], \tag{36}$$

$$\beta_L = (1 - \eta_1), \qquad \beta_u = (1 + \eta_1).$$

Figure 7 shows the p.d.f. of the amplitudes of the transfer functions at the dimensionless frequencies $\omega/\omega_0 = 0.9$, 1, and 1.1. The p.d.f.'s in Figures 7(a), (b), and (c) correspond to the curves of Figure 3(a), while those in 7(e), (f) and (g) correspond to the curves in 3(b). The amplitudes are normalized with respect to the amplitudes of the mean system, A_f^0, at those frequencies. The means of the distributions are indicated by the solid lines. The distributions are seen to vary widely with frequency. The probabilities of the amplitudes lying in the 1σ band around the mean are also shown. We note that for the parameters chosen, the probability of having the normalized amplitude in the 1σ interval below the mean is at least twice that of having it in the 1σ interval above. The p.d.f.'s are greatly influenced by the value of ξ_0. For $\xi_0 = 2.5\%$ we have a bimodal distribution at $\omega/\omega_0 = 1$ [Figure 7(b)]. The corresponding distribution for $\xi_0 = 5\%$ shown in Figure 7(e) is unimodal. Monte Carlo simulations were done to verify these analytically obtained distributions. The sample size needed to come close to the analytical results was found to exceed 100,000. A typical Monte Carlo simulation result is shown superposed on the analytical result in Figure 7(b).

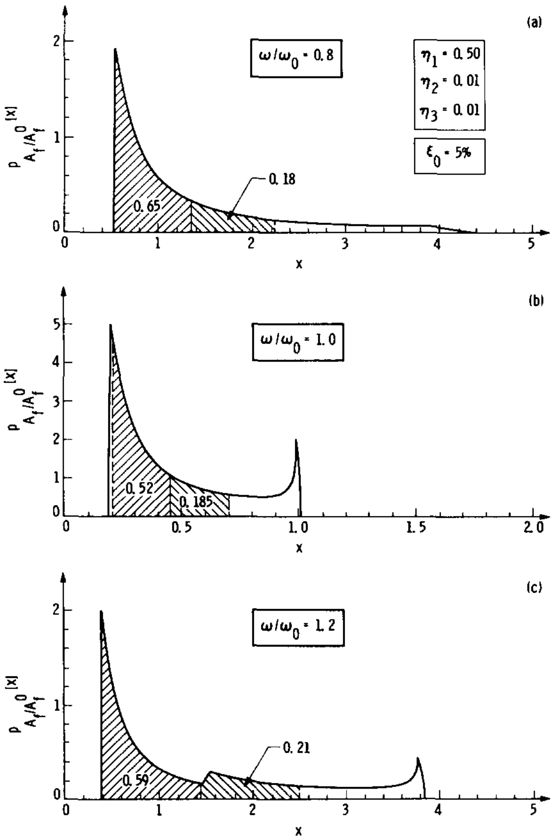

FIG. 8.

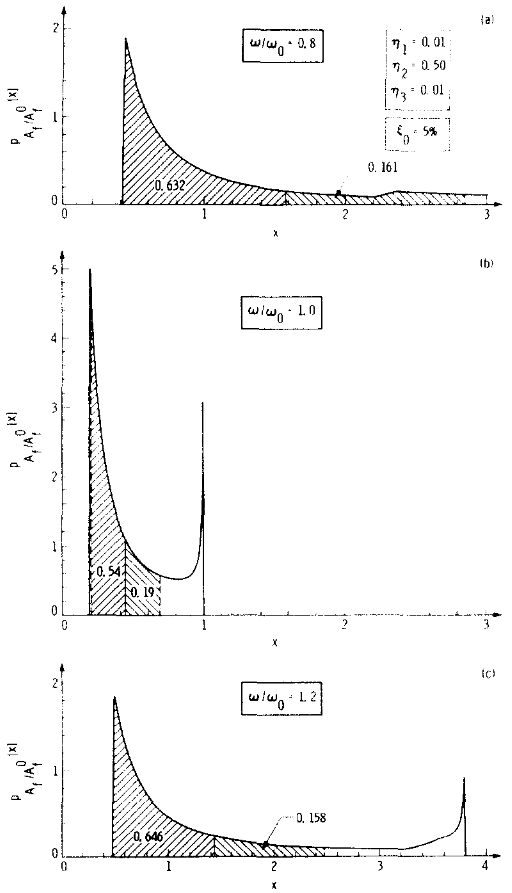

FIG. 9.

Figures 8, 9, and 10 show the distributions of the normalized amplitude A_f/A_f^0 for the parameters corresponding to Figures 4(a), (b), and (c), respectively. From these figures we observe qualitatively different effects that uncertainties in mass and stiffness have on the one hand, and uncertainties in damping have on the other, on the amplitude of the transfer function of the random system in the vicinity of $\omega/\omega_0 = 1$. The uncertainty in damping creates a distribution with a long tail, implying that amplitudes higher than $A_f^0(\omega/\omega_0 = 1)$ are likely. In fact, for the parameters chosen, the 1σ band above the mean captures a probability of about 0.2 (Figure 10). The

