Response of Uncertain Dynamic Systems. I

F. E. Udwadia

School of Civil Engineering
University of Southern California
Los Angeles, California 90089-1114

ABSTRACT

This paper and its sequel deal with the characterization of the response of a structure modeled by a single-degree-of-freedom system in which the mass, stiffness, and damping parameters are *only* imprecisely known. Using an information theoretic framework, closed form expressions are developed for the probability densities of the parameters that control the characteristics of the dynamic response, namely, the natural frequency of vibration, the damped natural frequency, and the percentage of critical damping. The manner in which these closed form expressions depend on the extent and nature of our knowledge about the uncertain parameters is highlighted. Some comparisons with perturbation solutions are also presented.

I. INTRODUCTION

In the last decade or two, an increasing amount of attention has been paid by engineers and scientists all over the world towards obtaining improved methods of analysis for structural and mechanical systems subjected to various types of dynamic loading environments. This work has mainly been prompted by the need for better prediction of the structural response of linear and nonlinear systems so that more economical and safer designs can be obtained.

The structural analyst/designer, however, has two basic sources of uncertainty to contend with, while striving to obtain economical and safer designs. The first is uncertain knowledge of the dynamic loading time histories that may be brought to bear at various (often uncertain) locations of the structure. The second is the basic uncertainty in the structural model of the system

itself.

To date, a large amount of effort has been expended in dealing with the first source of uncertainty. Various methods (e.g., [1–7]) are currently in use, such as the spectrum methods, Monte Carlo simulations, and statistical characterizations of system dynamics under suitable assumptions regarding the loading time histories. In fact, the whole field of stochastic processes has been brought to bear on the response characterization of systems subjected to random loads.

As much as the first area of uncertainty mentioned above has been perhaps overworked, the second has been neglected. At the risk of exaggerating, one may go so far as to say that in many cases the structural analyst totally ignores the fact that he may not have the correct model for the system whose dynamic response he is required to find. No amount of meticulous, time-consuming computation that he performs will yield satisfactory predictions of the system dynamics unless his model, to begin with, is in close approximation to reality. This the analyst does know. One may then ask why it is that the analyst often ignores uncertainties in the knowledge of his structural model. The answer, it appears, is threefold.

The first reason is that accurate, controlled testing of large-scale structures has become a prevalent field of study only in the last decade or so. Such refined experimental testing has brought to light the fact that the dynamic characteristics of large-scale systems are often quite different from those calculated (estimated) from design drawings. For instance, a large building structure may have as much as 30–40% of its total stiffness contributed by nonstructural components. The assessment of its stiffness distribution from structural design drawings (most of which don't even show any nonstructural components) could then lead to sizeable errors in the prediction of its response to dynamic loads. Even if an attempt to include the nonstructural elements were made, the analyst would soon be confronted with such bewildering questions as "How much does the specific configuration for water piping in the structure contribute to its stiffness?"

This brings us to the second reason. It is often difficult to quantify the level of uncertainty attached to the estimates of the parameter values. While it is known that uncertainties (say) in the stiffness estimates of various structural members do exist, the level of confidence that a designer may have is often difficult for him to gauge due to the multitude of sources that contribute to the uncertainty.

The last reason, which is perhaps the most important one, is that improved methods need to be developed so that the analyst can have a suitable framework to handle the uncertainties involved in the structural models. Few researchers [8–12] have addressed this last problem area, and the work done to date concentrates on studying the response of uncertain systems within the

framework of perturbation theory. This implicitly assumes "small" uncertainty—a condition which, in many cases, may be contrary to common experience in the field of structural engineering. It is the intent of this paper to extend this work and develop analytical techniques to handle the "large" uncertainties met with in the day-to-day design and analysis of structures.

Though it is more than likely that the uncertainties in the parameter values of structural systems will increase as they move into the nonlinear regimes of response, nonlinear systems are not addressed here. This is primarily because of: (a) the increased complexity of the problem, and (b) the fact that the nature of the nonlinearities could be different for different structural configurations and/or materials. As stated by Rosenberg [13], "everybody knows what a banana is, but a non-banana could be anything." Attention is focused in this study on the simplest generic uncertain structural system model—the single-degree-of-freedom system.

The presentation is provided in two parts. The first part deals with the problem formulation within the general framework of information theory and provides probabilistic descriptions of the parameters of central importance in the assessment of the dynamics of such a system. Some perturbation results are also included. The second part uses the results derived in the first part and deals with the response to dynamic loads while focusing mainly on systems whose parameters are described by uniformly distributed random variables. The manner in which the uncertainties in the system parameter values map into the uncertainties in the transfer function of the system and its time history of response has been deduced in closed form. Deterministic transient excitations and random stochastic excitations are studied. Two cases of forced excitation, occurring in the analysis of structures subjected to wind loads and to earthquakes, have been considered.

This two-paper series presents several new closed-form results dealing with the characterization of such uncertain dynamic systems, as well as their response to deterministic and stationary stochastic time histories of excitation. Graphical portrayals have been given to provide a deeper physical insight into the analytical results obtained.

II. PROBLEM FORMULATION

Consider a structural system whose response $x(t)$ is modeled by a single-degree-of-freedom linear oscillator through the relation

$$m\ddot{x} + c\dot{x} + kx = d(t), \tag{1}$$

where m, k, and c are the mass, stiffness, and damping of the oscillator. The function $d(t)$ constitutes the externally applied force time history. In the

earthquake loading situation, $x(t)$ may be thought of as the relative response of the oscillator with respect to its base, and $d(t) \triangleq - m\ddot{z}(t)$ as the inertial loading of the mass m caused by the base acceleration $\ddot{z}(t)$.

Most analytical techniques for determining the response $x(t)$ of a physical system which is modeled by Equation (1) implicitly assume, at some stage, that the parameters m, k, and c are precisely known. These parameter values can be thought of as representing a "point" in a three-dimensional space which describes the system, called "parameter space." Thus any analysis which is predicated upon such a deterministic system model yields a nominal or baseline design. Such a design is then relevant to but a single point in our suitably defined system parameter space. Inevitably, due to numerous sources of modeling errors (especially in constructing "equivalent" single-degree-of-freedom representations of large complex systems), the actual system model is found to be at some other point in parameter space. This fact often undermines any claim that the analytically deduced model characteristics and/or responses are accurate reflections of the actual system characteristics and/or responses and often necessitates much *ad hoc* "hedging" around the nominal analysis (design)—a practice that has almost become standard in most fields of engineering analysis. Clearly, the importance of such *ad hoc* procedures becomes greater in fields where uncertainties in the system parameters are larger, for the "hedging" primarily aims at trying to recover, often intuitively, the basic system properties that are lost or obscured in consequence of parameter uncertainties. Hence the abundant use of such practices in fields like soil engineering.

To circumvent the deficiencies of current design practices, it is first necessary to recognize that any model (even a large finite-element model) never encompasses the truth, but at best is a mathematical statement of what and *how much* is known. The system model must not only specify nominal values of system parameters, but must also contain an *admission* of prior ignorance regarding the possible deviations from these nominal values. Thus we are led towards a quantification of prior ignorance. A straightforward approach to this problem, it appears, would be to assign a probability distribution for each of the system parameters. However, the problem is a little more subtle than that, for one is seldom provided with a complete probability assignment based upon empirical data. Reliance on a highly limited set of available statistical information is thus usually necessitated. One is then forced to induce a complete probability model for the parameters, which is *consistent* with the data on hand but *which admits the greatest possible prior ignorance*, thus avoiding so to speak the need to "invent" data which do not exist in support of an *ad hoc* probability assignment.

The method of ascribing such a probability distribution which provides for a maximum of ignorance while including the complete statistical database

available at hand can be obtained by viewing the problem in an information-theoretic framework: one first defines a measure of prior information, the entropy, and then determines the probability assignment which maximizes the entropy subject to the constraints imposed on available data.

Consider the case where (say) the uncertain parameter k in the relation (1) is known *a priori* to lie between two values, say a and b. We shall assume that a and b are both finite. Denoting the probability density of k by $p_k(\ell)$, the *a priori* ignorance is described by the Shannon measure given by

$$J = - \int_a^b p_k(\ell) \ln p_k(\ell)\, d\ell. \tag{2}$$

If further a knowledge of say the first n moments of the distribution (e.g., the mean, variance, etc.) is available, then the maximally unpresumptive probability density function for k would be such as to maximize J subject to these n constraints. Thus, given the n constraints

$$\int_a^b \ell^i p_k(\ell)\, d\ell = d_i, \qquad i = 0, 1, \ldots, n, \tag{3}$$

where d_i, $i = 1, 2, \ldots, n$, are given and $d_0 = 1$, we use the technique of undetermined multipliers to maximize the functional

$$J = J + \sum_{i=0}^{n} \lambda_i \left[\int_a^b \ell^i p_k(\ell_i)\, d\ell_i - d_i \right].$$

The use of standard variational calculus then leads to

$$p_k(\ell) = \exp\left[-1 - \sum_{i=0}^{n} \lambda_i \ell^i \right], \tag{4}$$

where the multipliers λ_i, $i = 0, 1, \ldots, n$, can be found using the $n + 1$ equations of the set (3).

Perhaps the three most commonly occurring applications of the relation (4) arise when:

(a) *k is known to lie between two values, a and b, with no further information* ($n = 0$). In that case the most unpresumptive probability density function (p.d.f) consistent with this knowledge is a uniform distribution between a and b.

(b) *k is known to lie between two values, a and b, and its mean (nominal value) is known (n = 1)*. This yields a p.d.f. which, in general, is exponential. If further the mean is $(a + b)/2$, then the distribution reverts to the uniform distribution of case (a) above.

(c) *k is known to lie between two values, a and b, its mean (nominal value) is given by a finite number $(a + b)/2$, and its variance is known (n = 2)* and *less than* $\frac{1}{3}[(b - a)/2]^2$. This yields a p.d.f. which is a truncated Gaussian distribution.

In a similar manner various other distributions for uncertain parameters can be adduced, depending on the nature and quantity of the *a priori* data available. However, from a practical engineering standpoint, the situations represented by cases (a) and (c) above appear to be most common. Extensive use of these two types of density functions will be made in Section III. Figure 1 shows the density functions for the uncertain parameters m, k, and c which will be used in this model.
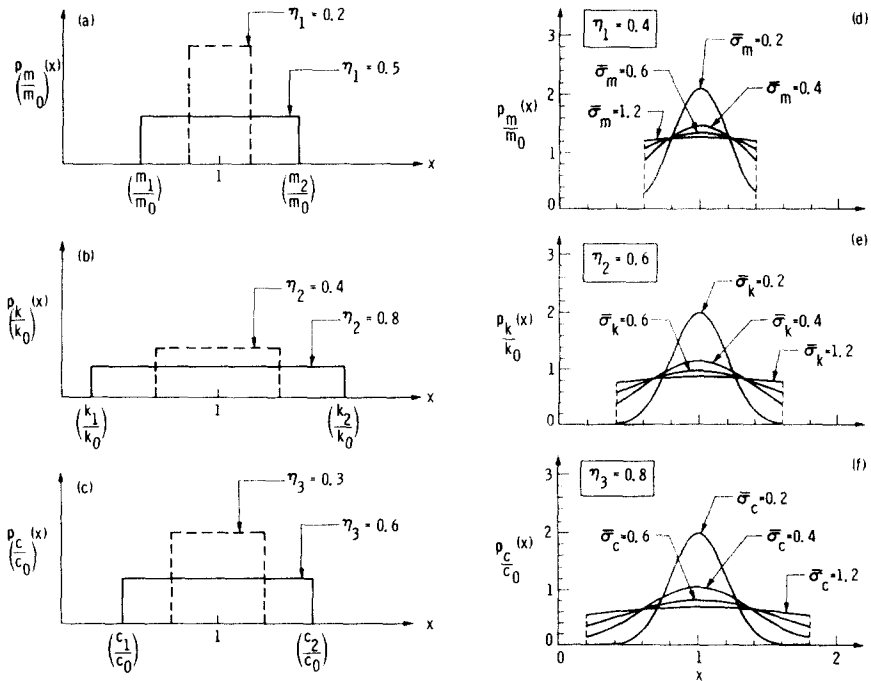

Fig. 1

III. STATISTICS OF SOME IMPORTANT OSCILLATOR CHARACTERISTICS

In this section, we shall describe the probability distributions that index the behavior of an oscillating system described by the relation (1). These properties are: the natural frequency of vibration, ω_n; the percentage of critical damping, ξ; and the damped natural frequency of vibration, $\omega_n\sqrt{1-\xi^2}$.

