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ON GENERAL NONLINEAR CONSTRAINED MECHANICAL SYSTEMS

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Abstract. This paper develops a new, simple, general, and explicit form of the equations of motion for general constrained mechanical systems that can have holonomic and/or nonholonomic constraints that may or may not be ideal, and that may contain either positive semi-definite or positive definite mass matrices. This is done through the replacement of the actual unconstrained mechanical system, which may have a positive semi-definite mass matrix, with an unconstrained auxiliary system whose mass matrix is positive definite and which is subjected to the same holonomic and/or nonholonomic constraints as those applied to the actual unconstrained mechanical system. A simple, unified fundamental equation that gives in closed-form both the acceleration of the constrained mechanical system and the constraint force is obtained. The results herein provide deeper insights into the behavior of constrained motion and open up new approaches to modeling complex, constrained mechanical systems, such as those encountered in multi-body dynamics.

1. Introduction. The understanding of constrained motion is an important area of analytical dynamics that has been worked on by numerous researchers. References [1]-[10] give a brief sampling of some of the researchers who have made substantial contributions; nevertheless, several questions remain unanswered at the present time. A significant problem in deriving the equation of motion for constrained mechanical systems arises when the mass matrix of the unconstrained mechanical system is singular. Since the mass matrix then does not have an inverse, standard methods for obtaining the constrained equations of motion, which usually rely on the invertability of the mass matrix, cannot be used. For example, the so-called fundamental equation developed by Udwadia and Kalaba [18] cannot be directly applied. Observing this, Udwadia and Phohomsiri [17] derived an explicit equation of motion for such systems with singular mass matrices. However, the structure of their explicit equation differs significantly from their so-called fundamental equation [18]. Recently, by using the concept of an unconstrained auxiliary system, Udwadia and Schutte [16] developed a simpler explicit equation of motion that has the same

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form as the so-called fundamental equation, and is valid for systems whose mass matrices may or may not be singular. In this paper we present a new alternative equation of motion for systems with positive definite and/or positive semi-definite mass matrices that is in many respects superior to that proposed in Ref. [16].

We consider in this paper an unconstrained auxiliary system that has a positive definite mass matrix instead of the actual unconstrained mechanical system whose mass matrix may be positive semi-definite. When subjected to the same 'given' force and the same constraints as in the actual unconstrained mechanical system, the unconstrained auxiliary system provides in closed form, at each instant of time, the acceleration of the actual constrained mechanical system. Furthermore, by augmenting the 'given' force that is acting on the actual unconstrained mechanical system we obtain from the auxiliary system the proper constraint force acting on the actual unconstrained mechanical system. In short, the auxiliary system obtained herein gives the equation of constrained motion of the actual mechanical system in closed form, whether or not the mass matrix is singular. The results obtained herein are more general, simpler, and computationally much more efficient than those in Ref. [16]. Also, the proofs are simpler, and more importantly, the results lead to deeper insights into the nature of constrained motion of mechanical systems.

We briefly point out the importance of being able to formulate correctly the constrained equations of motion for mechanical systems whose mass matrices are positive semi-definite. When a minimum number of coordinates is employed to describe the (unconstrained) motion of mechanical systems, the corresponding set of Lagrange equations usually yields mass matrices that are non-singular [12]. One might thus consider that systems with singular mass matrices are not common in classical dynamics. However, in modeling complex multi-body mechanical systems, it is often helpful to describe such systems with *more* than the minimum number of required generalized coordinates. And in such situations, the coordinates are then not independent of one another, often yielding systems with positive semi-definite mass matrices. Thus, in general, singular mass matrices can and do arise when one wants more flexibility in modeling complex mechanical systems. The reason that more than the minimum number of generalized coordinates are usually not used in the modeling of complex multi-body systems, though this could often make the modeler's task much simpler, is that they result in singular mass matrices, and to date systems with such matrices have been difficult to handle within the Lagrangian framework. We give an important example dealing with the rotational dynamics of a rigid body in this paper showing how singular mass matrices can appear in the modeling of mechanical systems.

This is the reason it is useful to obtain in closed form the general, explicit equations of motion for constrained mechanical systems whose mass matrices may or may not be singular. Since such systems normally arise when modeling *large-scale*, *complex* mechanical systems in which the modeler seeks to substantially facilitate his/her work by using more than the minimum number of coordinates to describe the system, it is also important to keep an eye on the computational efficiency of the equations so obtained.

2. System Description of General Constrained Mechanical Systems. It is useful to conceptualize the description of a constrained mechanical system, S, in a three-step procedure. We do this in the following way:

First, we describe the so-called unconstrained mechanical system in which the coordinates are all independent of each other. We do that by considering an unconstrained mechanical system whose motion at any time t can be described, using Lagrange's equation, by

$$M(q,t)\ddot{q} = Q(q,\dot{q},t), \tag{2.1}$$

with the initial conditions

$$q(t=0) = q_0, \dot{q}(t=0) = \dot{q}_0, \tag{2.2}$$

where q is the generalized coordinate n-vector; M is an n by n matrix that can be either positive semi-definite ($M \geq 0$) or positive definite ($M > 0$) at each instant of time; and Q is an n-vector, called the 'given' force, which is a known function of q, $\dot{q}$, and t. We shall often refer to the system described by equation (2.1) as the *unconstrained mechanical system* S.

Second, we impose a set of constraints on this unconstrained description of the system. We suppose that the unconstrained mechanical system is now subjected to the m constraints given by

$$\varphi_i(q,\dot{q},t) = 0, \quad i = 0, 1, 2...., m, \tag{2.3}$$

where $r \leq m$ equations in the equation set (2.3) are functionally independent. The set of constraints described by (2.3) includes all the usual varieties of holonomic and/or nonholonomic constraints, and then some. We shall assume that the initial conditions (2.2) satisfy these m constraints. Therefore, the components of the n-vectors q_0 and $\dot{q}_0$ cannot all be independently assigned. We further assume that the set of constraints (2.3) is smooth enough so that we can differentiate them with respect to time t to obtain the relation

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t), \tag{2.4}$$

where A is an m by n matrix whose rank is r, and b is an m-vector. We note that each row of A arises by appropriately differentiating one of the m constraint equations.

Using the information in the previous two steps, in the last step we bring together the description of motion of the constrained mechanical system as

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q^c(q,\dot{q},t), \tag{2.5}$$

where Q^c is the constraint force n-vector that arises to ensure that the constraints (2.4) are satisfied at each instant in time. Thus, equation (2.5) describes the motion of the actual constrained mechanical system, S. In what follows we shall suppress the arguments of the various quantities unless required for clarity.