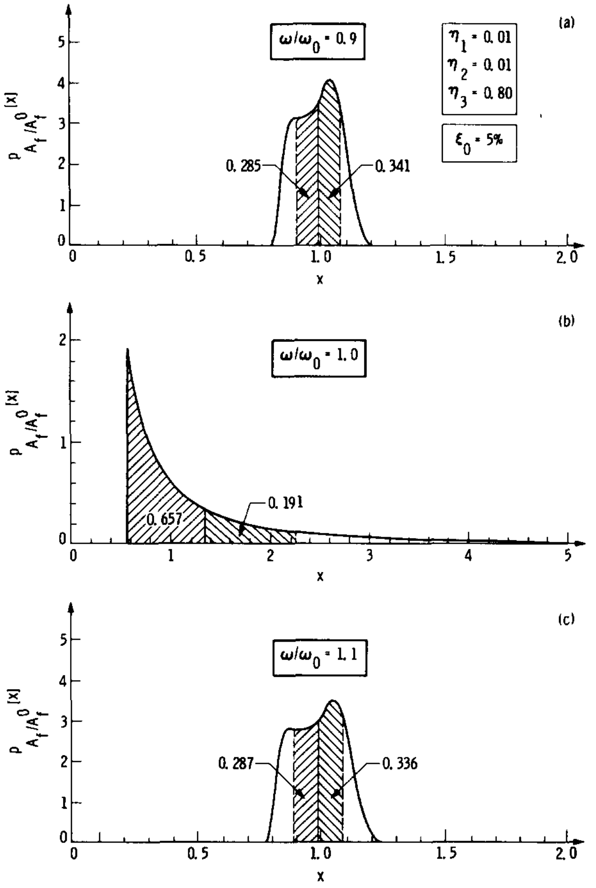

FIG. 10.

corresponding distributions for uncertainties in m and k, however, show that the probability of the normalized amplitude of the transfer function, A_f/A_f^0, being greater than unity is negligibly small [Figures 8(b) and 9(b)]. The value of A_f^0 at $\omega/\omega_0 = 1$ is thus an upper bound on the value of the amplitude A_f for the random system at that frequency. The reason for this is as follows: Consider the expression for the amplitude of the transfer function [look at the integrand in Equation (13)]. At $\omega = \omega_0$, the parameters $k = k_0$ and $m = m_0$ cause the first bracket under the radical sign in Equation (13) to be zero. With $\omega = \omega_0$, any other values of k and m cause that bracket to be nonzero and consequently the amplitude to be less than A_f^0.

The phases of the transfer functions $H_{e,f}(\omega)$ are given by

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right). \tag{37}$$

The expected value of the phase angle then becomes, for $\omega \neq 0$,

$$\Phi(\omega) \triangleq E[\phi(\omega)] = \frac{1}{V} \sum_{s_l=1}^{2} \sum_{s_j=1}^{2} \sum_{s_i=1}^{2} \varepsilon_{s_i s_j s_l} \psi(m_{s_i}, k_{s_j}, c_{s_l}), \tag{38}$$

where ε_{pqr} and V are defined in (6) and (8), and

$$\psi(m_s, k_j, c_l) = \frac{1}{4\omega^3}\left[\tfrac{2}{3}ab^2 - \tfrac{8}{3}b^3 \tan^{-1}\left(\frac{a}{b}\right) - 2b(a^2+b^2)\tan^{-1}\left(\frac{b}{a}\right) + a\left(\frac{a^2}{3} - b^2\right)\ln(a^2+b^2)\right] \tag{39}$$

with $a = k_j - m_s\omega^2$ and $b = c_l\omega$. Clearly,

$$\Phi(0) = 0. \tag{40}$$

As with the amplitudes of the transfer function, it is easy to show that the p.d.f. of the phase angle at the dimensionless frequency $\Omega \triangleq \omega/\omega_0$ is dependent only on the scatters η_1, η_2, and η_3 in the values of m, k, and c and on the percentage of critical damping, ξ_0, of the system with the mean properties m_0, k_0, and c_0.

Figures 11 and 12 show the expected value of the phase Φ of the uncertain dynamic systems corresponding to those of Figures 3 and 4 (and 5 and 6). The phases for the system with the mean properties ϕ^0 are also shown. The p.d.f. for the phase angle can be obtained as

$$f_{\phi/\phi^0}(\bar{\phi}) = \phi^0(1+\tan^2\hat{\phi}) \times \int_{\beta_L}^{\beta_u}\int_{\alpha_L}^{\alpha_u} \frac{\beta}{2\xi_0\Omega^3} f_{m/m_0}\left(\frac{\alpha-\beta}{\Omega^2}\right) f_{c/c_0}\left(\frac{\beta\tan\hat{\phi}}{2\xi_0\Omega}\right) f_{k/k_0}(\alpha)\, d\alpha\, d\beta, \tag{41}$$

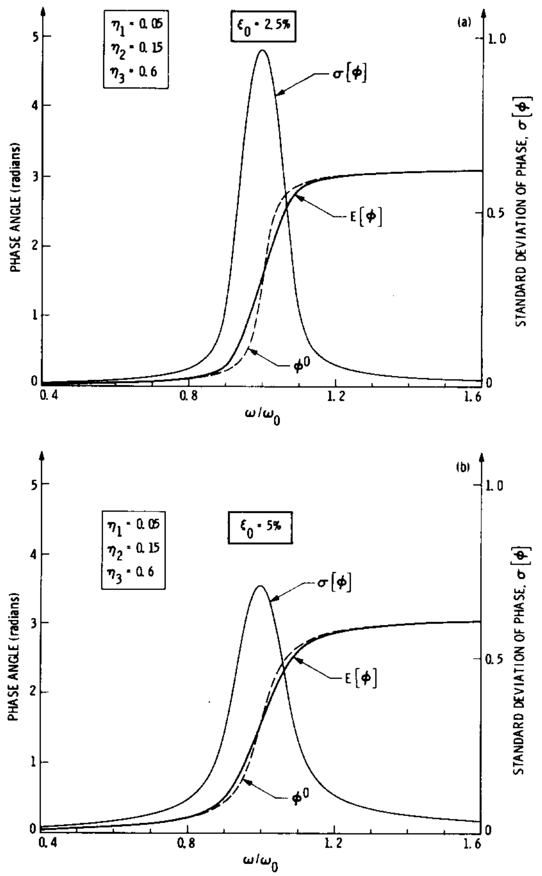

FIG. 11.