A. *Probability Density of ω_n*

As mentioned earlier, the uncertain knowledge of k and m leads to a probability assignment on k and m depending on the available *a priori* information. We shall assume, for lack of better knowledge, that the uncertainties in k and m are independent of each other. This implies that the adduced probability densities of k and m are also then independent of each other. Using the distributions of case (a) and case (c) of Section I (which, as discussed, imply corresponding levels of *a priori* knowledge of k and m), the distribution of $\omega_n \triangleq \sqrt{k/m}$ is found below.

(i) *k and m Uniformly Distributed.* Given that k is an uncertain parameter which lies in the range $0 < k_1 \leqslant k \leqslant k_2 < \infty$ and m another independent, uncertain parameter that lies in the range $0 < m_1 \leqslant m \leqslant m_2 < \infty$ [Figure 1(a) and (b)], one can define nominal values k_0 and m_0 through the relations

$$
\begin{aligned}
m_2 &= m_0 + \alpha, & k_2 &= k_0 + \beta, \\
m_1 &= m_0 - \alpha, & k_1 &= k_0 - \beta.
\end{aligned} \tag{5}
$$

The parameters α and β serve to quantify the uncertainties in m and k about their respective nominal (mean) values. We next normalize the various parameters with respect to $\omega_0 \triangleq \sqrt{k_0/,m_0}$ as follows:

$$
\begin{aligned}
\xi_{11} &= \frac{\sqrt{k_1/m_1}}{\omega_0}, & X &= \frac{\sqrt{k/m}}{\omega_0} = \frac{\omega}{\omega_0}, \\[2ex]
\xi_{12} &= \frac{\sqrt{k_1/m_2}}{\omega_0}, & \eta_1 &= \frac{\alpha}{m_0}, \\[2ex]
\xi_{21} &= \frac{\sqrt{k_2/m_1}}{\omega_0}, & \eta_2 &= \frac{\beta}{k_0}, \\[2ex]
\xi_{22} &= \frac{\sqrt{k_2/m_2}}{\omega_0}, & \bar{\eta} &= \frac{\eta_1}{\eta_2}.
\end{aligned} \tag{6}
$$

The normalized scatters η_1 and η_2 naturally satisfy the relation $0 \leqslant \eta_1, \eta_2 \leqslant 1$. The probability density of X, the dimensionless natural frequency of the oscillator normalized with respect to the natural frequency of the system with mean properties, in addition to depending on η_1 and η_2, also depends upon the ratio of the normalized scatters, $\bar{\eta}$. We note that $\bar{\eta} \geqslant 1$ implies $\xi_{22} \leqslant \xi_{11}$, and $\bar{\eta} \leqslant 1$ implies $\xi_{11} \leqslant \xi_{22}$. After some algebra, these distributions are given by the following (see Appendix 1):

(a) for $\bar{\eta} \leqslant 1$,

$$
p_X(x) = \begin{cases}
0, & x \leqslant \xi_{12}, \\[2mm]
\dfrac{1}{4\eta_1\eta_2}\left[(1+\eta_1)^2 x - (1-\eta_2)^2 \dfrac{1}{x^3}\right], & \xi_{12} < x \leqslant \xi_{11}, \\[2mm]
\dfrac{1}{\eta_2}x, & \xi_{11} < x < \xi_{22}, \quad (7a) \\[2mm]
\dfrac{1}{4\eta_1\eta_2}\left[(1+\eta_2)^2 \dfrac{1}{x^3} - (1-\eta_1)^2 x\right], & \xi_{22} \leqslant x < \xi_{21}, \\[2mm]
0, & \xi_{21} \leqslant x,
\end{cases}
$$

and

(b) for $\bar{\eta} \geqslant 1$,

$$
p_X(x) = \begin{cases}
0, & x \leqslant \xi_{12}, \\[2mm]
\dfrac{1}{4\eta_1\eta_2}\left[(1+\eta_1)^2 x - (1-\eta_2)^2 \dfrac{1}{x^3}\right], & \xi_{12} < x \leqslant \xi_{22}, \\[2mm]
\dfrac{1}{\eta_1 x^3}, & \xi_{22} < x < \xi_{11}, \quad (7b) \\[2mm]
\dfrac{1}{4\eta_1\eta_2}\left[(1+\eta_2)^2 \dfrac{1}{x^3} - (1-\eta_1)^2 x\right], & \xi_{11} \leqslant x \leqslant \xi_{21}, \\[2mm]
0, & \xi_{21} \leqslant x.
\end{cases}
$$

Though the distributions of k and m are symmetric about their respective means, the distribution of the normalized natural frequency X is not. As would be expected, the expressions depend only on the normalized scatters, η_1 and η_2. However, the two distributions, depending on the parameter $\bar{\eta}$, differ substantially in their functional forms in the central region, where X

lies between ξ_{11} and ξ_{22}. For $\bar{\eta} > 1$ the p.d.f. in this region is a *cubically decreasing* function, while for $\bar{\eta} < 1$ the corresponding p.d.f. is a *linearly increasing* one. For the case when $\bar{\eta} = 1$ (i.e. $\xi_{11} = \xi_{22}$) this central region in the p.d.f. disappears. Also, we note that the functional forms for the p.d.f. in the two cases are identical for the regions which fall outside this central zone.

Figure 2(a) illustrates the effect of uncertainties in the *a priori* knowledge of k as it affects the p.d.f. of $\sqrt{k/m}$. The value of η_1 is taken to be 0.5, and η_2 is varied between 0 and 0.7. When $\eta_2 = 0$, we have $\beta = 0$, so that the stiffness k is assumed to be exactly known *a priori*. It is seen that for this case, $\xi_{11} = \xi_{21}$ and $\xi_{12} = \xi_{22}$. For $\bar{\eta} \geqslant 1$, the p.d.f. is solely represented by the cubically *decreasing* expression, $(\eta_1 x^3)^{-1}$, given in Equation (7b) for the range $\xi_{22} < x < \xi_{11}$. As η_2 increases, the central region of the x domain decreases gradually while the two outer regions become more and more prominent, as indicated by a widening of the two side slopes. When $\bar{\eta} = 1$, the central region ceases to exist, and $\xi_{11} = \xi_{22}$. Either one of the relations (7a) or (7b) can be used to get $p_X(x)$ in that case. Lastly, for $\bar{\eta} < 1$, a linearly *increasing* p.d.f. in the central region is observed for $\eta_2 = 0.7$. Figure 2(b) shows the effect of uncertainties in the *a priori* knowledge of m when η_2 is maintained at a constant value of 0.5. The p.d.f. gradually changes from a linearly increasing function (for $\eta_1 = 0$) to a piecewise continuous function whose central region is a decreasing cubic (for $\eta_1 = 0.7$). The mode of the p.d.f. of X for $\bar{\eta} < 1$ occurs at $x > 1$; the mode of the p.d.f. of X for $\bar{\eta} > 1$ occurs at $x < 1$.

It is interesting to note that though the functional form for the p.d.f. of X depends on whether $\bar{\eta}$ is greater than or less than unity, its expected value does not. In fact, using the expressions (7a) and (7b), the expected value can be found, after some algebra, to be

$$E[X] = \frac{4}{3}\left\{ \frac{2 + \sqrt{1 - \eta_2^2}}{\left[(1 + \eta_1)^{1/2} + (1 - \eta_1)^{1/2}\right]\left[(1 + \eta_2)^{1/2} + (1 - \eta_2)^{1/2}\right]} \right\}. \quad (8)$$

Similarly, the standard deviation of the normalized variable X for all $\bar{\eta}$ is given by

$$\sigma[X] = \left\{ \frac{1}{2\eta_1} \ln\left(\frac{1 - \eta_1}{1 - \eta_1}\right) - E^2[X] \right\}^{1/2}. \quad (9)$$

The first term on the right-hand side of (9) is the second moment of the p.d.f. of the normalized frequency X. It is interesting to note that its value depends
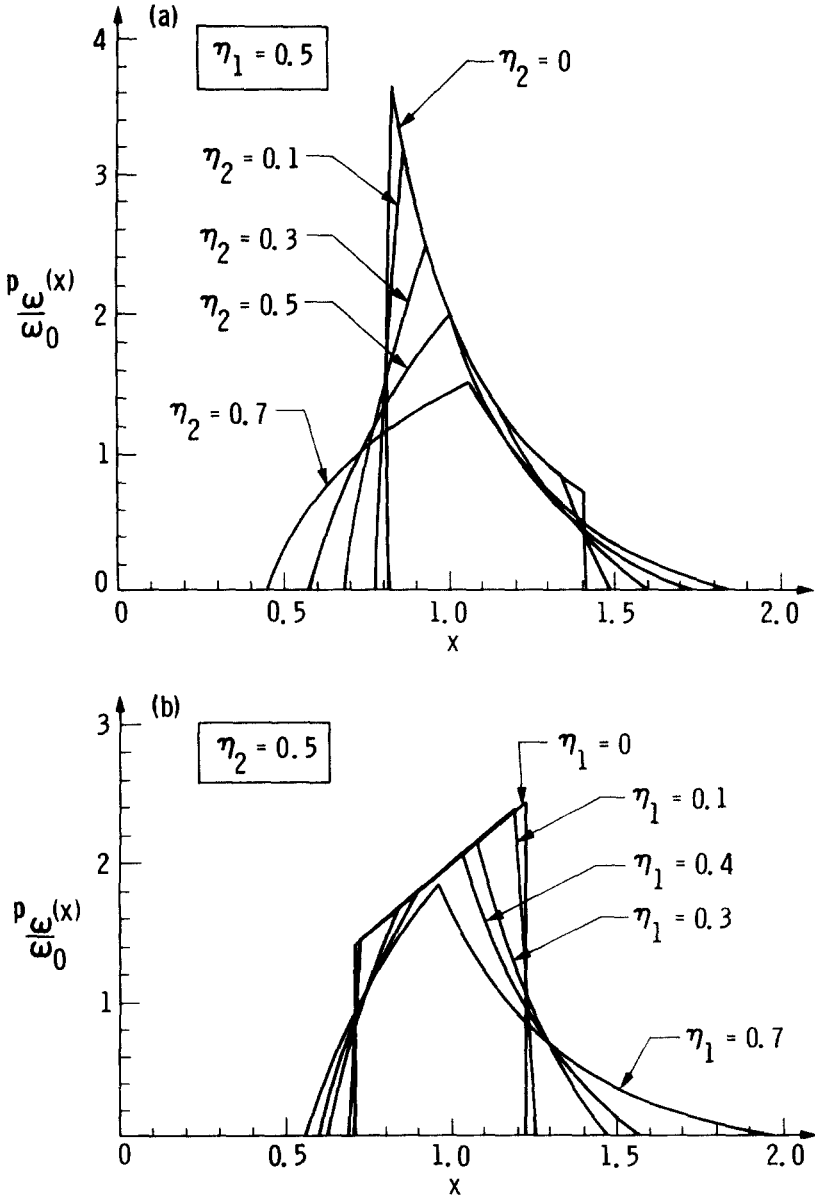
Fig. 2

only on η_1, the uncertainty in our knowledge of the mass parameter, and not on η_2.