Equation (2.3) provides the kinematical conditions related to the constraints. We now look at the dynamical conditions. The work done by the forces of constraints under virtual displacements at any instant of time t can be expressed as [21]

$$v^T(t)Q^c(q,\dot{q},t) = v^T(t)C(q,\dot{q},t), \tag{2.6}$$

where $C(q,\dot{q},t)$ is an n-vector describing the nature of the non-ideal constraints which is determined by physical observation and/or experimentation, and the virtual displacement vector, *v(t)*, is any non-zero n-vector that satisfies [20]

$$A(q,\dot{q},t)v = 0. \tag{2.7}$$

When the mass matrix M in equation (2.1) is positive definite, the explicit equation of motion of the constrained mechanical system S is given by the so-called fundamental equation [22]

$$\ddot{q} = a + M^{-1/2}B^{+}(b - Aa) + M^{-1/2}(I - B^{+}B)M^{-1/2}C, \tag{2.8}$$

where $a = M^{-1}Q$, $B = AM^{-1/2}$, and the superscript "+" denotes the Moore-Penrose (MP) inverse of a matrix [9], [13], [19]. We note that equation (2.8) is valid (i) whether or not the equality constraints (2.3) are holonomic and/or nonholonomic, (ii) whether or not they are nonlinear functions of their arguments, (iii) whether or not they are functionally dependent, (iv) and whether or not the constraint force is non-ideal. We note that the constrained mechanical system S is completely described through the knowledge of the matrices M and A, and the column vectors Q, b, and C. The latter four are functions of q, $\dot{q}$, and t, while the elements of the matrix M are, in general, functions of q and t. In what follows, we shall also denote the acceleration of the constrained system given in equation (2.8), $\ddot{q}_s(=\ddot{q})$.

However, when the unconstrained mechanical system given by (2.1) is such that the matrix M is singular, the above equation cannot always be applied since the matrix $M^{-1/2}$ may not exist. In that case, equation (2.8) needs to be replaced by equation [17]

$$\ddot{q} = \begin{bmatrix} (I - A^{+}A)M \\ A \end{bmatrix}^{+} \begin{bmatrix} (Q + C) \\ b \end{bmatrix} := \hat{M}^{+} \begin{bmatrix} (Q + C) \\ b \end{bmatrix}, \tag{2.9}$$

under the proviso that the rank of the matrix $\hat{M}^T = [M \mid A^T]$ is n. This rank condition is a necessary and sufficient condition for the constrained mechanical system to have a unique acceleration – a consequence of physical observation of the motion of classical mechanical systems.

However, the form of equation (2.9) when M is positive definite is noticeably different from the form of the so-called fundamental equation (2.8). A unified equation of motion that is applicable to both these situations is presented in Udwadia and Schutte [16]. They considered an auxiliary system that has a positive definite mass matrix, which is subjected to the same constraint conditions as the actual mechanical system that has a singular mass matrix. This positive definite mass matrix of the auxiliary system is expressed as $M + \alpha^2 A^{+}A$, where α is any non-zero real number and again the superscript "+" denotes the Moore-Penrose (MP) inverse of the matrix [16]. However, the use of the Moore-Penrose (MP) inverse of the matrix in $M + \alpha^2 A^{+}A$ makes it difficult to handle analytically and expensive to compute, especially when the row and column dimensions of A are large.

In this paper we uncover a new general equation of motion for constrained mechanical systems by instead using the augmented mass matrix, $M_{A^TG} = M + \alpha^2 A^T G A$, which is simpler, more general and lends itself equally to use by the so-called fundamental equation (2.8). The function $\alpha(t)$ is an arbitrary, nowhere-zero, sufficiently smooth (C^2) real function of time, and $G(q,t) = N^T(q,t)N(q,t)$ is any arbitrary m by m positive definite matrix whose elements are sufficiently smooth functions (C^2) of the arguments. Thus greater generality, simpler results, and greater computational efficiency are herein achieved. Furthermore, the proofs of the various results are much simpler than in Ref. [16].

3. Explicit Equations of Motion for General Constrained Mechanical Systems. From physical observation, the acceleration of a system in classical dynamics under a given set of forces and under a given set of initial conditions is known to be uniquely determinable. As shown in Ref. [17] a necessary and sufficient condition for this to occur is that the rank of the matrix $\hat{M}^T = [M \mid A^T]$ is n. We shall therefore assume throughout this paper that for the constrained systems we consider herein, the matrices M and A are such that this condition is always satisfied. Thus we assume that the actual constrained mechanical system under consideration is appropriately mathematically modeled and the resulting acceleration of the system can be uniquely found.

3.1. Positive Definiteness of the Augmented Mass Matrices.

Lemma 3.1. *Let $M \geq 0$, let $\alpha(t)$ be an arbitrary, nowhere-zero, sufficiently smooth (C^2) real function of time, and let $G(q,t) = N^T(q,t)N(q,t)$ be any m by m positive definite matrix [4] whose elements are sufficiently smooth functions (C^2) of the arguments. The n by n augmented mass matrix $M_{A^TG} := M + \alpha^2 A^T GA$ is positive definite at each instant of time if and only if the n by $n+m$ matrix $\hat{M}^T = [M \mid A^T]$ has rank n at each instant of time.*

Proof. (a) Consider any fixed instant of time. Assume that $\hat{M}$ has rank n; we shall prove that the augmented mass matrix $M_{A^TG} := M + \alpha^2 A^T GA$ is positive definite at that instant. We first observe that the matrix M_{A^TG} is symmetric since M is symmetric as is $A^T GA$.

Since the column space of the matrix A is identical to the column space of αA,

$$n = rank(\hat{M}) = rank(\begin{bmatrix} M \\ A \end{bmatrix}) = rank(\begin{bmatrix} M \\ \alpha A \end{bmatrix}) = rank([M \mid \alpha A^T]). \tag{3.1}$$

We shall denote by $Col(X)$ the column space of the matrix X. Since $Col(M) = Col(M^{1/2})$, and $Col(A^T) = Col(A^T N^T)$ because N is nonsingular, we get

$$n = rank([M \mid \alpha A^T]) = rank([M^{1/2} \mid \alpha A^T N^T]) := rank(\tilde{M}), \tag{3.2}$$

where we have denoted $\tilde{M} := [M^{1/2} \mid \alpha A^T N^T]$.

Next, we consider the augmented mass matrix M_{A^TG}. It can be expressed as

$$\begin{aligned} M_{A^TG} =& M + \alpha^2 A^T GA = M + \alpha^2 A^T N^T NA \\ =& [M^{1/2} \mid \alpha A^T N^T] \begin{bmatrix} M^{1/2} \\ \alpha NA \end{bmatrix} \\ =& \tilde{M}\tilde{M}^T \geq 0. \end{aligned} \tag{3.3}$$

Thus, the n by n matrix M_{A^TG} must at least be positive semi-definite. But from (3.2), $rank(\tilde{M}) = n$, hence M_{A^TG} is positive definite.

(b) Consider any fixed instant of time. Assume that $M_{A^TG} := M + \alpha^2 A^T GA$ is positive definite; we shall prove that $\hat{M}$ has full rank n at that instant.

From (3.3) and the assumption that $M_{A^TG} > 0$, we have $M_{A^TG} = \tilde{M}\tilde{M}^T > 0$, so that $\tilde{M}\tilde{M}^T$ has rank n, and hence $rank(\tilde{M}) = n$.