where

$$\alpha_L = \max\left[(1-\eta_2), \beta + (1-\eta_1)\Omega^2\right], \qquad \alpha_u = \min\left[1+\eta_2, \beta + (1+\eta_1)\Omega^2\right],$$

$$\beta_L = \max\left[1 - \eta_2 - (1+\eta_1)\Omega^2, (1-\eta_3)\frac{2\xi_0\Omega}{\tan\hat{\phi}}\right],$$

$$\beta_L = \min\left[1 + \eta_2 - (1-\eta_1)\Omega^2, (1+\eta_3)\frac{2\xi_0\Omega}{\tan\hat{\phi}}\right],$$

$$\hat{\phi} = \left(\bar{\phi}\phi_0\right). \tag{42}$$

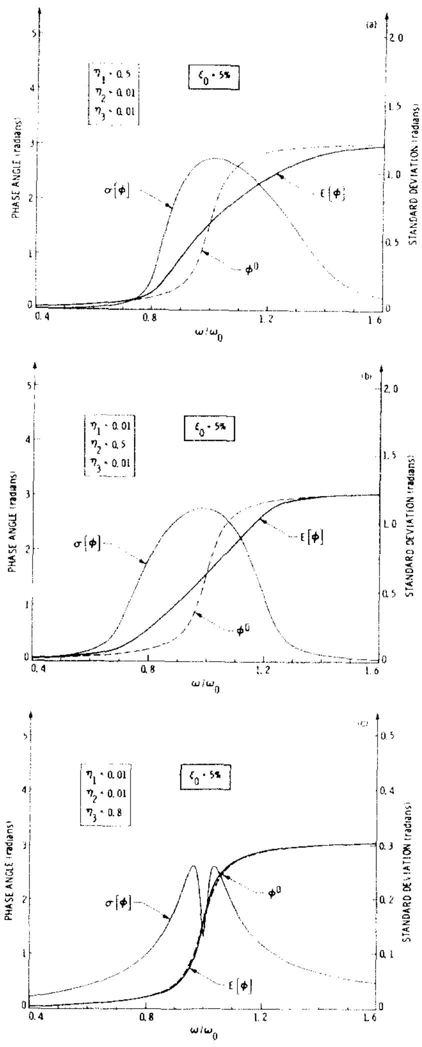

FIG. 12.

The p.d.f.'s of the phase of the random systems at specific frequencies normalized with respect to the phase of the system with the mean properties at those frequencies are shown in Figures 13 to 16. The density functions are symmetric for $\omega/\omega_0 = 1$ about their mean values. The 1σ bands above and below the distribution means are also shown. We note that for $\omega/\omega_0 \neq 1$, the probability for the normalized phase to be in the 1σ band below the mean

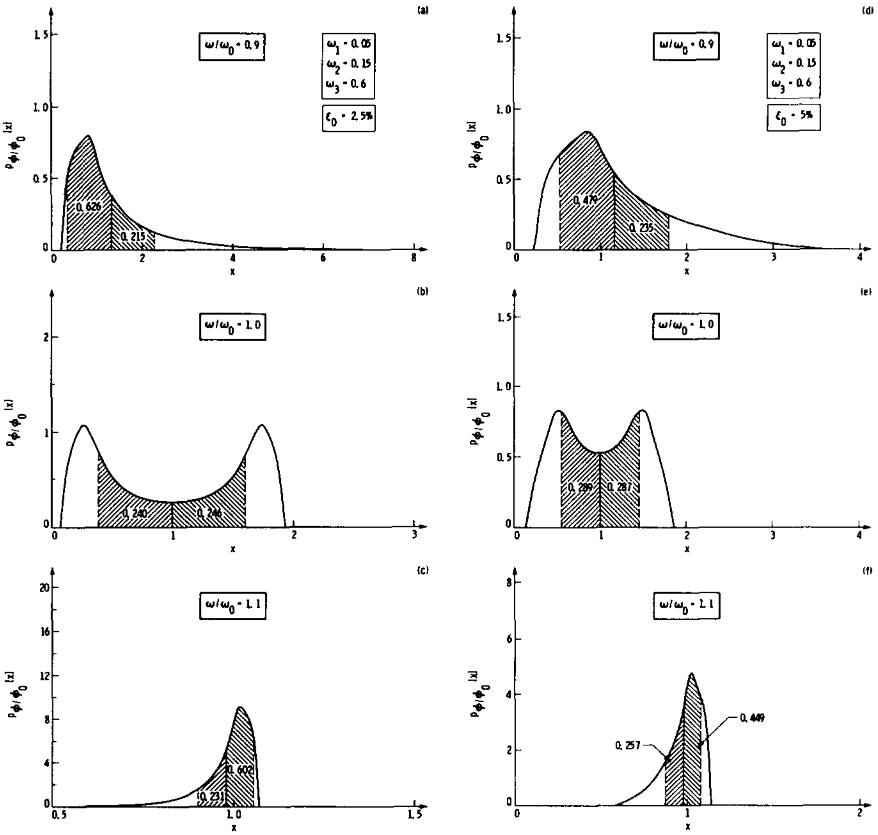

FIG. 13.

can be widely different from its probability of lying in the 1σ band above the mean. These probability values are shown in the figures.

III. TRANSIENT RESPONSE

We are now in a position to study the response of the random system given by Equation (1) when subjected to a transient, though deterministic, excitation. Assuming that the system starts from rest,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)D(\omega)e^{i\omega t}\,d\omega, \tag{43}$$