In most applications in structural engineering, the values of η_1 and η_2 do not exceed about 0.1 and 0.2. In soil engineering, however, uncertainties in η_2 often are as large as 0.5 to 0.7. In the study of fluid-structure interaction, where the quantification of the hydrodynamic effects of "added mass" may become uncertain for turbulent flow regimes, values of η_1 as high as 0.7 to 0.8 may be encountered.

We next consider the situation where the mean, the standard deviation, and the range of values that the parameters m and k can take are known *a priori*. This leads in a natural manner, from the information-theoretic point of view, to the truncated Gaussian distributions of Figure 1(d) and (e).

(ii) *k and m are Gaussian Distributed.*   We assume that k and m are two uncertain, independent parameters whose ranges of variation are known *a priori* so that $0 < k_1 \leqslant k \leqslant k_2 < \infty$ and $0 < m_1 \leqslant m \leqslant m_2 < \infty$. Knowledge of their nominal values $[m_0 \triangleq (m_1 + m_2)/2$ and $k_0 \triangleq (k_1 + k_2)/2]$ and of the *a priori* variances σ_k and σ_m of k and m about their respective mean values would then lead to truncated Gaussian distributions from the information-theoretic viewpoint. The p.d.f.'s of m and k can be expressed as

$$p_{m/m_0}(x) = \begin{cases} \dfrac{A}{\sqrt{2\pi}\,\bar{\sigma}_m} \exp\left[-\dfrac{1}{2}\left(\dfrac{x-1}{\bar{\sigma}_m}\right)^2\right], & 1-\eta_1 < x \leqslant 1+\eta_1, \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

and

$$p_{k/k_0}(x) = \begin{cases} \dfrac{B}{\sqrt{2\pi}}\dfrac{1}{\bar{\sigma}_k} \exp\left[-\dfrac{1}{2}\left(\dfrac{x-1}{\bar{\sigma}_k}\right)^2\right], & 1-\eta_2 < x \leqslant 1+\eta_2, \\ 0 & \text{otherwise,} \end{cases} \quad (11)$$

where, as in (6),

$$\eta_1 = \frac{m_2 - m_1}{2m_0}, \qquad \eta_2 = \frac{k_2 - k_1}{2k_0}.$$

The normalized variances are defined by,

$$\bar{\sigma}_m = \sigma_m/m_0 \quad \text{and} \quad \bar{\sigma}_k = \sigma_k/k_0.$$

The constants A and B which normalize the p.d.f.'s are given by

$$A = \left[1 - 2\,\mathrm{erf}(-\eta_1/\bar{\sigma}_m) \right]^{-1},$$

$$B = \left[1 - 2\,\mathrm{erf}(-\eta_2/\bar{\sigma}_k) \right]^{-1}, \tag{12}$$

where the error function is defined by

$$\mathrm{erf}(*) \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{*} \exp\left(-\frac{t^2}{2} \right) dt.$$

Figure 1(d) shows the variation in the p.d.f.'s of m/m_0 for various values of $\bar{\sigma}_m$ and η_1. The parameter η_1 controls the width of the region for which the p.d.f. is nonzero, while the parameter $\bar{\sigma}_m$ controls the peakedness of the distribution about the mean value. Thus both these parameters, in a sense, jointly control the scatter of m about its mean value. A similar situation arises for the distribution of k [Figure 1(e)].

Using the nomenclature

$$X = \omega/\omega_0,$$

$$\bar{A} = \frac{AB}{\pi} \left(\bar{\sigma}_k \bar{\sigma}_m x^3 \right)^{-1},$$

$$p = \bar{\sigma}_k^{-2} + \left(\bar{\sigma}_m x^2 \right)^{-2},$$

$$q = \bar{\sigma}_k^{-2} + \left(\bar{\sigma}_m x \right)^{-2},$$

$$s = \bar{\sigma}_k^{-2} + \bar{\sigma}_m^{-2}, \tag{13}$$

$$r = q/p$$

$$r_1 = \bar{A} \exp\left[-\frac{1}{2}\left(s - \frac{q^2}{p} \right) \right],$$

$$q_1 = \sqrt{2\pi}\, q p^{-3/2},$$

$$\left[G(*\,;y) \right]_{a,b} = G(*\,;b) - G(*\,;a),$$

after considerable algebra, the p.d.f. of the dimensionless frequency is obtained, for various values of $\bar{\eta} \triangleq \eta_1 / \eta_2$, as follows: For $\bar{\eta} \leqslant 1$,

$$p_X(x) = \begin{cases} 0, & x \leqslant \xi_{12}, \\ [G]_{a_1, a_2}, & \xi_{12} < x \leqslant \xi_{11}, \\ [G]_{a_3, a_4}, & \xi_{11} < x < \xi_{22}, \\ [G]_{a_5, a_6}, & \xi_{22} \leqslant x < \xi_{21}, \\ 0, & \xi_{21} \leqslant x, \end{cases} \qquad (14a)$$

where

$$G(\bar{\sigma}_m, \bar{\sigma}_k, \eta_1, \eta_2; y) = r_1 \left\{ q_1 \operatorname{erf}(\sqrt{p}\, y) - \frac{1}{p} \exp\left(-\frac{py^2}{2} \right) \right\} \qquad (15)$$

and

$$a_1 = (1 - \eta_2) - r, \qquad a_2 = x^2(1 + \eta_1) - r,$$

$$a_3 = x^2(1 - \eta_1) - r, \qquad a_4 = a_2, \qquad (16a)$$

$$a_5 = a_3, \qquad a_6 = (1 + \eta_2) - r.$$

Similarly, for $\bar{\eta} \geqslant 1$,

$$p_X(x) = \begin{cases} 0, & x \leqslant \xi_{12}, \\ [G]_{b_1, b_2}, & \xi_{12} < x \leqslant \xi_{22}, \\ [G]_{b_3, b_4}, & \xi_{22} < x < \xi_{11}, \\ [G]_{b_5, b_6}, & \xi_{11} \leqslant x < \xi_{21}, \\ 0, & \xi_{21} \leqslant x, \end{cases} \qquad (14b)$$

where G is as defined in equation (15), and

$$b_i = a_i, \qquad i = 1, 2, 5, 6,$$

$$b_3 = (1 - \eta_2) - r, \quad \text{and} \quad b_4 = (1 + \eta_2) - r. \qquad (16a)$$

We note that the p.d.f. of X, the normalized natural frequency, is solely

dependent on the dimensionless parameters η_1, η_2, $\bar{\sigma}_m$, and $\bar{\sigma}_k$ which determine the scatters of the distributions of k and m.

For fixed values of η_1 and η_2, the truncated Gaussian distributions of m and k tend towards uniform distributions as $\bar{\sigma}_m$ and $\bar{\sigma}_k$ tend to infinity. Thus the uniform distribution turns out to be a special member of the family of truncated Gaussian distributions. In the case of uniform distributions, it is easily shown that the scatters η_1 and η_2 are linearly related to the respective standard deviations and are given by

$$\eta_1 = \sqrt{3}\,\bar{\sigma}_m \quad \text{and} \quad \eta_2 = \sqrt{3}\,\bar{\sigma}_k.$$

So that reasonable comparisons in the distributions of the dimensionless frequency X between the Gaussian and the uniform case can be adduced for the limiting conditions stated above, we introduce, for the Gaussian case, the parameters η_1^* and η_2^* and obtain the distributions of X for various values of η_i^*, $i = 1,2$, where η_i^* are defined in terms of the normalized deviations $\bar{\sigma}_k$ and $\bar{\sigma}_m$ by the relations

$$\eta_1^* = \sqrt{3}\,\bar{\sigma}_m \quad \text{and} \quad \eta_2^* = \sqrt{3}\,\bar{\sigma}_k.$$

Figure 3 illustrates the p.d.f. of ω/ω_0 when $\eta_1 = 0.14$, $\eta_2 = 0.5$, and $\eta_1^* = 0.5$. The change in the p.d.f. of ω/ω_0 as η_2^*, the normalized standard deviation of the parameter k, changes from 0.1 to 1.3 is illustrated in Figure 3(c). As the uncertainty in the knowledge of the stiffness k increases with η_2^* [see Figure 3(b)], the density function of ω/ω_0 changes from one which is almost symmetrical about the line $x = \omega/\omega_0 = 1$ to one whose mode shifts significantly towards the *higher* frequencies. For large values of η_2^*, the distribution of k becomes almost uniform, and the density of ω/ω_0 tends towards a linear variation as predicted by Equation 7(a).

The variation in the density of ω/ω_0 for various values of η_1^* holding η_2^* fixed at a value of 0.5 is shown in Figure 4(c). The small value of η_1 chosen causes the p.d.f. of ω/ω_0 to be only mildly affected by variation of the parameter η_1^*.

Figures 5 and 6 show the p.d.f.'s of ω/ω_0 when $\eta_2 = 0.2$ and $\eta_1 = 0.5$, indicating a situation in which the uncertainty in the mass m may be higher than in the stiffness parameter k. Such patterns of uncertainty, as stated before, often occur in fluid-structure interaction problems, where the added-mass effects may become difficult to assess in an appropriate fashion for certain flow regimes. Figure 5(c) shows the effect of changes in η_1^* on the p.d.f. of ω/ω_0. For very small values of η_1^*, the p.d.f may again be reasonably
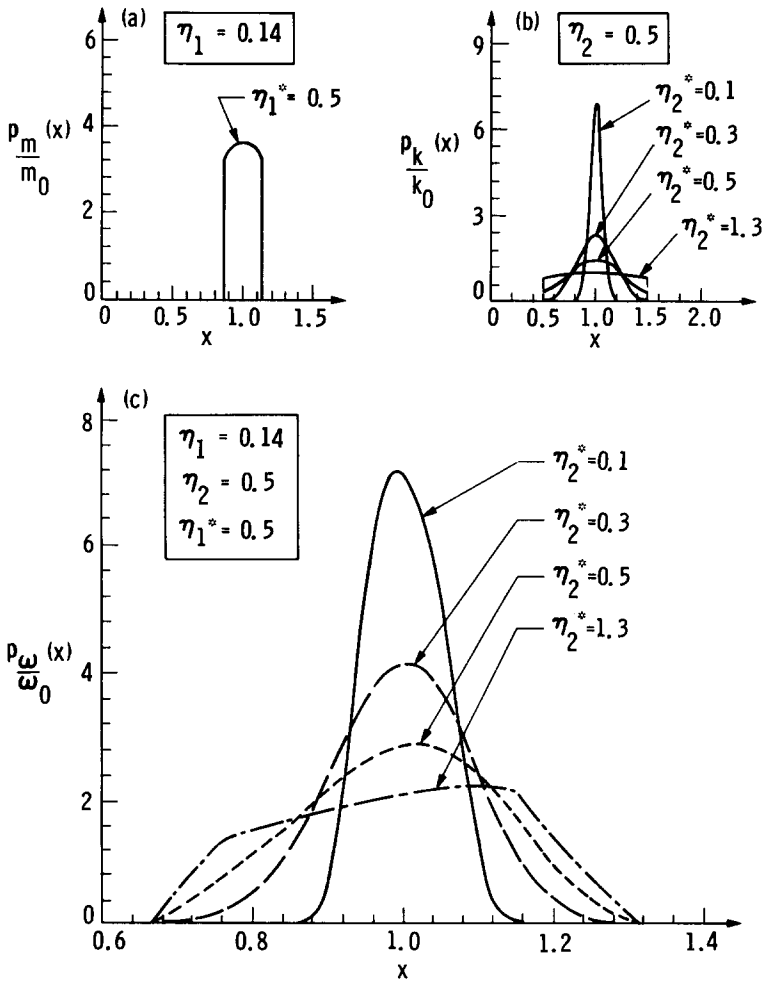
FIG. 3

approximated as being symmetrical about the line $\omega/\omega_0 = 1$. For higher values of η_1^*, the distribution becomes increasingly asymmetrical, eventually moving towards the cubic variation in the "central region" as predicted by Equation 7(b). With increasing values of η_1^*, the mode of the distribution now moves towards the *lower* frequencies when η_1, η_2, η_2^* are fixed.