Since elementary row operations do not change the rank of a matrix, we find that

$$\begin{aligned} n = rank(\tilde{M}^T) = rank(\begin{bmatrix} M^{1/2} \\ \alpha NA \end{bmatrix}) \\ = rank(\begin{bmatrix} M^{1/2} \\ NA \end{bmatrix}) = rank([M^{1/2} \mid A^T N^T]). \end{aligned} \tag{3.4}$$

And also since $Col(M) = Col(M^{1/2})$ and $Col(A^T) = Col(A^T N^T)$, we have

$$\begin{aligned} rank([M^{1/2} \mid A^T N^T]) &= rank([M \mid A^T]) \\ &= rank([M \mid A^T]^T) \\ &= rank(\hat{M}). \end{aligned} \tag{3.5}$$

Hence, $rank(\hat{M}) = n$, and the proof is therefore complete. □

3.2. Explicit Equation for Constrained Acceleration. Having the auxiliary mass matrix M_{A^TG}, which has been proved to be always a positive definite matrix, we are now ready to begin implementing the explicit equations of motion for the constrained acceleration of the system S that may have a positive semi-definite mass matrix, $M \geq 0$. We begin by proving a useful result that will be used many times from here on.

Lemma 3.2. *Let A^+ denotes the Moore-Penrose (MP) inverse of the m by n matrix A, then*

$$(I - A^+ A)A^T = 0. \tag{3.6}$$

Proof.

$$\begin{aligned} (I - A^+A)A^T = A^T - A^+AA^T &= A^T - (A^+A)^T A^T \\ &= A^T - A^T(A^T)^+ A^T = 0. \end{aligned} \tag{3.7}$$

In the second equality above, we have used the fourth Moore-Penrose (MP) condition (see [3], [13], [19]) and in the last equality, we have used the first MP condition. This yields the stated result. □

Recall that our actual mechanical system has a mass matrix M that may be positive-semi definite, and since $M^{-1/2}$ does not exist we encounter difficulty in finding the acceleration of the constrained mechanical system when using the fundamental equation (see equation (2.8)). However, we note from Lemma 3.1 that the matrix M_{A^TG} is always positive definite when $\hat{M}$ has rank n. Moreover, this rank condition is a check that our mathematical model appropriately describes a given physical system, since in all physical systems in classical mechanics the acceleration must be uniquely determinable. Were we then to use this matrix M_{A^TG} (instead of M) as the mass matrix of an 'appropriate' unconstrained auxiliary system, subjected to the same constraints as the actual unconstrained mechanical system, we would encounter no difficulty in using the fundamental equation (2.8) to obtain the acceleration of this constrained auxiliary system, since the mass matrix of this auxiliary system is positive definite! Our aim then is to define this unconstrained auxiliary system in the 'appropriate' manner so that the resultant constrained acceleration it yields upon application of the fundamental equation *always* coincides with the acceleration of our actual constrained mechanical system. We now proceed to show that this indeed can be done, and we demonstrate how to accomplish this.

Consider any unconstrained mechanical system S,

(i) whose equation of motion is described by equation (2.1) where the n by n mass matrix M may be positive semi-definite or positive definite, and whose initial conditions are given in relations (2.2),

(ii) which is subjected to the m constraints given by equation (2.4) (or equivalently by equation (2.3)) that are satisfied by the initial conditions q_0 and $\dot{q}_0$ as described by equation (2.2), and

(iii) which is subjected to the non-ideal constraint that is prescribed by the n-vector $C(q, \dot{q}, t)$ as in equation (2.6).

Recall that we shall always subsume that the actual mechanical system S has the property that $\hat{M}^T = \left[M \mid A^T\right]$ has rank n at each instant of time.

Consider further an unconstrained auxiliary system S_{A^TG} that has

(1) an augmented mass matrix given by

$$M_{A^TG} = M(q, t) + \alpha^2(t)A^T(q, \dot{q}, t)G(q, t)A(q, \dot{q}, t) > 0, \tag{3.8}$$

where $\alpha(t)$ is any sufficiently smooth function (C^2 would be sufficient) of time that is nowhere zero, and $G(q, t) = N^T(q, t)N(q, t)$ is any m by m positive definite matrix with its elements sufficiently smooth functions (C^2 would be sufficient) of its arguments, and

(2) an augmented 'given' force defined by

$$Q_{A^TG,z}(q, \dot{q}, t) = Q(q, \dot{q}, t) + A^T(q, \dot{q}, t)G(q, t)z(q, \dot{q}, t) \tag{3.9}$$

where $z(q, \dot{q}, t)$ is any arbitrary, sufficiently smooth m-vector,

(3) so that the equation of motion of this unconstrained auxiliary system is given by

$$\begin{aligned} M_{A^TG}(q, t)\ddot{q} &= Q(q, \dot{q}, t) + A^T(q, \dot{q}, t)G(q, t)z(q, \dot{q}, t) \\ &:= Q_{A^TG,z}(q, \dot{q}, t). \end{aligned} \tag{3.10}$$

Similar to the conceptualization stated in section 2, the system described by equation (3.10) is referred to as the *unconstrained auxiliary system* S_{A^TG}.

(4) We shall subject this unconstrained auxiliary system S_{A^TG} to (a) the same initial conditions, and (b) the same constraints, which the unconstrained mechanical system S is subjected to, as described in items (ii) and (iii) above.

We note that the unconstrained auxiliary system S_{A^TG} differs from the unconstrained mechanical system S in that at each instant of time (a) it has an augmented mass matrix M_{A^TG} and (b) it is subjected to an augmented 'given' force $Q_{A^TG,z}$. Furthermore, in item (2) above, z is arbitrary and therefore it can be chosen to be identically zero.

The two unconstrained systems S and S_{A^TG} when subjected to the same set of constraints (both ideal and/or non-ideal at each instant of time) and the same set of initial conditions yield, correspondingly, what we shall call the constrained mechanical system and the constrained auxiliary system.

Result 1. The acceleration of the constrained mechanical system S obtained by considering the unconstrained mechanical system and its constraints as described by (i)-(iii), is identical with, and directly obtained from, the explicit acceleration of the constrained auxiliary system S_{A^TG} obtained by considering the unconstrained auxiliary system and its constraints as described by (1)-(4).