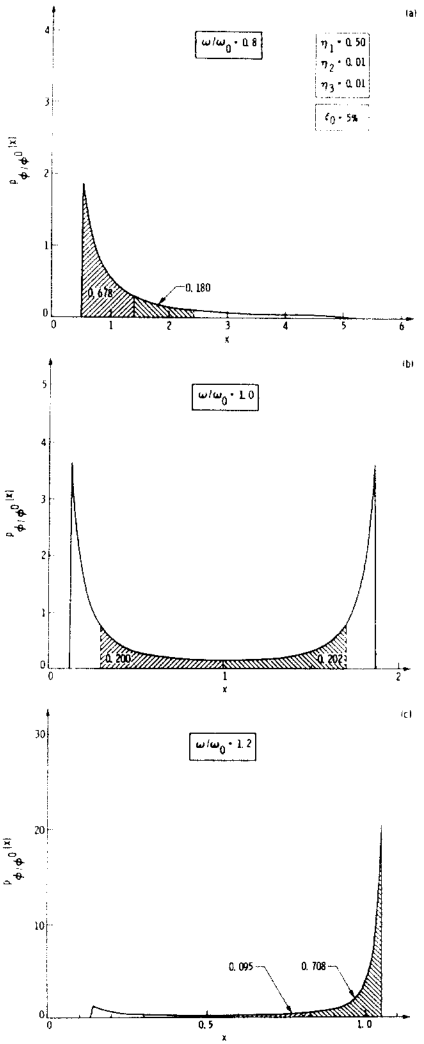
(a)
$\omega/\omega_0 = 0.8$
$\eta_1 = 0.50$
$\eta_2 = 0.01$
$\eta_3 = 0.01$
$\zeta_0 = 5\%$
0.678
0.180
(b)
$\omega/\omega_0 = 1.0$
0.200
0.202
(c)
$\omega/\omega_0 = 1.2$
0.095
0.708
x

FIG. 14.

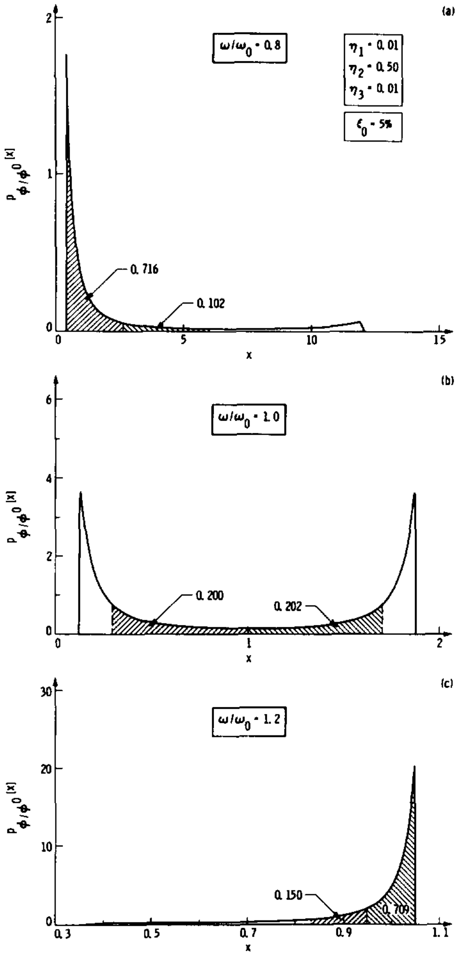

FIG. 15.

so that

$$E\left[x_f(t)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} E\left[H_f(\omega)\right] D(\omega) e^{i\omega t}\, d\omega. \tag{44}$$

The relations (7)–(12) explicitly provide $E[H_f(\omega)]$. Similarly, the expected value of the relative response when the oscillator is subjected to a base

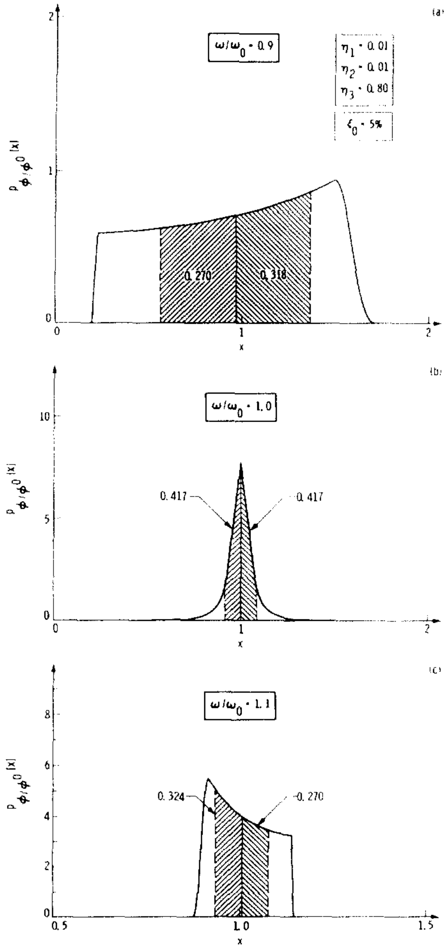

FIG. 16.

acceleration $\ddot{z}(t)$ is

$$E[x_e(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} E[H_e(\omega)]\,\ddot{z}(\omega)e^{i\omega t}\,d\omega, \tag{45}$$

where $E[H_e(\omega)]$ is obtained from (7) and (9). The autocorrelation $R_{e,f}(t_1, t_2)$

of the response then becomes

$$R_f(t_1,t_2) = E\left[x_f(t_1)x_f(t_2)\right] = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[H_f(\omega_1)H_f^*(\omega_2)\right] \times D(\omega_1)D^*(\omega_2)e^{i(\omega_1 t_1-\omega_2 t_2)}\,d\omega_1\,d\omega_2 \tag{46}$$

and

$$R_e(t_1,t_2) = E\left[x_e(t_1)x_e(t_2)\right] = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[H_e(\omega_1)H_e^*(\omega_2)\right] \times \ddot{Z}(\omega_1)\ddot{Z}^*(\omega_2)e^{i(\omega_1 t_1-\omega_2 t_2)}\,d\omega_1\,d\omega_2. \tag{47}$$

Here we have made use of the fact that $x_{e,f}(t)$ are real functions of time.

The expressions for $E[H_{e,f}(\omega_1)H_{e,f}^*(\omega_2)]$, when obtained, can be used in the inverse double transformation represented by (46) and (47). For $\omega_1=\omega_2=\omega$, we have

$$E\left[H_{e,f}(\omega)H_{e,f}^*(\omega)\right] = E\left[A_{e,f}^2(\omega)\right]. \tag{48}$$

Closed-form expressions for this expectation are provided in Equations (18) and (19) for forced excitation, and (25) and (26) for earthquake loading.