Figure 6 shows the effect of varying η_2^*, holding η_1, η_2, and η_1^* constant, with the value of $\bar{\eta} > 1$. Comparing these results with those of Figure 3 (for which $\bar{\eta} < 1$), we observe that shape of the density function critically depends
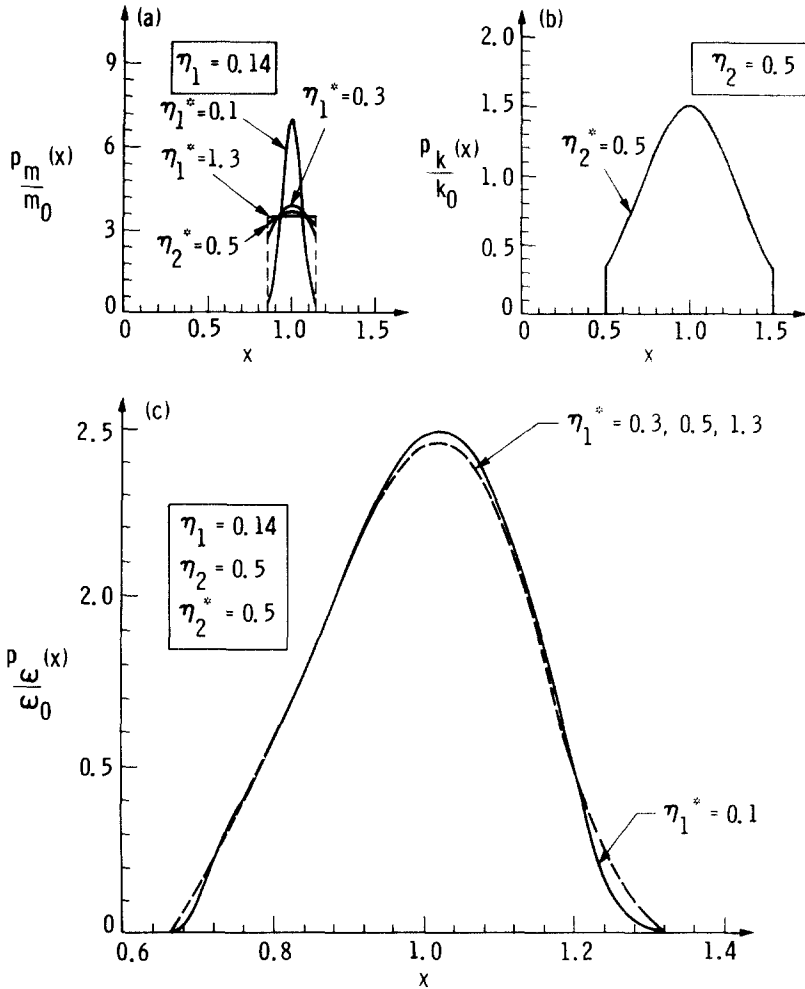
FIG. 4

on the parameter set $S_N \triangleq \{ \eta_1, \eta_2, \eta_1^*, \eta_2^* \}$. In fact, as Figure 7 illustrates, a wide variety of distributions can be obtained for various values of the elements of S_N. Here we have chosen $\eta_1 = \eta_2 = 0.5$ so that the "relative scatters" in the uncertainties of m and k for each of the curves are controlled only by η_1^* and η_2^* respectively. Curve A represents the situation when the uncertainty in knowledge of m is large ($\eta_1^* = 2$) compared to that in k ($\eta_2^* = 0.1$). The distribution resembles the uniform distribution with the
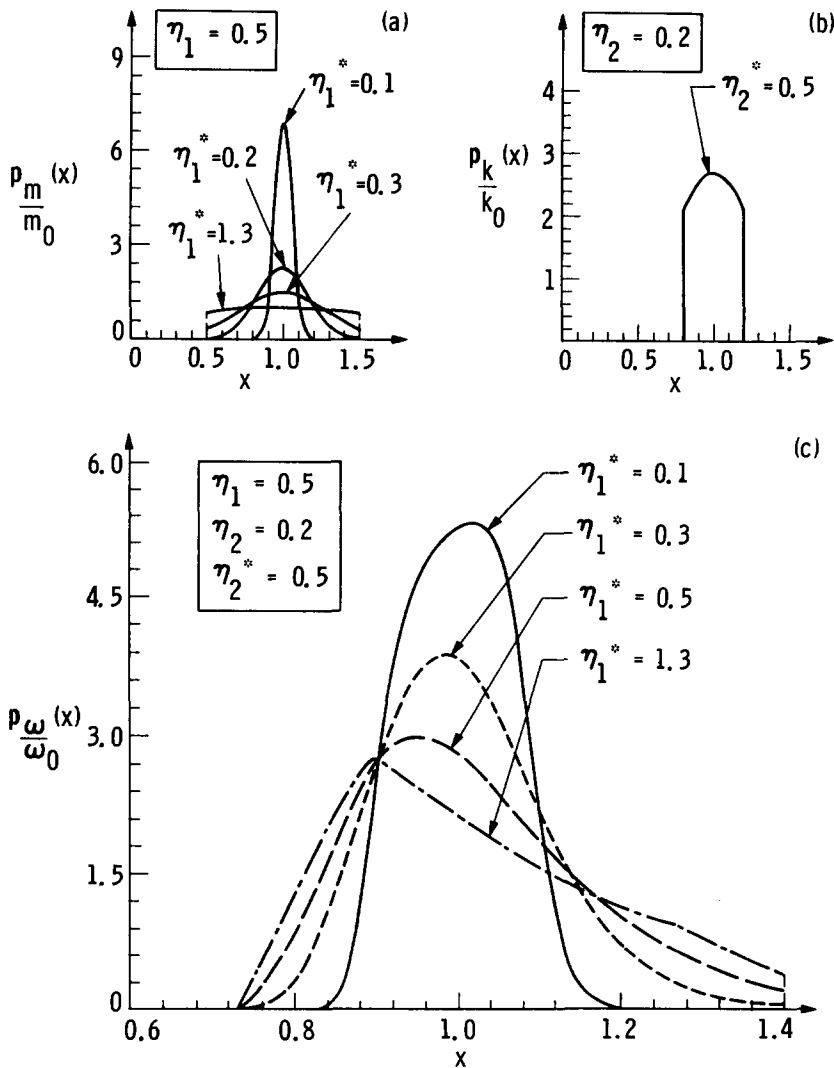
Fig. 5

cubically declining function in the central region, as indeed it should. As the uncertainty in k increases and that in m decreases, the density function gradually changes. Curve D depicts the case (again similar to the uniform distribution with smoothed-out corners) when the uncertainty in k is much larger than in m, yielding a linearly increasing density function in the central
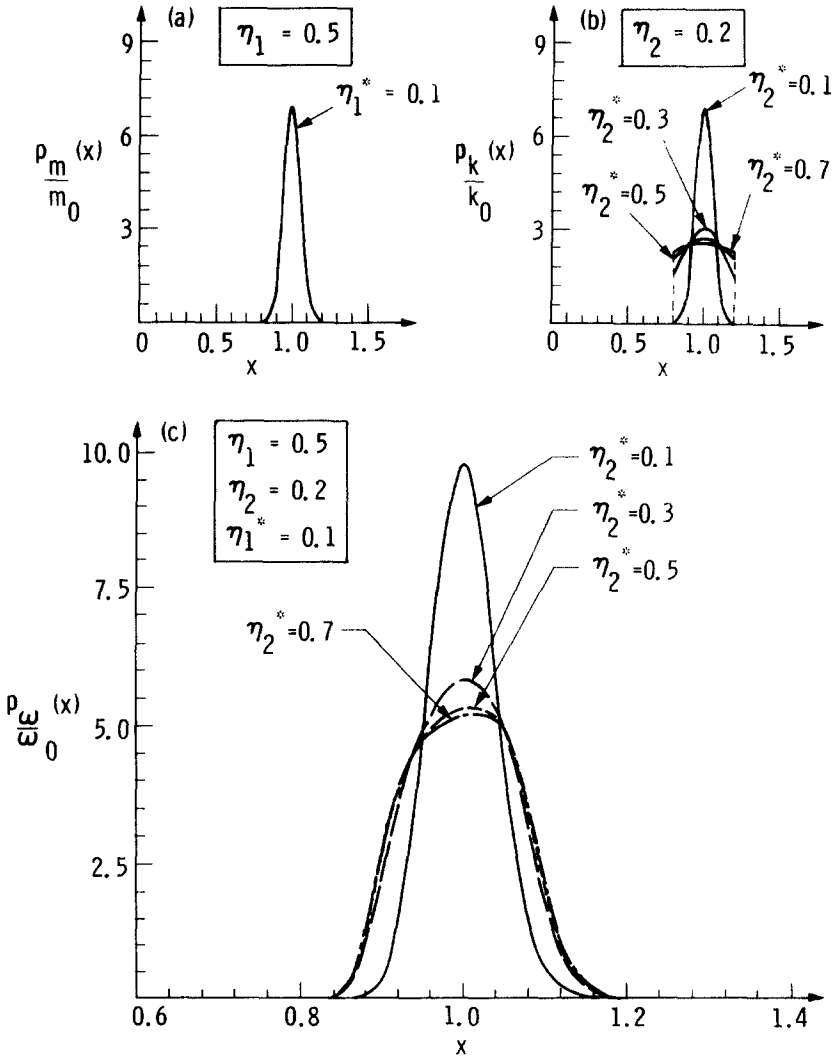
FIG. 6

region. Using different values of η_1^* and η_2^*, a wide variety of density functions result.