Proof. As shown in Ref. [17], the acceleration, $\ddot{q}_s$, of the constrained mechanical system S is described by the equation (see equation (2.9))

$$\ddot{q}_s = \begin{bmatrix} (I - A^+A)M \\ A \end{bmatrix}^+ \begin{bmatrix} (Q+C) \\ b \end{bmatrix} := \bar{M}^+ \begin{bmatrix} (Q+C) \\ b \end{bmatrix}, \tag{3.11}$$

while the acceleration, $\ddot{q}_{s_{A^TG}}$, of the constrained auxiliary system S_{A^TG} is given by (2.9)

$$\begin{aligned} \ddot{q}_{s_{A^TG}} &= \begin{bmatrix} (I - A^+A)M_{A^TG} \\ A \end{bmatrix}^+ \begin{bmatrix} Q_{A^TG,z} + C \\ b \end{bmatrix} \\ &:= \bar{M}_A^+ \begin{bmatrix} Q + A^TGz + C \\ b \end{bmatrix}. \end{aligned} \tag{3.12}$$

Let us consider first the term $(I - A^+A)M$ of (3.11). Post-multiplication of both sides of (3.6) by $\alpha^2 GA$, yields

$$\alpha^2(I - A^+A)A^TGA = 0, \tag{3.13}$$

so that

$$\begin{aligned} (I - A^+A)M =& (I - A^+A)M + \alpha^2(I - A^+A)A^TGA \\ =& (I - A^+A)(M + \alpha^2A^TGA) \\ =& (I - A^+A)M_{A^TG}. \end{aligned} \tag{3.14}$$

Using (3.14) in equation (3.11) thus yields

$$\ddot{q}_s = \begin{bmatrix} (I - A^+A)M_{A^TG} \\ A \end{bmatrix}^+ \begin{bmatrix} Q+C \\ b \end{bmatrix} := \bar{M}_A^+ \begin{bmatrix} Q+C \\ b \end{bmatrix}. \tag{3.15}$$

We note that the acceleration of the constrained system is still the same even though the mass matrix $M \geq 0$ is replaced with the augmented mass matrix $M_{A^TG} > 0$ (see equations (3.11) and (3.15)).

Pre-multiplying and post-multiplying both sides of (3.6) by M_{A^TG} and Gz respectively, we have

$$M_{A^TG}(I - A^+A)A^TGz = 0. \tag{3.16}$$

Noting that for any matrix X, $X^+ = (X^TX)^+X^T$ [13], from equation (3.15), we have

$$
\begin{aligned}
\ddot{q}_s =& \bar{M}_A^+ \begin{bmatrix} Q + C \\ b \end{bmatrix} \\
&= \left[\bar{M}_A^T \bar{M}_A\right]^+ \left[M_{A^TG}(I - A^+A) \mid A^T\right] \begin{bmatrix} Q + C \\ b \end{bmatrix} \\
&= \left[\bar{M}_A^T \bar{M}_A\right]^+ \left[M_{A^TG}(I - A^+A)(Q + C) + A^T b\right] \\
&= \left[\bar{M}_A^T \bar{M}_A\right]^+ [M_{A^TG}(I - A^+A)(Q + C) \\
&\qquad\qquad + M_{A^TG}(I - A^+A)A^T G z + A^T b] \\
&= \left[\bar{M}_A^T \bar{M}_A\right]^+ [M_{A^TG}(I - A^+A)(Q + A^T G z + C) + A^T b] \\
&= \left[\bar{M}_A^T \bar{M}_A\right]^+ [M_{A^TG}(I - A^+A) \mid A^T] \begin{bmatrix} Q + A^T G z + C \\ b \end{bmatrix} \\
=& \bar{M}_A^+ \begin{bmatrix} Q + A^T G z + C \\ b \end{bmatrix} = \ddot{q}_{s_{A^TG}}.
\end{aligned} \tag{3.17}
$$

The fourth equality above follows from equation (3.16) and the last from equation (3.12) This proves the claim. □

Since we know that at each instant of time the acceleration of the constrained mechanical system S is the same as that of the constrained auxiliary system S_{A^TG} (see equation (3.17)), and also that the augmented mass matrix of the system S_{A^TG} is positive definite, we can directly apply the so-called fundamental equation (2.8) to the unconstrained auxiliary system described by (3.10) to get $\ddot{q}_{s_{A^TG}}$ and therefore $\ddot{q}_s$ explicitly as [22]

$$
\begin{aligned}
\ddot{q}_s = a_{A^TG,z} &+ M_{A^TG}^{-1/2} B_{A^TG}^+ (b - A a_{A^TG,z}) \\
&+ M_{A^TG}^{-1/2} (I - B_{A^TG}^+ B_{A^TG}) M_{A^TG}^{-1/2} C,
\end{aligned} \tag{3.18}
$$

where

$$M_{A^TG} = M + \alpha^2 A^T G A > 0, \tag{3.19}$$

$$Q_{A^TG,z} = Q + A^T G z, \tag{3.20}$$

$$a_{A^TG,z} = M_{A^TG}^{-1} Q_{A^TG,z} = M_{A^TG}^{-1} Q + M_{A^TG}^{-1} A^T G z, \tag{3.21}$$

and

$$B_{A^TG} = A M_{A^TG}^{-1/2}. \tag{3.22}$$

Remark 1. We know that when $\hat{M}$ has rank n, the acceleration, $\ddot{q}_s$, of the constrained mechanical system S is unique and is explicitly given by equation (3.11). And since we have shown that $\ddot{q}_s = \ddot{q}_{s_{A^TG}}$ at each instant of time, the acceleration of the constrained auxiliary system must be independent of the arbitrary (nowhere-zero) scalar function $\alpha(t)$, the arbitrary m-vector $z(t)$, and the arbitrary (positive definite) matrix $G(q,t)$, provided each of these three entities is a sufficiently smooth (C^2) function of their arguments.

Remark 2. Since α, z and G are arbitrary as just stated, we can further particularize equation (3.18) by setting $\alpha \equiv 1$, $z \equiv 0$, $G \equiv I_m$ in describing our unconstrained auxiliary system. Thus this unconstrained auxiliary system now has a simple augmented mass matrix $M + A^T A$, and it is subjected to the same 'given' force as the unconstrained mechanical system S. This unconstrained auxiliary system, when subjected to the same constraints (kinematical and dynamical) as those placed on S, yields the acceleration of the constrained mechanical system S, given by

$$\begin{aligned} \ddot{q}_s = a_{A^T} &+ M_{A^T}^{-1/2} B_{A^T}^{+} (b - A a_{A^T}) \\ &+ M_{A^T}^{-1/2} (I - B_{A^T}^{+} B_{A^T}) M_{A^T}^{-1/2} C, \end{aligned} \tag{3.23}$$

where

$$\begin{aligned} M_{A^T} &= M + A^T A > 0, \\ a_{A^T} &= M_{A^T}^{-1} Q, \end{aligned}$$

and

$$B_{A^T} = A M_{A^T}^{-1/2}.$$

3.3. Explicit Equation for Constraint Force. So far, we have developed an unconstrained auxiliary system $S_{A^T G}$ which always has a positive definite mass matrix, and we have used it in the so-called fundamental equation (2.8) to directly yield the acceleration of the constrained mechanical system S. We now further explore whether the constraint force Q^c acting on the unconstrained mechanical system S (that is brought into play by the presence of the constraints (ii) and (iii) described earlier in Section 3.2) can be directly adduced from the equation of motion of the constrained auxiliary system $S_{A^T G}$. To show this, we begin by putting forward a useful result.

Lemma 3.3.

$$M_{A^T G}^{1/2} B_{A^T G}^{+} A M_{A^T G}^{-1} A^T = A^T, \tag{3.24}$$

where $M_{A^T G}$ is defined in equation (3.8) and $B_{A^T G}$ is defined in equation (3.22).