Also one can show that for $\omega_1 \neq \omega_2$,

$$E\left[H_f(\omega_1)H_f^*(\omega_2)\right] = \frac{1}{V_{12}}\left[F_f(\omega_1) - F_f(-\omega_2)\right], \tag{49}$$

where

$$F_f(\omega) = \sum_{s_i=1}^{2}\sum_{s_j=1}^{2}\sum_{s_l=1}^{2}\varepsilon_{s_i s_j s_l}t_f\left(m_{s_i},k_{s_j},c_{s_l}\right), \tag{50}$$

$$t_f\left(m_s,k_j,c_l\right) = \int^{c_l}\ln\left(k_j - m_s\omega^2 + ic\omega\right)\ln\left(m_s\alpha - ic\right)dc - i\omega\int^{m_s}\ln\left(k_j - m\omega^2 + ic_l\omega\right)\ln\left(m\alpha - ic_l\right)dm,$$

$$\alpha = \omega_1 - \omega_2, \tag{51}$$

$$V_{12} = \left[\left(\omega_1^2-\omega_2^2\right)\left(1-\frac{\omega}{\alpha}\right)(k_2-k_1)(c_2-c_1)(m_2-m_1)\right]^{-1}. \tag{52}$$

The expression for $E[H_e(\omega_1)H_e^*(\omega_2)]$ when $\omega_1 \neq \omega_2$ is more difficult to obtain in closed form. Though it is amenable to further simplification, it is best left as a double integral as follows:

$$E\left[H_e(\omega_1)H_e^*(\omega_2)\right] = F(k_2, \omega_1) - F(k_1, \omega_1) - F(k_2, -\omega_2) + F(k_1, -\omega_2), \quad (53)$$

where

$$F(k_j, \omega) = \frac{1}{V(\omega_1 + \omega_2)} \int_{c_1}^{c_2} \int_{m_1}^{m_2} \frac{m^2}{m\alpha - ic} \ln\left(k_j - m\omega^2 + ic\omega\right) dm\, dc. \quad (54)$$

The expressions (49) through (54) can be computed quite accurately and quickly. For numerical computations, we use the FFT to obtain $D(\omega)$ or $\ddot{Z}(\omega)$, multiply by the analytically obtained values of $E[H_e(\omega)]$ or $E(H_f(\omega)]$ given by (7), (9), and (10), and use the FFT again to obtain the inverse transform of their product. This yields the expected value of the time response of the system as expressed by (44) and (45). To obtain the variance of the response as a function of time, we first obtain $E[x_{e,f}^2(t)]$ by setting $t_1 = t_2$ in (46) and (47). For details of the use of the one- and two-dimensional FFTs, we refer the reader to Reference [2]. It should be noted that in the use of the discrete Fourier transform, great economies can be obtained by realizing that $E[H(\omega_i)H^*(\omega_j)] = E[H(\omega_i)H(-\omega_j)]$.

IV. NUMERICAL EXAMPLE

Consider a structure, subjected to a transient base acceleration $\ddot{z}(t)$, modeled by a single-degree-of-freedom system whose parameters m, k, and c are only approximately known [Figure 17(a)]. Say m is believed to lie somewhere between 0.95 and 1.05, k is believed to lie between 85 and 115, and c is believed to lie between 0.4 and 1.6. We assume, of course, that these quantities are provided in the appropriate units. Then the system with the mean properties defined by the triplet (m_0, k_0, c_0) is $(1, 100, 1)$. The undamped fundamental frequency ω_0 corresponding to this system is 10 rad/sec, and the percentage of critical damping, ξ_0, is 5%. The parameters

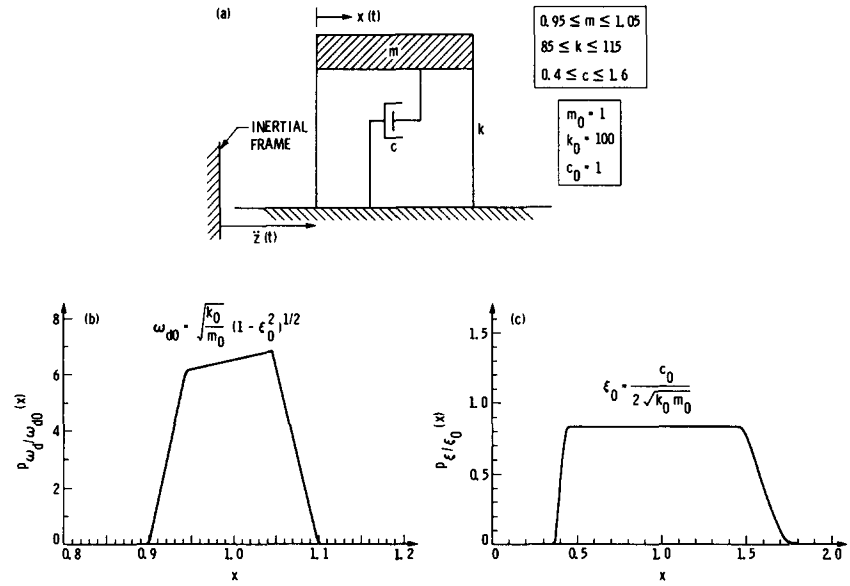

FIG. 17.

chosen and the levels of uncertainty are consistent with what one often comes across in structural analysis.

In the absence of any further information, the maximally unpresumptive distributions will be uniform distributions. The corresponding probability densities of ω_d (the damped natural frequency of vibration of the random system) and ξ (the percentage of critical damping) are obtained from the analytical results of paper I and are shown in Figures 17(b) and (c).