(iii) *Expected Values and Some Perturbation Results.* Figure 8(a) shows the relationship (8) plotted for various values of η_1 and η_2 ranging from 0 to 1. As seen, the expected value of the normalized frequency, when k and m
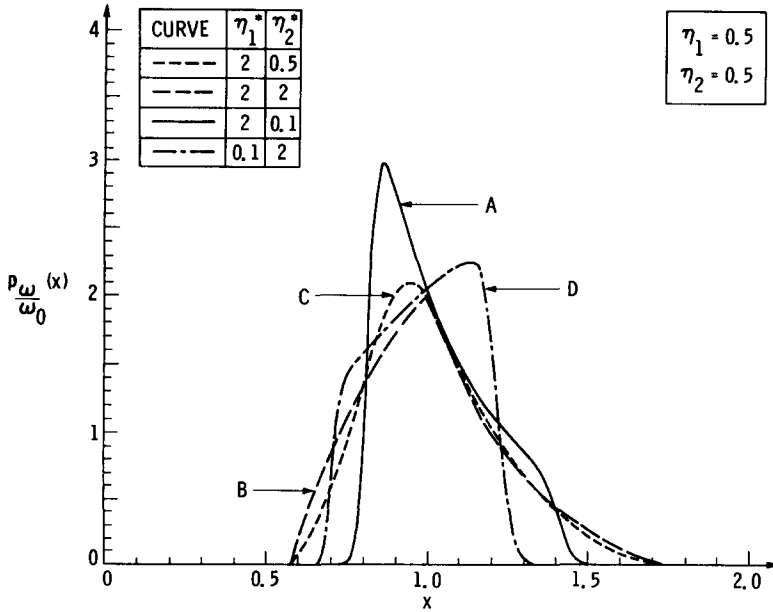
CURVE	η_1^*	η_2^*
----	2	0.5
---	2	2
——	2	0.1
—·—	0.1	2

$\eta_1 = 0.5$
$\eta_2 = 0.5$

FIG. 7

are uniformly distributed, decreases as η_2, the normalized scatter in the stiffness, increases, and increases as η_1 increases. As η_1 and η_2 tend to zero, $E[X] \to 1$, so that $E[\sqrt{k/m}] \to \omega_0$. The distributions of m and k then approach delta functions. We note that for the widely different p.d.f's shown in Figure 2, whose shapes critically depend on the relative values of η_1 and η_2, the expected values of ω are within $\pm 5\%$ of the value of ω_0 as long as η_1 and η_2 are less than 0.5. However, the use of ω_0 as an estimate of $E[\sqrt{k/m}]$ may lead to substantial error (in an absolute sense) if ω_0 is large.

The expression (8) can further be expanded in powers of η_1 and η_2 to yield

$$E[X] = \frac{1 - \eta_2^2/6}{(1 - \eta_1^2/8)(1 - \eta_2^2/8)} + O(\eta_1^4, \eta_2^4).$$ (17)

Neglecting terms in fourth powers of η_1 and η_2, (17) indicates that the frequency of the system having the mean parameter values (i.e. ω_0) is an *overestimate* of the expected value of natural frequency of the random system
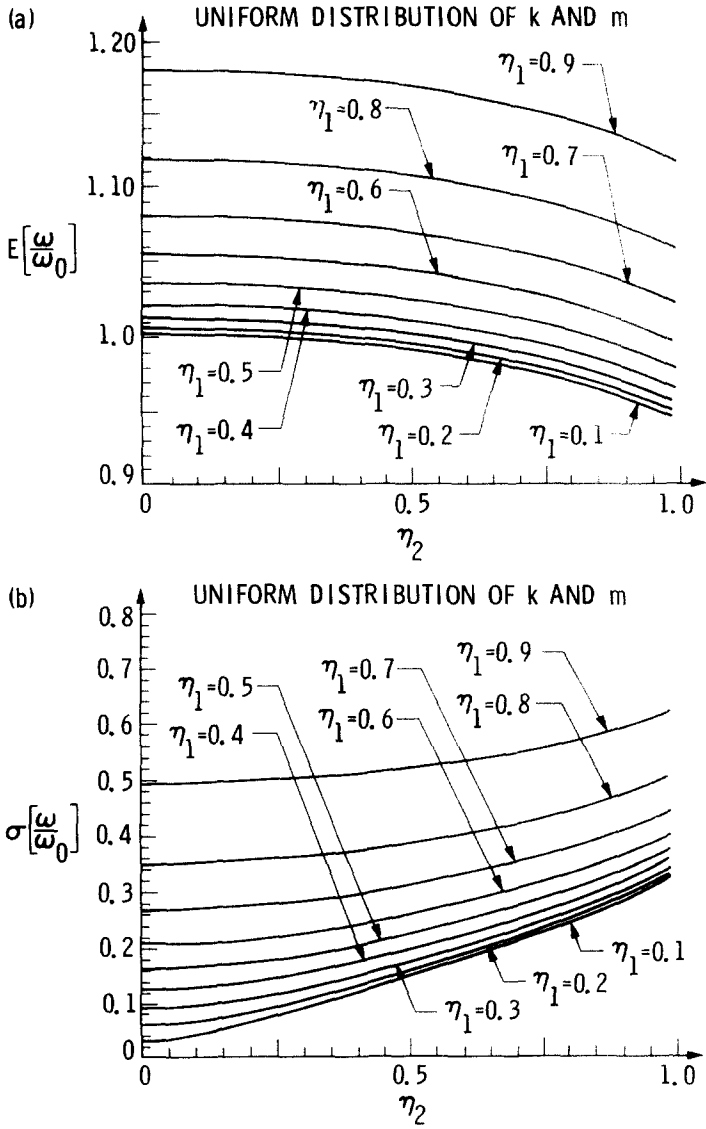
(a) UNIFORM DISTRIBUTION OF k AND m

(b) UNIFORM DISTRIBUTION OF k AND m

Fig. 8

provided $\eta_2 > \sqrt{3}\,\eta_1$. Furthermore,

$$\frac{\partial E}{\partial \eta_1} \simeq \frac{\eta_1}{4} \frac{1 - \eta_2^2/24}{\left(1 - \eta_1^2/8\right)^2} \quad \text{and} \quad \frac{\partial E}{\partial \eta_2} \simeq \frac{-\eta_2}{12} \frac{1}{1 - \eta_1^2/8}. \tag{18}$$

If η_1 and $\eta_2 \ll 1$, then to $O(\eta_1^2, \eta_2^2)$, the expected value of the normalized frequency is approximately three times as sensitive to variations in η_1 as it is to variations in η_2. Observe that increases in η_1 tend to increase $E[X]$, while increases in η_2 tend to lower it.

Whereas perturbation approaches do not yield results directly for the p.d.f.'s they can be used to estimate the expected value of X. Referring to deviations of k from k_0 as Δk and those of m from m_0 as Δm, we have

$$\frac{\sqrt{k/m}}{\sqrt{k_0/m_0}} = \left(1 + \frac{\Delta k}{k_0}\right)^{1/2} \left(1 + \frac{\Delta m}{m_0}\right)^{-1/2}$$

$$= \left[1 + \frac{1}{2}\frac{\Delta k}{k_0} - \frac{1}{8}\left(\frac{\Delta k}{k_0}\right)^2 + O\left(\frac{\Delta k}{k_0}\right)^3 + \cdots\right]$$

$$\times \left[1 - \frac{1}{2}\frac{\Delta m}{m_0} + \frac{3}{8}\left(\frac{\Delta m}{m_0}\right)^2 - O\left(\frac{\Delta m}{m_0}\right)^3 + \cdots\right].$$
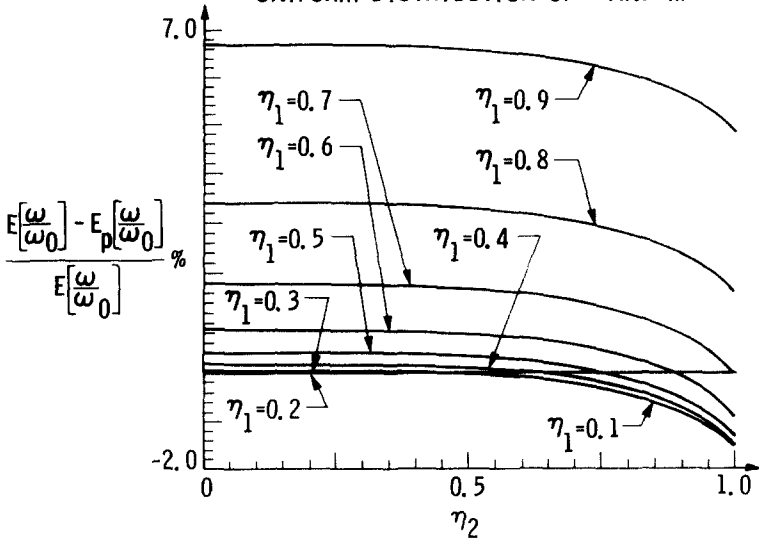
Taking expected values, the perturbation result yields

$$E_p[X] \cong 1 - \frac{1}{8}\left(\frac{\sigma_k}{k_0}\right)^2 + \frac{3}{8}\left(\frac{\sigma_m}{m_0}\right)^2 \tag{19}$$

where we have assumed that the scatters m and k about the means are small enough to be adequately represented by uniform distributions. The relation (19) indicates that the scatter in k tries to reduce the expected value of X below unity while the scatter in m attempts to raise it beyond unity.

Comparing the second and third terms on the right-hand side of (19), we note again that though $\bar{\sigma}_m$ for many mechanical systems is considerably less than $\bar{\sigma}_k$, its influence on the expected value of X is three times as large.

Figure 9(a) shows the percentage difference between the expected value of X for the uniform distribution and the result obtained by the perturbation analysis. As seen, the expected values obtained from the perturbation approach are reasonably accurate for $\eta_1, \eta_2 < 0.2$.

(a)   COMPARISON OF EXACT AND PERTURBATION RESULTS
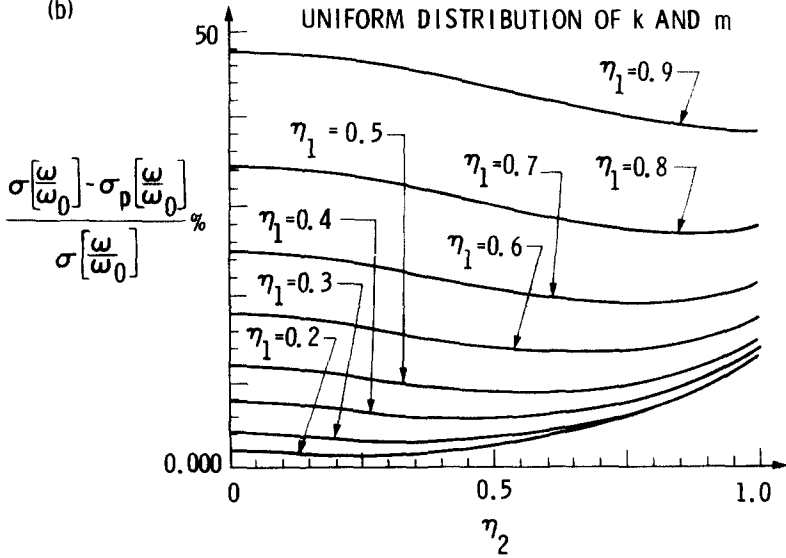UNIFORM DISTRIBUTION OF k AND m

$$\frac{E\left[\frac{\omega}{\omega_0}\right] - E_p\left[\frac{\omega}{\omega_0}\right]}{E\left[\frac{\omega}{\omega_0}\right]} \%$$

$\eta_1 = 0.7$ $\eta_1 = 0.9$

$\eta_1 = 0.6$ $\eta_1 = 0.8$

$\eta_1 = 0.5$ $\eta_1 = 0.4$

$\eta_1 = 0.3$

$\eta_1 = 0.2$ $\eta_1 = 0.1$

η_2

(b)   UNIFORM DISTRIBUTION OF k AND m

$$\frac{\sigma\left[\frac{\omega}{\omega_0}\right] - \sigma_p\left[\frac{\omega}{\omega_0}\right]}{\sigma\left[\frac{\omega}{\omega_0}\right]} \%$$

$\eta_1 = 0.9$

$\eta_1 = 0.5$ $\eta_1 = 0.7$ $\eta_1 = 0.8$

$\eta_1 = 0.4$ $\eta_1 = 0.6$

$\eta_1 = 0.3$

$\eta_1 = 0.2$

η_2

Fig. 9

Similar series expansions can be carried out to obtain the standard deviation of X using a perturbation approach. After some algebra, this yields

$$\sigma_p(X) = \frac{1}{2} \sqrt{\left(\frac{\sigma_k}{k_0}\right)^2 + \left(\frac{\sigma_m}{m_0}\right)^2}, \tag{20}$$

where again we assume that $\Delta m / m_0$ and $\Delta k / k_0$ are sufficiently small to approximate the distributions of m and k by uniform distributions.