Proof.

$$\begin{aligned} M_{A^T G}^{1/2} B_{A^T G}^{+} A M_{A^T G}^{-1} A^T &= M_{A^T G}^{1/2} B_{A^T G}^{+} A M_{A^T G}^{-1/2} M_{A^T G}^{-1/2} A^T \\ &= M_{A^T G}^{1/2} B_{A^T G}^{+} B_{A^T G} B_{A^T G}^T \\ &= M_{A^T G}^{1/2} (B_{A^T G}^{+} B_{A^T G})^T B_{A^T G}^T \\ &= M_{A^T G}^{1/2} B_{A^T G}^T (B_{A^T G}^T)^{+} B_{A^T G}^T \\ &= M_{A^T G}^{1/2} B_{A^T G}^T \\ &= A^T. \end{aligned} \tag{3.25}$$

In the third equality above, we have used the fourth MP condition and in the fifth we have used the first MP condition. □

From equation (2.5) we know that once we obtain the constrained acceleration $\ddot{q} (= \ddot{q}_s)$ from equation (3.18) of the mechanical system S, we can determine the constraint force Q^c acting on the unconstrained mechanical system S (described by equation (2.1)) at each instant of time from the relation

$$Q^c = M\ddot{q} - Q = M\ddot{q}_s - Q. \tag{3.26}$$

Alternatively, consider the equation of motion of the constrained auxiliary system S_{A^TG}, which can be obtained by pre-multiplying both sides of the equation (3.18) by M_{A^TG}. We have

$$\begin{aligned} M_{A^TG}\ddot{q}_s &= Q + A^TGz + M_{A^TG}^{1/2}B_{A^TG}^{+}(b - Aa_{A^TG,z}) \\ &\qquad + M_{A^TG}^{1/2}(I - B_{A^TG}^{+}B_{A^TG})M_{A^TG}^{-1/2}C \\ &:= Q_{A^TG,z} + Q_{A^TG,z}^c, \end{aligned} \tag{3.27}$$

where the constraint force acting on the unconstrained auxiliary system S_{A^TG} (described by equation (3.10)) is denoted by,

$$Q_{A^TG,z}^c = M_{A^TG}^{1/2}B_{A^TG}^{+}(b - Aa_{A^TG,z}) + M_{A^TG}^{1/2}(I - B_{A^TG}^{+}B_{A^TG})M_{A^TG}^{-1/2}C. \tag{3.28}$$

We notice from equation (3.27) that under the same set of constraints (both ideal and non-ideal) as those acting on the unconstrained mechanical system S, the constraint force acting on the unconstrained auxiliary system S_{A^TG} is $Q_{A^TG,z}^c$; the explicit expression for this force is given by equation (3.28).

We now explore the connection between Q^c and $Q_{A^TG,z}^c$, our aim being to obtain Q^c explicitly from $Q_{A^TG,z}^c$. We now claim that this can indeed be done by appropriately choosing the m-vector $z(t)$ which has so far been left arbitrary.

To show this, we begin by considering only the third member on the right-hand side of the first equality of equation (3.27). Expanding it, we have

$$\begin{aligned} &M_{A^TG}^{1/2}B_{A^TG}^{+}(b - Aa_{A^TG,z}) \\ &= M_{A^TG}^{1/2}B_{A^TG}^{+}(b - AM_{A^TG}^{-1}[Q + A^TGz]) \\ &= M_{A^TG}^{1/2}B_{A^TG}^{+}(b - Aa_{A^TG} - AM_{A^TG}^{-1}A^TGz) \\ &= M_{A^TG}^{1/2}B_{A^TG}^{+}(b - Aa_{A^TG}) - M_{A^TG}^{1/2}B_{A^TG}^{+}AM_{A^TG}^{-1}A^TGz \\ &= M_{A^TG}^{1/2}B_{A^TG}^{+}(b - Aa_{A^TG}) - A^TGz. \end{aligned} \tag{3.29}$$

Notice that we have denoted $a_{A^TG} := M_{A^TG}^{-1}Q$ in the second equality, and used relation (3.24) in the last equality of the above equation. Using equation (3.29) in the third member on the right-hand side of equation (3.27) yields

$$\begin{aligned} M_{A^TG}\ddot{q}_s &= Q + M_{A^TG}^{1/2}B_{A^TG}^{+}(b - Aa_{A^TG}) \\ &\qquad + M_{A^TG}^{1/2}(I - B_{A^TG}^{+}B_{A^TG})M_{A^TG}^{-1/2}C \\ &:= Q + Q_{A^TG}^c, \end{aligned} \tag{3.30}$$

where

$$a_{A^TG} = M_{A^TG}^{-1}Q, \tag{3.31}$$

and

$$Q_{A^TG}^c = M_{A^TG}^{1/2}B_{A^TG}^{+}(b - Aa_{A^TG}) + M_{A^TG}^{1/2}(I - B_{A^TG}^{+}B_{A^TG})M_{A^TG}^{-1/2}C. \tag{3.32}$$

Equation (3.30) shows that the acceleration $\ddot{q}_s$ of the constrained mechanical system S is given by

$$\ddot{q}_s = M^{-1}_{A^TG}[Q + Q^c_{A^TG}], \tag{3.33}$$

and that it is indeed independent of the arbitrary m-vector $z(t)$ as remarked in the previous sub-section. Furthermore, equating the right-hand sides of equation (3.27) with (3.30) (both of which equal $M_{A^TG}\ddot{q}_s$), we get

$$Q_{A^TG,z} + Q^c_{A^TG,z} = Q + A^TGz + Q^c_{A^TG,z} = Q + Q^c_{A^TG}, \tag{3.34}$$

so that from the last equality we have

$$Q^c_{A^TG} = Q^c_{A^TG,z} + A^TGz. \tag{3.35}$$

We now prove the following result:

Result 2. When the unconstrained mechanical system, S, and the unconstrained auxiliary system, S_{A^TG} , have the same initial conditions and when they are each subjected to the same (ideal and non-ideal) constraints, with the choice of the m-vector,

$$z(t) = \alpha^2(t)b(q, \dot{q}, t), \tag{3.36}$$

where $\alpha(t)$ is a nowhere-zero, sufficiently smooth function of time, the constraint force acting on the unconstrained auxiliary system S_{A^TG} is the same as the constraint force acting on the unconstrained mechanical system S at each instant of time. In short,

$$Q^c = Q^c_{A^TG,\alpha^2b}. \tag{3.37}$$

Proof. From equation (3.30), we know that at each instant of time

$$\begin{aligned} Q^c_{A^TG} &= M_{A^TG}\ddot{q}_s - Q \\ &= (M + \alpha^2A^TGA)\ddot{q}_s - Q \\ &= M\ddot{q}_s - Q + \alpha^2A^TGA\ddot{q} \\ &= Q^c + \alpha^2A^TGb. \end{aligned} \tag{3.38}$$

In the last equality above, we have used equations (3.26) and (2.4). Substituting equation (3.38) in equation (3.35), we get

$$Q^c = Q^c_{A^TG,z} + A^TGz - \alpha^2A^TGb, \tag{3.39}$$

which is the general result that relates the constraint force Q^c acting on the unconstrained mechanical system S to the constraint force $Q^c_{A^TG,z}$ acting on the unconstrained auxiliary system S_{A^TG}, at each instant of time. Note that in equation (3.39) $\alpha(t)$ is any arbitrary nowhere-zero scalar function of time, the m-vector $z(t)$ is any arbitrary sufficiently smooth function of time, and $G(q,t)$ is any positive definite matrix whose elements are continuous functions of its arguments. Finally, using (3.39), when $z = \alpha^2b$, the result follows. □

Therefore, the force of constraint Q^c acting on the unconstrained mechanical system S can also be directly obtained from the force of constraint $Q^c_{A^TG,z}$ acting on the unconstrained auxiliary system S_{A^TG} by the appropriate selection of the m-vector $z(t) = \alpha^2(t)b(q, \dot{q}, t)$.