The relative response of the system to a deterministic transient base acceleration is shown in Figure 18. The base acceleration is obtained by using amplitude-modulated white noise [Figure 18(c)]. The solid line in Figure 18(a) represents the expected value of the response, $E[x_e(t)]$, as obtained by Equation (45), using the analytical expression for $E[H_e(\omega)]$ from (7) and (9). The computations are performed using the FFT [Equation (45)]. The variance of the response is obtained by using Equation (47) with $t_1 = t_2$, where $E[H_e(\omega_1)H_e^*(\omega_2)]$ is obtained using the analytical expressions derived. In the figure is shown the 1σ band below and above the mean. We note that the base excitation stops around 6 seconds and the response beyond that consists of damped forced vibrations of the system. Unlike the undamped case, the variance tends to zero as $t \to \infty$. Figure 18(b) shows the response of the system with the mean properties. Comparing this response with $E[x_e(t)]$

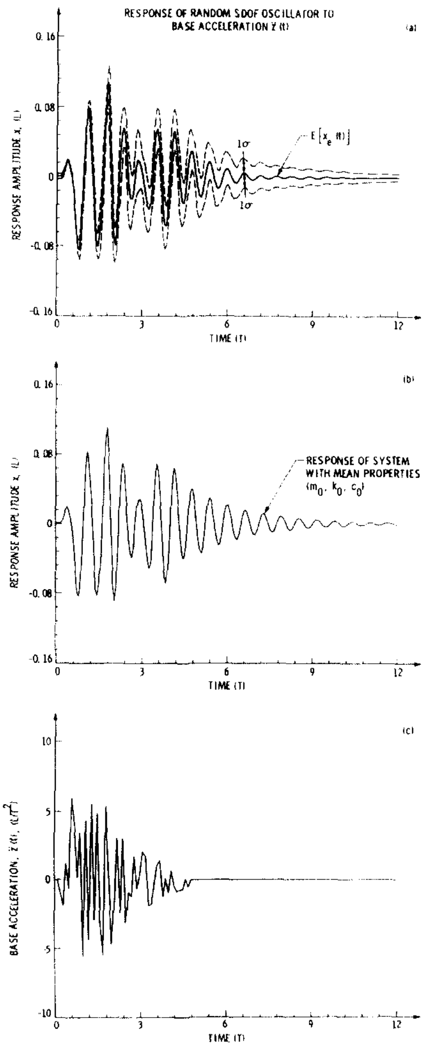

FIG. 18.

[Figure 18(a)], we observe that the expected response of the random system is of a shorter duration (has greater effective damping, so to say) than the response of the deterministic system.

Figure 19 shows the response of the random system to an impulsive base acceleration. The 1σ band on either side of the expected response is also indicated. From the magnitudes of the standard deviation of the response, we

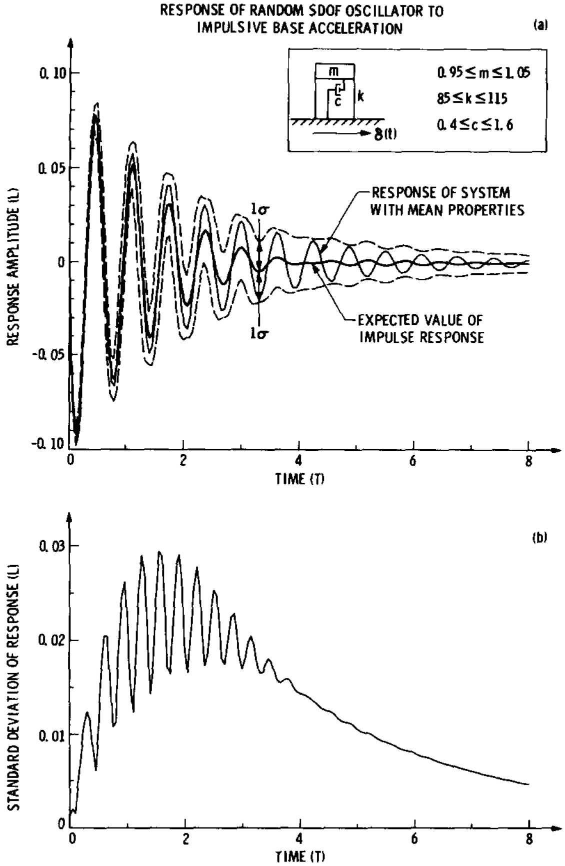

FIG. 19.

see that the uncertainty in the system parameters could critically affect its response amplitude and therefore the safety of the structure.

V. RESPONSE TO RANDOM EXCITATION

Consider now a stationary stochastic process $y(t)$ with mean μ and autocorrelation $R(\tau) = E[y(t+\tau)y(t)]$. Consider the systems defined by

$$m\ddot{x}_f + c\dot{x}_f + kx_f = y(t), \tag{55a}$$

$$m\ddot{x}_e + C\dot{x}_e + kx_e = -my(t). \tag{55b}$$

Their response is

$$x_{e,f}(t)=\int_{-\infty}^{\infty}h_{e,f}(\alpha)y(t-\alpha)\,d\alpha, \tag{56}$$

where $h_f(t)$ is the impulse response and $h_e(t)=-mh_f(t)$.

We shall assume that the random functions $y(t)$ are generated through a process independent of those that generate the random variables m, k, and c. This assumption is generally valid, as the base acceleration or forced excitation is usually not linked to the system parameters. We now need to consider two sorts of expectations: one, the ensemble average taken at any time t over the random variables $y(t)$; the other, the average taken over the probability distributions of the random parameters m, k, and c in parameter space. We shall denote these expected values as E_s and E_p, respectively. Thus

$$E_s\left[x_{e,f}(t)\right]=\int_{-\infty}^{\infty}h_{e,f}(\alpha)E_s[y(t-\alpha)]\,d\alpha, \tag{57}$$

$$E_p\left[E_s\left[x_{e,f}(t)\right]\right]=\int_{-\infty}^{\infty}E_p\left[h_{e,f}(\alpha)\right]E_s[y(t-\alpha)]\,d\alpha. \tag{58}$$

We note that the operators E_s and E_p commute. We shall denote $E(*)=E_p[E_s[*]]$. Thus

$$E\left[x_{e,f}(t)\right]=\mu\int_{-\infty}^{\infty}E_p\left[h_{e,f}(\alpha)\right]d\alpha=\mu E\left[H_{e,f}(0)\right]. \tag{59}$$