Figure 8(b) shows the variation of $\sigma(X)$ with η_1 and η_2 for the uniform distribution [Equation (9)]. As observed, the value of $\sigma(X)$ is less than 0.3 for η_1 and η_2 less than 0.5. However, the perturbation results provide large errors in the estimates of $\sigma(X)$, as seen from Figure 9(b), even for $\eta_1, \eta_2 < 0.1$.

B. Probability Density of ξ

The damping parameter c in the modeling of dynamic systems is the one that is generally known with the lowest degree of certainty. It is often assessed on the basis of past experience and is generally inferred through analogy with similar systems on which dynamic test data may be available. Let us normalize the percentage of critical damping ξ with respect to the percentage for a system which has the mean parameter values. Thus if the viscous damping c is a random variable lying in the range (c_1, c_2), so that $-\infty < c_1 \leqslant c \leqslant c_2 \leqslant \infty$, and if

$$c_1 = c_0(1 - \eta_3), \qquad c_2 = c_0(1 + \eta_3), \tag{21}$$

then we consider the random variable

$$\bar{\xi} = \xi / \xi_0,$$

where

$$\xi = \frac{c}{2\sqrt{km}} \quad \text{and} \quad \xi_0 = \frac{c_0}{2\sqrt{k_0 m_0}}.$$

Then the density of $\bar{\xi}$ can be expressed in terms of the density of the normalized random variable c/c_0 by the relation

$$p_{\bar{\xi}}(x) = \int_{(1-\eta_3)/x}^{(1+\eta_3)/x} \delta p_{c/c_0}(\delta X) p_{\bar{r}}(\delta) \, d\delta, \tag{22}$$

where $\bar{\nu}$ is the random variable $\sqrt{km}\,/\sqrt{k_0 m_0}$. Thus if a closed-form expression for $p_{\bar{\nu}}(\delta)$ is available, the integration can be performed, yielding the density function of $\bar{\xi}$.

(i) *k, m, and c Are Uniformly Distributed* [Figure 1(a),(b),(c)]. For a uniform distribution, using the notation in Equation (6), we get, after considerable algebra,

(a) for $\bar{\eta} \leqslant 1$

$$
p_{\bar{\nu}}(\delta) = \begin{cases}
0, & \delta \leqslant \varepsilon_{11}, \\
\varepsilon_0 \delta \ln(\delta/\varepsilon_{11}), & \varepsilon_{11} \leqslant \delta \leqslant \varepsilon_{21}, \\
\frac{1}{2}\varepsilon_0 \delta \ln\!\left(\dfrac{1+\eta_1}{1-\eta_1}\right), & \varepsilon_{21} < \delta < \varepsilon_{12}, \\
\varepsilon_0 \delta \ln(\varepsilon_{22}/\delta), & \varepsilon_{12} \leqslant \delta < \varepsilon_{22}, \\
0, & \varepsilon_{22} \leqslant \delta,
\end{cases}
\tag{23a}
$$

and

(b) for $\bar{\eta} \geqslant 1$

$$
p_{\bar{\nu}}(\delta) = \begin{cases}
0, & \delta \leqslant \varepsilon_{11}, \\
\varepsilon_0 \delta \ln(\delta/\varepsilon_{11}), & \varepsilon_{11} < \delta \leqslant \varepsilon_{12}. \\
\frac{1}{2}\varepsilon_0 \delta \ln\!\left(\dfrac{1+\eta_2}{1-\eta_2}\right), & \varepsilon_{12} < \delta < \varepsilon_{21}. \\
\varepsilon_0 \delta \ln(\varepsilon_{22}/\delta), & \varepsilon_{21} \leqslant \delta < \varepsilon_{22}, \\
0, & \varepsilon_{22} \leqslant \delta,
\end{cases}
\tag{23b}
$$

where

$$
\varepsilon_0 = \frac{1}{\eta_1 \eta_2}
$$

and

$$
\varepsilon_{ij} = \left(\frac{m_i k_j}{m_0 k_0}\right)^{1/2}, \qquad i = 1, 2, \quad j = 1, 2.
\tag{24}
$$

We note that ε_{ij} depends solely on the normalized scatters η_1 and η_2. For $\bar{\eta} = 1$, that is $\eta_1 = \eta_2$, the values of ε_{12} and ε_{21} become identical and the central portion of the probability density function for $\bar{\nu}$ which lies between ε_{12} and ε_{21} disappears in (23).

Thus the probability density function of the normalized percentage of critical damping $\bar{\xi}$ depends on η_1, η_2, and η_3. For a given set of values of η_1, η_2 and η_3, the expression (22) can be obtained in closed form by integration over the various regions, in which the probability distributions are now analytically known. When c/c_0 is uniformly distributed, (22) simplifies to

$$p_{\xi/\xi_0}(x) = \frac{1}{2\eta_3} \int_{(1-\eta_3)/x}^{(1+\eta_3)/x} \delta p_{\bar{\nu}}(\delta) \big[H(\delta x - 1 + \eta_3) - H(\delta x - 1 - \eta_3) \big] \, d\delta$$

(22a)

where H is the Heaviside function. However, it appears more efficacious to perform this integration numerically so that arbitrary values of η_1, η_2, and η_3 can be easily handled.

Figure 10 shows the probability distribution of the normalized percentage of critical damping for two situations. In the first, the uncertainty in the mass parameter is low ($\eta_1 = 0.05$) while that in the stiffness parameter is relatively large ($\eta_2 = 0.2$). We note that for $\eta_3 > 0.25$, a commonly occurring situation in structural systems, the probability density of $\bar{\xi}$ is quite flat for $0.5\xi_0 \leqslant \xi \leqslant 1.5\xi_0$. As seen in the lower part of the figure, this flattening becomes even more important as the uncertainties in m and k become larger and comparable to each other. This result has rather serious implications in the modeling of dynamic systems. The graphs clearly indicate that uncertainties in the viscous damping parameter of reasonable (perhaps even what would be considered low) amounts ($\eta_3 \lesssim 0.20$), when coupled with large uncertainties in the mass and stiffness parameter values, could lead to large uncertainties in the normalized percentage of critical damping, $\bar{\xi}$.

(ii) *$k, m,$ and c are Truncated Gaussian Distributed.* Let the distributions of m and k be given by (10)–(12) and that of c be given by

$$P_{c/c_0}(x) = \begin{cases} \dfrac{C}{\sqrt{2\pi}\,\bar{\sigma}_c} \exp\left[-\dfrac{1}{2}\left(\dfrac{x-1}{\bar{\sigma}_c} \right)^2 \right], & 1 - \eta_3 < x < 1 + \eta_3, \\ 0 & \text{otherwise,} \end{cases}$$
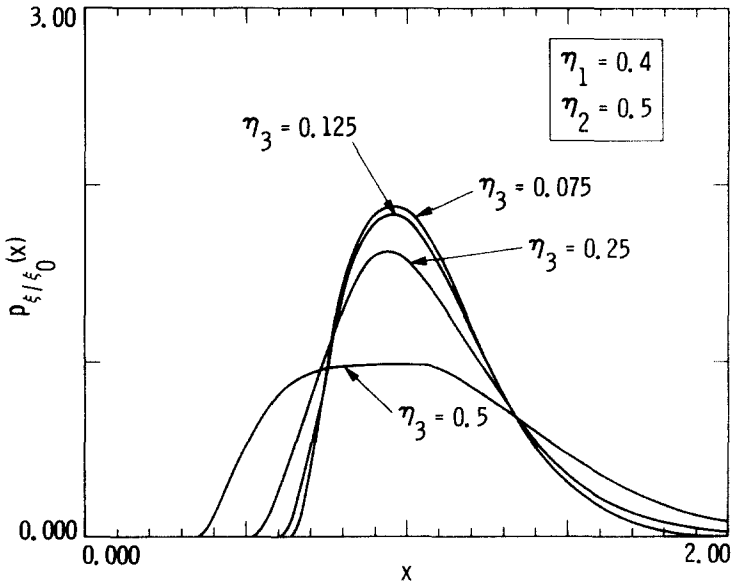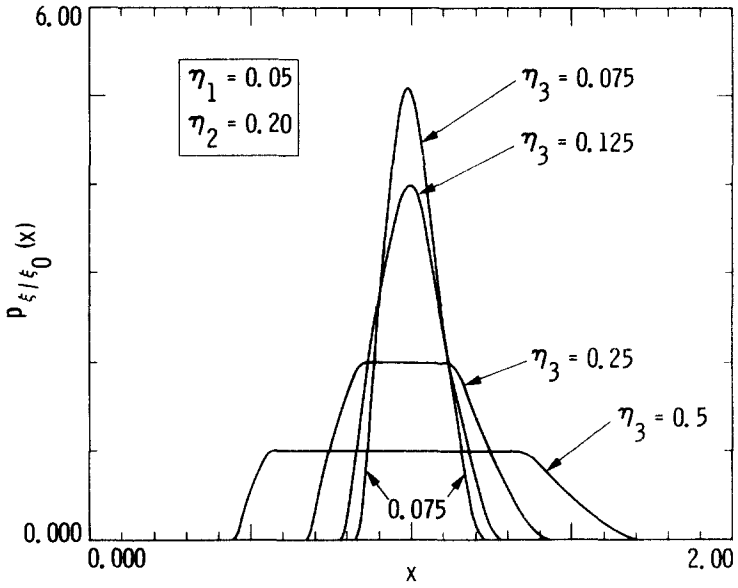
(25)

Fig. 10

where,

$$C = \left[1 - 2\,\text{erf}\left(-\frac{\eta_3}{\bar{\sigma}_c} \right) \right]^{-1},$$

(26)

$$\bar{\sigma}_c = \sigma_c / c_0,$$

σ_c being the standard deviation of c. Again use can be made of Equation (20) if $p_{\bar{\nu}}(\delta)$ is obtained. After considerable algebra, this density is found as follows:

(a) for $\eta \leqslant 1$,

$$p_{\bar{\nu}}(\delta) = \begin{cases} 0, & \delta \leqslant \varepsilon_{11}, \\[2mm] 2\delta \int_{1-\eta_1}^{\delta^2/(1-\eta_2)} \frac{1}{s} p_{k/k_0}\left(\frac{\delta^2}{s} \right) p_{m/m_0}(s)\, ds, & \varepsilon_{11} < \delta \leqslant \varepsilon_{21}, \\[3mm] 2\delta \int_{1-\eta_1}^{1+\eta_1} \frac{1}{s} p_{k/k_0}\left(\frac{\delta^2}{s} \right) p_{m/m_0}(s)\, ds, & \varepsilon_{21} < \delta < \varepsilon_{12}, \\[3mm] 2\delta \int_{\delta^2/(1+\eta_2)}^{1+\eta_1} \frac{1}{s} p_{k/k_0}\left(\frac{\delta^2}{s} \right) p_{m/m_0}(s)\, ds, & \varepsilon_{12} \leqslant \delta < \varepsilon_{22}, \\[3mm] 0, & \varepsilon_{22} \leqslant \delta, \end{cases}$$

(27a)

and

(b) for $\eta \geqslant 1$

$$p_{\bar{\nu}}(\delta) = \begin{cases} 0, & \delta < \varepsilon_{11}, \\[2mm] 2\delta \int_{1-\eta_1}^{\delta^2/(1-\eta_2)} \frac{1}{s} p_{k/k_0}\left(\frac{\delta^2}{s} \right) p_{m/m_0}(s)\, ds, & \varepsilon_{11} \leqslant \delta < \varepsilon_{12}, \\[3mm] 2\delta \int_{1-\eta_2}^{1+\eta_2} \frac{1}{s} p_{k/k_0}(s)\, p_{m/m_0}\left(\frac{\delta^2}{s} \right)\, ds, & \varepsilon_{12} \leqslant \delta < \varepsilon_{21}, \\[3mm] 2\delta \int_{\delta^2/(1+\eta_2)}^{1+\eta_1} \frac{1}{s} p_{k/k_0}\left(\frac{\delta^2}{s} \right) p_{m/m_0}(s)\, ds, & \varepsilon_{21} \leqslant \delta < \varepsilon_{22}, \\[3mm] 0, & \varepsilon_{22} \leqslant \delta, \end{cases}$$

(27b)

where $p_{k/k_0}(x)$ and $p_{m/m_0}(x)$ are defined in Equations (10) and (11).

We note that the expressions in Equation (27) are in fact applicable to arbitrary truncated p.d.f.'s for k and m as long as the variables are indepen-dent. The ε_{ij} are as defined in (24).

Once again the integral of Equation (22) yields the probability density of $\bar{\xi}$. For the truncated Gaussian case this is generally difficult to evaluate in closed form. We note that the normalized percentage of critical damping depends only on the six parameters $\{\eta_1, \eta_2, \eta_3, \eta_1^*, \eta_2^*, \eta_3^*\}$, where

$$\eta_1^* = \sqrt{3}\,\bar{\sigma}_m, \qquad \eta_2^* = \sqrt{3}\,\bar{\sigma}_k, \quad \text{and} \quad \eta_3^* = \sqrt{3}\,\bar{\sigma}_c.$$

Figures 11 and 12 show the density functions for two different parameter sets. In Figure 11 we note that the densities resemble those of Figure 10 except that the functions have smoothed-out corners. Changes in the distribu-tion with η_3^* are shown. Figure 12 shows the small influence on the distribution of $\bar{\xi}$ that the uncertainty in η_2^* causes when η_3 and η_3^* are relatively large.

C. Probability Density of ω_d

Consider the general situation in which the probability density of ω_d is to be determined, given that the random variables m/m_0, k/k_0, and c/c_0 are independent of each other. The truncated distributions of m/m_0, k/k_0, and c/c_0 may be taken to be arbitrary, but of known form. If the random system is assumed to be oscillatory (i.e. $c_1 \leqslant 2\sqrt{k_2 m_2}$), then using the transformation of variables

$$y_1 = m,$$

$$y_2 = k/m,$$

$$y_3 = \omega_d = \left[\frac{k}{m} - \left(\frac{c}{2m} \right)^2 \right]^{1/2},$$

(28)

the p.d.f. of the normalized damped frequency defined as

$$\bar{\omega}_d = \frac{\omega_d}{\omega_{d0}}, \qquad \text{where} \quad \omega_0 = \omega_0\sqrt{1 - \xi_0^2},$$
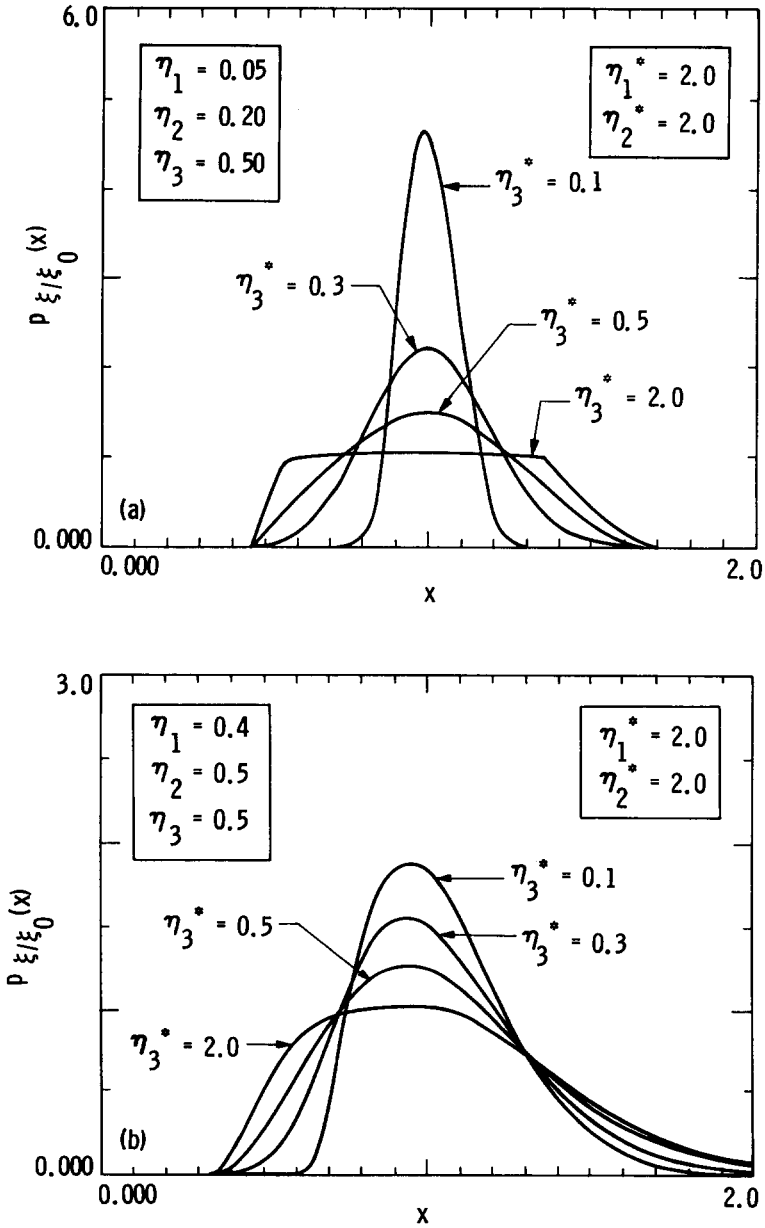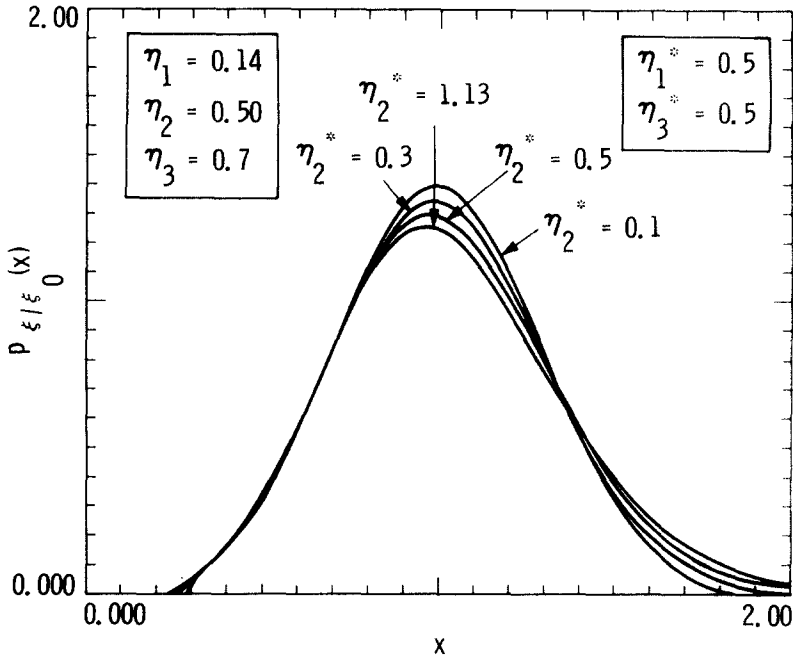
FIG. 11

$$\text{Fig. 12}$$

can be expressed after considerable algebra as

$$p_{\bar{\omega}_d}(x) = \int_{b_1}^{b_2} \int_{a_1}^{a_2} \frac{y_1^2 x \left(1 - \xi_0^2\right)/\xi_0}{\sqrt{y_2 - \left(1 - \xi_0^2\right)x^2}}$$

$$\times p_{m/m_0}(y_1) p_{k/k_0}(y_1 y_2) p_{c/c_0}\left(\frac{y_1}{\xi_0} \sqrt{y_2 - \left(1 - \xi_0^2\right)x^2} \right) dy_1 \, dy_2. \quad (29)$$

We note that the p.d.f. of $\bar{\omega}_d$ is not merely a function of η_i, η_i^*, $i = 1, 2, 3$, the parameters that define the scatters of the distributions of m, k, and c, but is also a function of ξ_0, the percentage of critical damping related to the oscillator with the mean parameter values.