We have now shown that if one would like to derive the constrained equations of motion of a general mechanical system S that has either a positive semi-definite or positive definite mass matrix, which is subjected to the kinematical constraints $A\ddot{q} = b$ and the non-ideal dynamical constraints described by the n-vector $C(q, \dot{q}, t)$ (under the proviso that matrix $\hat{M}$ has rank n), one could obtain the (explicit) constrained equation of motion of the mechanical system S by following the three-step conceptualization of constrained motion as follows in terms of a new unconstrained auxiliary system [14]:

(i) Description of the unconstrained auxiliary system:
   (a) Replace the mass matrix $M \geq 0$ of the actual unconstrained mechanical system S as given in equation (2.1) with the augmented mass matrix M_{A^TG} as given in equation (3.8);
   (b) Choose $z = \alpha^2 b$ and replace the 'given' force Q acting on the actual unconstrained mechanical system S with the augmented 'given' force $Q_{A^TG,\alpha^2 b}$ as defined in equation (3.9);
   (c) Use the augmented mass matrix described in (a) and the augmented 'given' force described in (b) to obtain equation (3.10) which describes the unconstrained auxiliary system S_{A^TG};

(ii) Description of the constraints: Subject this unconstrained auxiliary system to the same set of constraints (both ideal and non-ideal) and initial conditions as the actual unconstrained mechanical system S;

(iii) Description of the constrained auxiliary system: Apply the so-called fundamental equation [18],[22] (see equation (3.27)) to the unconstrained auxiliary system described in (i) above, which is subjected to the constraints described in (ii).

The resulting equation of motion of this constrained auxiliary system has the following two important features:

(1) The explicit acceleration of the constrained auxiliary system S_{A^TG}, obtained by using equation (3.18), is the same, at each instant of time, as the explicit acceleration of the constrained mechanical system S obtained by using equation (2.9), and
(2) at each instant of time, the constraint force Q^c (see equation (3.26)) acting on the unconstrained mechanical system S (because of the presence of the constraints imposed on it) is the same as the constraint force $Q^c_{A^TG,\alpha^2 b}$ (see equation (3.28)) acting on the unconstrained auxiliary system S_{A^TG}, which is described by equation (3.10).

We are thus led to the somewhat surprising conclusion: the dynamics of the actual constrained mechanical system S are *completely mimicked* by the dynamics of the above-mentioned constrained auxiliary system S_{A^TG}.

Lastly, we point out that if one were interested *only* in obtaining the acceleration at each instant of time of the constrained mechanical system S, one can use *any* arbitrary m-vector $z(t)$ to obtain the augmented 'given' force $Q_{A^TG,z}$ (see equation (3.20)) and then use equation (3.18). For simplicity, Occam's razor would then suggest that we might prefer to take $z(t) \equiv 0$. Furthermore, as pointed out in item (i), part (b), above, if one were, *in addition*, also interested in finding the correct constraint force acting on the unconstrained mechanical system S, one would need to choose $z(t) = \alpha^2(t)b(q, \dot{q}, t)$ and use equations (3.21) and (3.28). Clearly then, when the constraints are such that $b(q, \dot{q}, t) \equiv 0$, the choice of $z \equiv 0$ in the augmented 'given' force (3.9) in the description of the unconstrained auxiliary system (3.10) is automatically selected, and the 'given' force on both the actual unconstrained system and the unconstrained auxiliary system become identical. In that case, the use of equation (3.27) yields, at each instant of time, the correct acceleration of the constrained mechanical system S as well as the correct force of the constraint acting on the unconstrained mechanical system S.

The approach of the above three-step conceptualization of constrained motion by utilizing the auxiliary system is summarized in Table 1. This table schematically shows how one generates the auxiliary system S_{A^TG} from the actual given mechanical system S. Step 1 deals with the description of the unconstrained system S and the corresponding unconstrained auxiliary system S_{A^TG}. Instead of using the mass matrix M of the given mechanical system S that may or may not be singular, we use the mass matrix M_{A^TG} for the auxiliary system S_{A^TG} which is positive definite under the proviso that $\hat{M}$ has full rank. In addition, we also augment the given force Q of the actual mechanical system S with the term A^TGz in defining the unconstrained auxiliary system. Then the unconstrained acceleration of the auxiliary system can be written as $a_{A^TG,z} = M_{A^TG}^{-1}Q_{A^TG,z}$ while the unconstrained acceleration is undefined, as shown, in the case where the mass matrix is singular for the unconstrained mechanical system.

In Step 2, while describing of the constraints we apply the same set of (ideal and non-ideal) constraints to the auxiliary system S_{A^TG} as applied to the actual mechanical system S.

In Step 3, we can obtain the explicit equation for the constrained acceleration of the actual mechanical system by using the unconstrained auxiliary system and the constraints defined in the previous two steps and applying the so-called fundamental equation [18], [22] (see equation (3.18)). The fundamental equation also explicitly gives the constraint force on the actual mechanical system when using the m-vector $z = \alpha^2 b$ in our definition of the unconstrained auxiliary system (see equations (3.21) and (3.28)).

4. Equations for Rotational Motion of a Rigid Body Using Quaternions.

We show in this example how the results obtained in this paper can be directly applied to get the quaternion equations of rotational motion for rigid bodies in a simple and direct manner. When considering the rotational dynamics of rigid bodies, the use of quaternions removes singularity problems that inevitably arise when using Euler angles. However, the quaternion 4-vector describing a physical rotation is constrained to have unit norm, and hence the equations of motion in terms of quaternions can be considered as constrained equations of motion.

Consider a rigid body that has an absolute angular velocity, $\omega \in \mathbf{R}^3$, with respect to an inertial coordinate frame. The components of this angular velocity with respect to its body-fixed coordinate frame whose origin is located at the body's center of mass are denoted by ω_1, ω_2, and ω_3. Let us assume, without loss of generality, that the body-fixed coordinate axes attached to the rigid body are aligned along its principal axes of inertia, where the principal moments of inertia are given by $J_i > 0$, i=1,2,3. The rotational kinetic energy of the rigid body is then simply

$$T = \frac{1}{2}\omega^T J\omega = 2\dot{u}^T E^T JE\dot{u} = 2u^T \dot{E}^T J\dot{E}u, \tag{4.1}$$

Table 1.