The expressions for $E[H_{e,f}(0)]$ are given in (9a) and (10a). Also,

$$E_s\left[x_{e,f}(t_1)x_{e,s}(t_2)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h_{e,f}(\alpha)h_{e,f}(\beta)E_s \\ \times\left[y(t_1-\alpha)y(t_2-\beta)\right]d\alpha\,d\beta, \tag{60}$$

so that

$$E\left[x_{e,f}(t_1)x_{e,f}(t_2)\right]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E_p\left[h_{e,f}(\alpha)h_{e,f}(\beta)\right] \\ \times R_{yy}(t_1-t_2+\alpha-\beta)\,d\alpha\,d\beta. \tag{61}$$

Thus $E[x_{e,f}(t_1)x_{e,f}(t_2)]$ is a function of $\tau=t_1-t_2$ and the process $x_{e,f}(t)$ is stationary (in the wide sense).

Denoting,

$$\begin{aligned}\bar{R}_{x_{e,f}x_{e,f}}(\tau) &= E\left[x_{e,f}(t)x_{e,f}(t+\tau)\right]\\ S_{yy}(\omega) &= \int_{-\infty}^{\infty} R_{yy}(\tau)e^{-i\omega\tau}\,d\tau,\\ \bar{S}_{x_{e,f}x_{e,f}}(\omega) &= \int_{-\infty}^{\infty} \bar{R}_{x_{e,f}x_{e,f}}(\tau)e^{-i\omega\tau}\,d\tau,\end{aligned} \tag{62}$$

we have from (61)

$$\bar{S}_{x_{e,f}x_{e,f}} = E_p\left[H_{e,f}(\omega)H^*_{e,f}(\omega)\right]S_{yy}(\omega). \tag{63}$$

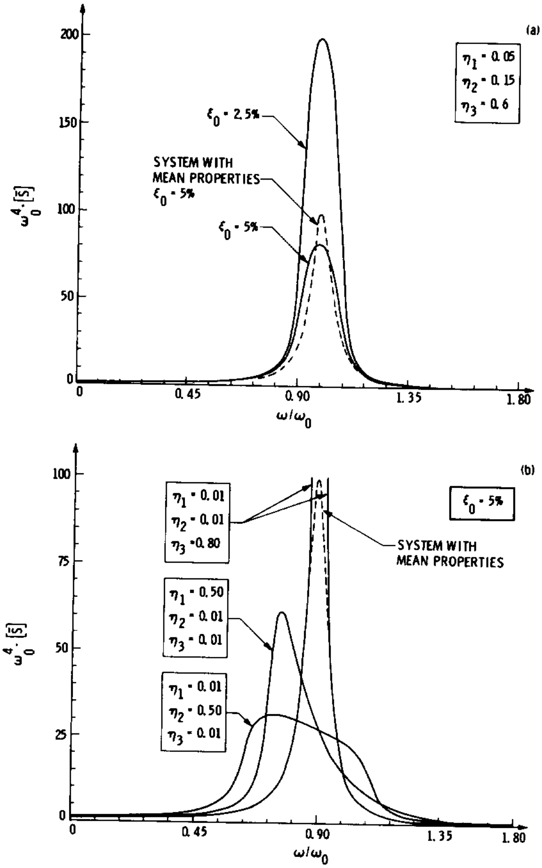

FIG. 20.

Thus given the power spectrum of the input, $S_{yy}(\omega)$, the power spectrum of the output is obtained by multiplying by $E[|H_{e,f}(\omega)|^2]$, a quantity which was obtained in closed form through the relations (18)–(20) and (25)–(28). It should be noted that $\bar{S}$ corresponds to the autocorrelation function $E_p\{E_s[x(t)x(t+\tau)]\}$. It may be thought of as the power spectrum of the output averaged over all possible systems $\{m_i, k_j, c_l\}$ over the volume of parameter space that the random system encompasses.

The power spectrum of the random system's response when multiplied by ω_0^4 can be shown to be dependent on the power spectrum of the input and the dimensionless parameters ξ_0, η_1, η_2, η_3, and ω/ω_0. Figure 20 shows the power spectrum of the response for the random systems considered in Figures 5 and 6, respectively, when subjected to a white-noise base acceleration $[S_{yy}(\omega)=1]$. We note that the expected spectrum of the response is quite different from that obtained by using a deterministic system with its mean properties m_0, c_0, and k_0. The closed-form expressions obtained in Equations (18)–(20) and (25)–(28) can be used in (63) to yield the expected power spectrum of the response to any given input spectrum $S_{yy}(\omega)$.

VI. CONCLUSIONS

In this paper I have investigated the response of an uncertain single-degree-of-freedom oscillator whose mass, damping, and stiffness parameters (m, k, and c) are only known imprecisely. Using the framework established in paper I [1], I have used uniform distributions for these three parameters as being maximally unpresumptive. The response to both base excitations and forced excitations has been studied.

(1) Closed-form expressions for the statistics of the frequency-dependent transfer function of such an uncertain system are provided. The statistics of the normalized transfer functions are shown to depend only on the nondimensional parameters η_1, η_2, and η_3 which provide the extent of uncertainty in our knowledge of the mass, stiffness, and damping parameters, and on ξ_0, the percentage of critical damping for the system with the mean properties. The effect of uncertainties in the individual parameters m, k, and c on the statistics of the transfer function are graphically portrayed. Monte Carlo simulations are used to verify the analytically obtained results.

(2) Using the closed-form expressions for the statistics of the transfer function, the statistics of the response to transient deterministic excitation is obtained. A numerical example is provided to illustrate the manner in which the uncertainty in the parameter values of mass, stiffness, and damping map into the time-varying uncertainty in the system's response. For levels of uncertainty which are found in common engineering practice it is shown that

the response uncertainties may be quite high, making the need for such analyses important in the design and analysis of critical structures such as nuclear reactors and spacecraft. The expected response of the system is seen to be widely different from the response of the system with the mean properties even for levels of modest uncertainty.