The ranges of integration in (29) for y_1 and y_2 extend over all regions for which the indicated p.d.f.'s are nonzero. Thus for the truncated distributions of Figure 1, the limits of the integration can be replaced by a_1 to a_2 for the parameter y_1 and by b_1 to b_2 for the parameter y_2. The values of these limits
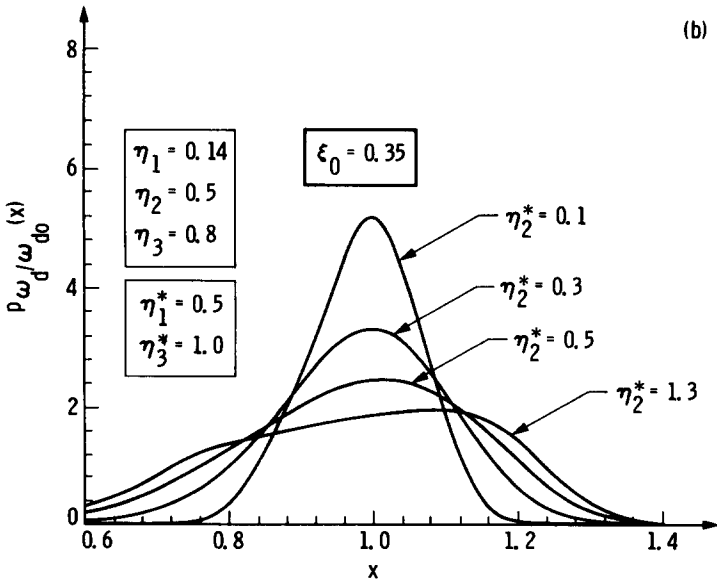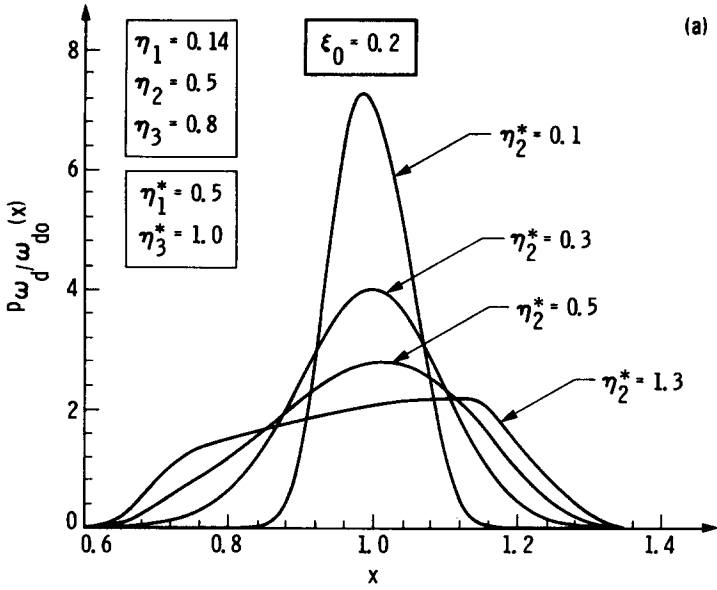
FIG. 13

then become

$$a_1 = 1 - \eta_1, \qquad a_2 = 1 + \eta_1,$$

$$b_1 = \max\left[x^2(1 - \xi_0^2), \frac{1 - \eta_2}{1 + \eta_1}\right];$$

$$b_2 = \min\left\{\frac{1 + \eta_2}{1 - \eta_1}, x^2(1 - \xi_0^2) + \left[\frac{\xi_0(1 + \eta_3)}{(1 - \eta_1)}\right]^2\right\}. \tag{30}$$

Numerical integration can be performed to yield $p_{\bar{\omega}_d}(x)$ from (29) and (30), using either uniform distributions of m, k, and c or Gaussian distributions or combinations thereof.

Figure 13 indicates the p.d.f.'s obtained for the truncated Gaussian distributions used before in Figure 3 with the parameters η_3 and η_3^* set respectively to 0.8 and 1.0. We note that the general shape of the distributions for the damped and the undamped normalized frequency are similar. For values of ξ_0 that are less than 5%, the distributions appear to be almost the same as for the undamped case. Figure 13(b) shows the distributions for $\xi_0 = 0.35$. Here a noticeable difference between the normalized damped and undamped frequency distributions is observed, especially at lower values of the normalized frequency. A similar situation is seen when Figure 14(a) and (b) are compared with Figure 5.

The fact that for $\xi_0 < 5\%$ the distributions of the normalized damped and undamped frequencies are identical (even for large values of η_3 and/or η_3^*) can be used to good advantage, for the closed-form solutions of Equations (7) and (13)–(16) are faster to compute than the double integrals in Equation (29).

IV. CONCLUSIONS

In this paper we have used an information-theoretic framework to pose the problem of a single-degree-of-freedom random system subjected to a time history of dynamic loading. Using, for the uncertain parameters, the least presumptive distributions which are consistent with the data generally available in engineering practice, closed-form expressions for the probability density functions of the normalized undamped natural frequency (ω/ω_0), the normalized damped frequency (ω_d/ω_{d0}), and the normalized percentage of critical damping (ξ/ξ_0) have been obtained. It is shown that these distributions depend solely on the dimensionless parameters that quantify our lack of
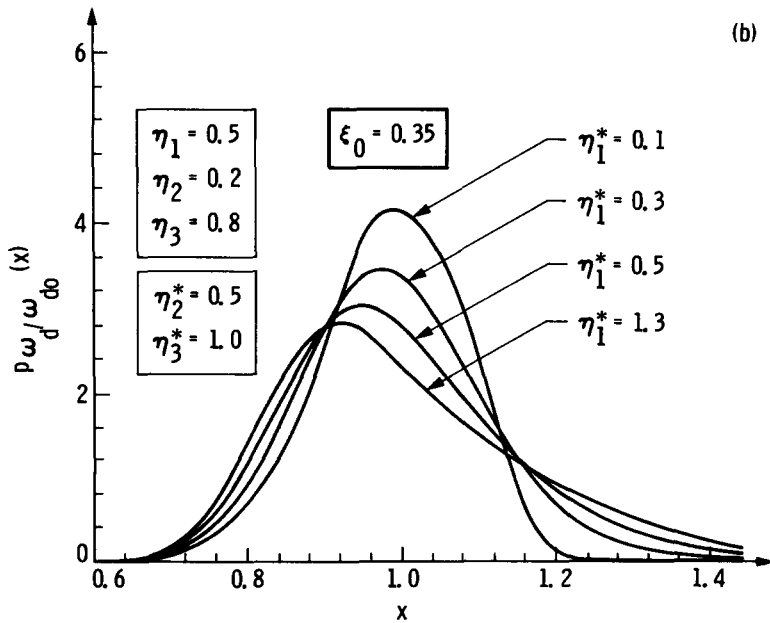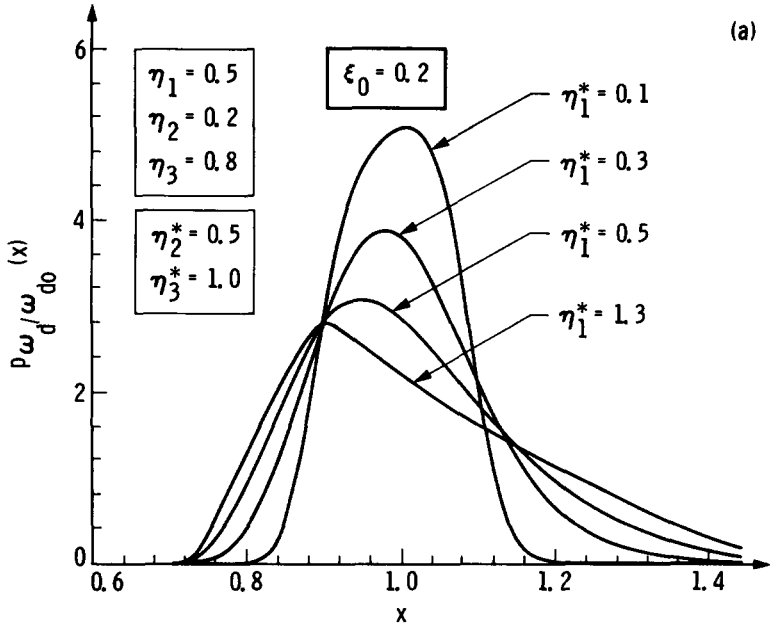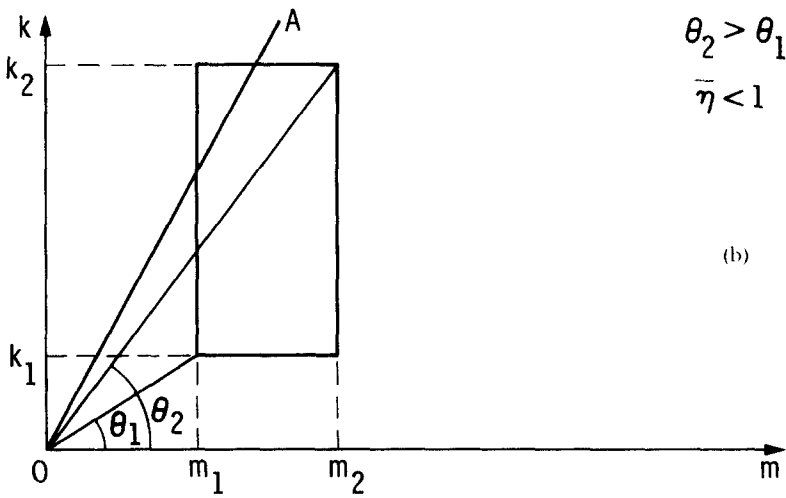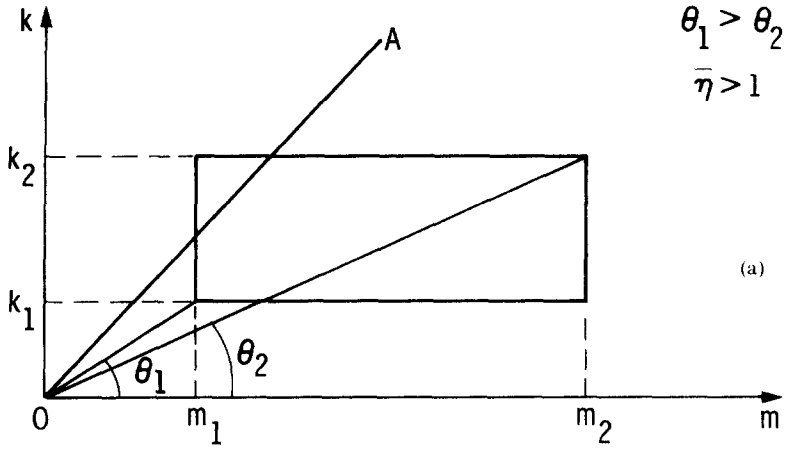
Fig. 14

FIG. 15

knowledge of the system; the distribution of ω_d/ω_{d0} depends, in addition, on the value of ξ_0.

The density functions for the normalized natural frequencies differ markedly in their qualitative nature, depending on the extent of uncertainty in the mass parameter relative to that (e.g. Figure 2) in the stiffness parameter. Also, for large uncertainties in the mass and/or stiffness parameters (η_1 and/or η_2 for the uniform distribution), the variance of the distribution of the percentage of critical damping may become large. This is a matter of some consequence in assessing the dynamic-response statistics of such a random oscillatory system.

The probability distributions obtained herein will, it is hoped, provide more insight into the system characteristics and will form the basis of obtaining the statistics of the response of the random system to dynamic loads. This will be taken up in the next paper.

APPENDIX

We sketch the derivation of the relations (7a) and (7b) here. Consider the random variables k and m to be in the intervals (k_1, k_2) and (m_1, m_2), respectively. We assume k_1 and m_1 to be greater than zero. The p.d.f. of $\sqrt{k/m}$ can now be obtained as follows: First we determine the probability that $k/m < x$. Let

$$P[k/m < x] = f(x).$$

Then, the p.d.f. is given by

$$p_{\sqrt{k/m}}(x) = 2xf'(x^2),$$

where the prime denotes differentiation with respect to the argument.

Figure 15 shows the two possible configurations in (k, m) space. In Figure 15(a), $k_2/m_2 < k_1/m_1$ (i.e. $\bar{\eta} > 1$), and in Figure 15(b), $k_2/m_2 > k_1/m_1$. The determination of $f(x)$ then reduces to determining the probability of the slope of line OA being less than x. Explicit expressions for this can now be easily found by integrating over the appropriate probability volumes. We leave the algebra to the reader.

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