System Descriptions	**Actual Mechanical System,** S	**Auxiliary System**, S_{A^TG}
Step 1: Description of Unconstrained System		
Mass Matrix	$M(q,t) \geq 0$ is an n by n matrix	$M_{A^TG} = M(q,t)$ $+\alpha^2(t)A^T(q,\dot{q},t)G(q,t)A(q,\dot{q},t) > 0$, $\alpha(t) \neq 0$ is an arbitrary function of time, $G(q,t) > 0$ is an arbitrary m by m matrix
Given Force	$Q(q,\dot{q},t)$	$Q_{A^TG,z}$ $= Q(q,\dot{q},t) + A^T(q,\dot{q},t)G(q,t)z(t)$, m-vector $z(t)$ is an arbitrary m-vector
Equation of motion	$M\ddot{q} = Q$	$M_{A^TG}\ddot{q}_{s_{A^TG}} = Q_{A^TG,z}$
Unconstrained Acceleration	$a = M^{-1}Q$ or Undefined	$a_{A^TG,z} = M^{-1}_{A^TG}Q_{A^TG,z}$
Step 2: Description of Constraints		
Description of Kinematical Constraints	$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t)$ A is an m by n matrix	$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t)$
Description of Non-ideal Constraints	$C(q,\dot{q},t)$	$C(q,\dot{q},t)$
Step 3: Description of Constrained System		
Equation of motion	$M\ddot{q} = Q + Q^c$	$M_{A^TG}\ddot{q}_{s_{A^TG}} = Q_{A^TG,z} + Q^c_{A^TG,z}$
Constrained Acceleration	$q =$ $\begin{bmatrix}(I - A^+A)M \\ A\end{bmatrix}^+ \begin{bmatrix}Q + C \\ b\end{bmatrix}$	$\ddot{q} = \ddot{q}_{s_{A^TG}} =$ $a_{A^TG,z} + M^{-1/2}_{A^TG}B^+_{A^TG}(b - Aa_{A^TG,z})$ $+M^{-1/2}_{A^TG}(I - B^+_{A^TG}B_{A^TG})M^{-1/2}_{A^TG}C$, $B_{A^TG} = AM^{-1/2}_{A^TG}$, $\alpha(t) \neq 0$ is an arbitrary function of time, $G(q,t) > 0$ is an arbitrary m by m matrix, $z(t)$ is an arbitrary m-vector
Constraint Force on Unconstrained System	$Q^c = M\ddot{q} - Q$	$Q^c = Q^c_{A^TG,\alpha^2b} =$ $M^{1/2}_{A^TG}B^+_{A^TG}(b - Aa_{A^TG,\alpha^2b})$ $+M^{1/2}_{A^TG}(I - B^+_{A^TG}B_{A^TG})M^{-1/2}_{A^TG}C$, $B_{A^TG} = AM^{-1/2}_{A^TG}$, $\alpha(t) \neq 0$ is an arbitrary function of time, $G(q,t) > 0$ is an arbitrary m by m matrix, $z(t) = \alpha^2(t)b(q,\dot{q},t)$

where $\omega = [\omega_1, \omega_2, \omega_3]^T$, $J = diag(J_1, J_2, J_3)$, and $u = [u_0, u_1, u_2, u_3]^T$ is the unit quaternion 4-vector that describes the rotation such that $\omega = 2E\dot{u} = -2\dot{E}u$, where

$$E = \begin{bmatrix} -u_1 & u_0 & u_3 & -u_2 \\ -u_2 & -u_3 & u_0 & u_1 \\ -u_3 & u_2 & -u_1 & u_0 \end{bmatrix}. \tag{4.2}$$

We note that the components of u are not independent and are constrained since the quaternion u must have unit norm to represent a physical rotation. Under the assumption, however, that these components are independent, one obtains the unconstrained equations of motion of the system using Lagrange's equations as

$$M\ddot{u} := 4E^T JE\ddot{u} = -8\dot{E}^T JE\dot{u} + \Gamma_u := Q, \tag{4.3}$$

where the 4-vector Γ_u in equation (4.3) represents the generalized impressed quaternion torque. The connection between the generalized torque 4-vector Γ_u and the physically applied torque 3-vector $\Gamma_B = [\Gamma_1, \Gamma_2, \Gamma_3]^T$, whose components Γ_i, i=1,2,3 are about the body-fixed axes of the rotating body, is known to be given by the relation

$$\Gamma_u = 2E^T \Gamma_B. \tag{4.4}$$

We note now that the 4 by 4 matrix $M = 4E^T JE$ in relation (4.3) of this unconstrained system is singular, since its rank is 3.

The unit norm constraint on the quaternion u requires that $u^T u = u_0^2 + u_1^2 + u_2^2 + u_3^2 = 1$, which yields $A = u^T$ and $b = -\dot{u}^T\dot{u}$. The 4 by 5 matrix $\hat{M}^T = [M \mid A^T]$ has full rank since

$$\begin{bmatrix} M \\ u^T \end{bmatrix} \begin{bmatrix} M & u \end{bmatrix} = \begin{bmatrix} 16E^T J^2 E & 0 \\ 0 & 1 \end{bmatrix} \tag{4.5}$$

is a symmteric matrix whose eigenvalues are 0, 1, $16J_1^2$, $16J_2^2$ and $16J_3^2$. Hence, by Lemma 3.1, the matrix $M_{A^TG} = M + \alpha^2 g(t) uu^T$ given in (3.8) is positive definite, where the arbitrary function $g(t) > 0$, $\forall t$ and we can choose α to be any positive constant. Using equation (3.18) we obtain using some algebra the generalized acceleration of the system given by

$$\ddot{u} = -\frac{1}{2}E^T J^{-1}\tilde{\omega} J\omega - \frac{1}{4}N(\omega)u + \frac{1}{2}E^T J^{-1}\Gamma_B, \tag{4.6}$$

where $\tilde{\omega}$ is the usual skew-symmetric matrix obtained from the 3-vector ω and $N(\omega)$ is the norm of ω.

5. Conclusions.

The main contributions of this paper are the following:

(i) In Lagrangian mechanics, describing mechanical systems with more than the minimum number of required coordinates is helpful in forming the equations of motion of complex mechanical systems since this often requires less labor in the modeling process. The reason that we do not usually use more coordinates than the minimum number is because in doing so we often encounter singular mass matrices and then standard methods for handling such constrained mechanical systems become inapplicable. For example, methods that rely on the invertability of the mass matrix cannot be used.