(3) The expected spectrum of the response of the system to stationary random excitation is obtained in closed form. It is shown that this expected spectrum could differ substantially from that for the system with the mean properties for commonly occurring levels of uncertainty met with in engineering design.

The results presented here are, to the best of my knowledge, new. They will aid the engineer/scientist in assessing the uncertainties in the response of such systems and, through that, their safety. The results obtained here are all the more significant because the levels of uncertainty commonly met with in actual engineering practice often make the application of perturbation techniques tenuous at best.

APPENDIX

Using the notation in [1],

$$E[x(t)] = \int_{I_\Omega} \left(x_0 \cos yt + v_0 \frac{\sin yt}{y} \right) p_\omega(y)\, dy, \tag{A.1}$$

where $p_\omega(y)$ is the p.d.f. of ω, and I_Ω is the interval over which $p_\omega(y)$ is nonzero. Integrating by parts, we have

$$E[x(t)] = \frac{1}{t} \int_{I_\Omega} \left[v_0 \cos yt \left(\frac{p(y)}{y} \right)' dy - x_0 \sin yt p'(y) \right] dy. \tag{A.2}$$

Thus, provided $[p(y)/y]'$ and $p'(y)$ are piecewise integrable functions over I_Ω,

$$E[x(t)] = O(1/t). \tag{A.3}$$

For uniformly distributed variables m, k, and c, by [1, Equation (7)], I_Ω can be split into three subintervals I_i, $i = 1,2,3$, within each of which $p_\omega(y)$ is of the form $a_i\omega + b_i/\omega^3$. For $\bar{\eta} \leqslant 1$, we get, using [1, Equation (7a)],

$$I_1 = (\xi_{12}\omega_0, \xi_{11}\omega_0), \qquad a_1 = \frac{1}{4\eta_1\eta_2\omega_0}(1+\eta_1)^2, \qquad b_1 = \frac{-1}{4\eta_1\eta_2\omega_0}(1-\eta_2)^2,$$

$$I_2 = (\xi_{11}\omega_0, \xi_{22}\omega_0), \qquad a_2 = \frac{1}{\eta_2\omega_0}, \qquad b_2 = 0, \tag{A.4}$$

$$I_3 = (\xi_{22}\omega_0, \xi_{21}\omega_0), \qquad a_3 = \frac{-1}{4\eta_1\eta_2\omega_0}(1-\eta_1)^2, \qquad b_3 = \frac{1}{4\eta_1\eta_2\omega_0}(1+\eta_2)^2,$$

For $\bar{\eta} > 1$, the intervals I_i, $i = 1,2,3$, and the parameters a_i, b_i, $i = 1,2,3$, can be similarly obtained from [1, Equation 7(b)].

Using (A.3), we have

$$E[x(t)] = \sum_{i=1}^{3} \left\{ a_i \frac{x_0 \cos yt}{t^2}\bigg|_{I_i^L}^{I_i^u} + \frac{b_i}{t}\int_{I_i}\left[\frac{3x_0 \sin yt}{y^4} - \frac{4v_0 \cos yt}{y^5}\right]dy \right\}, \tag{A.5}$$

where the first term is evaluated between the upper and lower limits I_i^L and I_i^u of each interval I_i, provided in (A.4).

The $E[x^2(t)]$ can similarly be expressed as

$$E[x^2(t)] = \frac{x_0^2}{2} + \frac{v_0^2}{2}\int_{I_\Omega}\frac{p(y)}{y^2}dy + \frac{x_0 v_0}{2t}\int_{I_\Omega}\cos 2yt\left(\frac{p(y)}{y}\right)' dy$$
$$- \frac{1}{4t}\int_{I_\Omega}\left[x_0^2 p'(y) - v_0^2\left(\frac{p(y)}{y^2}\right)'\right]\sin 2yt\, dy. \tag{A.6}$$

Again, if $p(y)/y^2$, $[p(y)/y]'$, and $[p(y)/y^2]'$ are piecewise integrable over I_Ω,

$$E[x^2(t)] = \frac{x_0^2}{2} + \frac{v_0^2}{2}\int_{I_\Omega}\frac{p(y)}{y^2}dy + O\left(\frac{1}{t}\right). \tag{A.7}$$

For the uniform distribution case we can divide the interval I_Ω into subintervals. (A.6) then reduces to

$$
\begin{aligned}
E\left[x^{2}(t)\right] = {} & \frac{x_0^2}{2} + v_0^2 \sum_{i=1}^{3} \frac{1}{2}\left(a_i \ln y - \frac{1}{4}\frac{b_i}{y^4}\right)\Bigg|_{I_i^L}^{I_i^u} \\
& + \sum_{i=1}^{3} \frac{a_i x_0^2 \cos 2yt}{8t^2}\Bigg|_{I_i^L}^{I_i^u} \\
& + \sum_{i=1}^{3} \left\{ \frac{v_0^2}{4t} \int_{I_i} \sin 2yt \left[a_i \ln y - \frac{5b_i}{y^6} \right] \right\} dy \\
& + \int_{i=1}^{3} \left\{ \frac{3}{4}\frac{x_0^2}{t} b_i \int_{I_i} \frac{\sin 2yt}{y^4}\, dy \right\} - \sum_{i=1}^{3} \frac{x_0 v_0}{2t} \int_{I_i} \frac{4b_i}{y^5} \cos 2yt\, dy. \qquad \text{(A.8)}
\end{aligned}
$$

Should the initial conditions x_0 and v_0 be independent random variables with known means and variances, then the expected response and the variance of response of the random system to random initial conditions would be obtained by replacing $E[v_0]$ and $E[x_0]$ for v_0 and x_0 in (A.2) and $E[v_0^2]$ and $E[x_0^2]$ for v_0^2 and x_0^2 in (A.6) under the presumption that x_0, v_0, m, and k are independent.

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