(ii) Under the proviso that $\hat{M}^T = [M \mid A^T]$ has rank n, in this paper a unified explicit equation of motion for a general constrained mechanical system has been developed irrespective of whether the mass matrix is positive definite or positive semi-definite (singular). This is accomplished by replacing the actual unconstrained mechanical system S with an unconstrained auxiliary system S_{A^TG}, which is obtained by adding $\alpha^2 A^T GA$ to the mass matrix M of the unconstrained mechanical system S, and adding $A^T Gz$ to the 'given' force Q acting on the unconstrained mechanical system S. The mass matrix, $M_{A^TG} = M + \alpha^2 A^T GA$, of this unconstrained auxiliary system S_{A^TG} is always positive definite irrespective of whether the mass matrix M is positive semi-definite ($M \geq 0$) or positive definite ($M > 0$). Thus, by applying the fundamental equation to this unconstrained auxiliary system, which is subjected to the same constraints (and initial conditions) as those imposed on the unconstrained mechanical system S, one directly obtains the acceleration of the constrained mechanical system S.

(iii) The restriction that $\hat{M}^T = [M \mid A^T]$ has full rank n, is not as significant a restriction in analytical dynamics as might appear at first sight, because it is a necessary and sufficient condition that the acceleration of the constrained system be uniquely determinable–a condition that is always satisfied in classical mechanics.

(iv) We show that the acceleration of the constrained mechanical system S so obtained through the use of the auxiliary system S_{A^TG} is independent of the arbitrarily prescribed: (1) nowhere-zero function $\alpha(t)$; (2) the m-vector $z(t)$; and (3) the positive definite matrix $G(q,t)$, provided these are sufficiently smooth (C^2) functions of their arguments. In the special case, when $\alpha(t) = 1$, $z(t) = 0$ and $G = I_m$, the unconstrained auxiliary system simplifies and is the same as the unconstrained mechanical system S except that the mass matrix of the auxiliary system is obtained by adding $A^T A$ to that of the unconstrained mechanical system S. Under identical constraints and initial conditions, the accelerations of the constrained auxiliary system and those of the constrained mechanical system are then identical, and the latter can then be obtained from the former. The fundamental equation [18], [22] gives the latter directly.

(v) The constraint force Q^c acting on the unconstrained mechanical system S (by virtue of the presence of the constraints) can be obtained directly from the constraint force $Q^c_{A^TG,z}$ acting on the unconstrained auxiliary system from the relation $Q^c = Q^c_{A^TG,z} + A^T Gz - \alpha^2 A^T Gb$. Furthermore, by choosing $z = \alpha^2 b$ when describing the unconstrained auxiliary system, we obtain the simpler result $Q^c = Q^c_{A^TG,\alpha^2 b}$.

(vi) The derivations of the results in this paper are simpler than those in Ref. [16]. The results obtained herein are shown to differ from those in Ref. [16] in two important respects. (1) They are simpler, because we do not use the generalized Moore-Penrose (MP) inverse of the matrix A in the determination of the unconstrained auxiliary system; instead, we simply use the transpose of A to describe the unconstrained auxiliary system. (2) They are more general because we can incorporate the arbitrary positive function $\alpha(t)$ and the arbitrary positive definite matrix $G(q,t)$ in the creation of our unconstrained auxiliary systems. Besides the simplicity and the aesthetic value that result from these differences, there are substantial practical benefits that accrue. Most importantly, the new equations provide a major improvement in terms of computational costs since the computation of the transpose of a matrix is near-costless compared to its generalized inverse; this difference in cost becomes increasingly important as the size of the computational model increases. Additionally, the flexibility in choosing $\alpha(t)$ and $G(q,t)$ becomes important from a numerical conditioning point of view, especially when dealing with large, complex multi-body systems.

(vii) The results in this paper point to deeper aspects of analytical mechanics and show that:

(a) given *any* constrained mechanical system S described by the matrices $M(q,t) \geq 0$ and $A(q,\dot{q},t)$, and the column vectors $Q(q,\dot{q},t)$, $b(q,\dot{q},t)$, and $C(q,\dot{q},t)$,

(b) there exists a kind of gauge invariance whereby there are *infinitely* many unconstrained systems with positive definite mass matrices given by

$$M_{A^TG} = M(q,t) + \alpha^2(t)A^T(q,\dot{q},t)G(q,t)A(q,\dot{q},t)$$

and 'impressed' forces given by

$$Q_{A^TG,z}(q,\dot{q},t) = Q(q,\dot{q},t) + A^T(q,\dot{q},t)G(q,t)z(q,\dot{q},t)$$

(c) which, when subjected to the same constraints (both holonomic and nonholonomic, ideal and non-ideal) as those on the given mechanical systems S, and when started with the same initial conditions as the given mechanical system S,

(d) will be *indistinguishable* in their motions from those of the given constrained mechanical system S.

The arbitrariness of the (nowhere-zero) function $\alpha(t)$ and that of the matrix $G(q,t) > 0$ ensures this gauge invariance.

REFERENCES

[1] P. Appell, *Sur une forme generale des equations de la dynamique*, C. R. Acad. Sci. III, **129** (1899), 459–460.

[2] P. Appell, *Example de mouvement d'un point assujeti a une liason exprimee par une relation non lineaire entre les composantes de la vitesse*, Rend. Circ. Mat. Palermo, **32** (1911), 48–50.

[3] N. G. Chataev, "Theoretical Mechanics," Mir Publications, Moscow, 1989.

[4] C. T. Chen, "Linear System Theory and Design," Oxford University Press, New York, 1999.

[5] P. A. M. Dirac, "Lectures in Quantum Mechanics," New York, Yeshiva University Press, 1964.

[6] C. Gauss, *Uber ein neues allgemeines grundgesetz der mechanik*, J. Reine Angew. Math., **4** (1829), 232–235.

[7] W. Gibbs, *On the fundamental formulae of dynamics*, Am. J. Math., **2** (1879), 49–64.

[8] H. Goldstein, "Classical Mechanics," Addison-Wesley, Reading, MA, 1981.

[9] F. Graybill, "Matrices with Applications in Statistics," Wadsworth and Brooks, 1983.

[10] G. Hamel, "Theoretische Mechanik: Eine Einheitliche Einfuhrung in die Gesamte Mechanik," Springer-Verlag, Berlin, New York, 1949.

[11] J. L. Lagrange, "Mechanique Analytique," Paris: Mme Ve Courcier, 1787.

[12] L. A. Pars, "A Treatise on Analytical Dynamics," Woodridge, CT: Oxbow Press, 1979.

[13] R. Penrose, *A generalized inverse of a matrices*, Proc. Cambridge Philos. Soc., **51** (1955), 406–413.

[14] A. Schutte and F. E. Udwadia, *New approach to the modeling of complex multi-body dynamical systems*, Journal of Applied Mechanics, **78** (2011), 11 pages.

[15] E. C. G. Sudarshan and N. Mukunda, "Classical Dynamics: A Modern Perspective," Wiley, New York, 1974.

[16] F. E. Udwadia and A. D. Schutte, *Equations of motion for general constrained systems in Lagrangian mechanics*, Acta Mechanica., **213** (2010), 111–129.

[17] F. E. Udwadia and P. Phohomsiri, *Explicit equations of motion for constrained mechanical systems with singular mass matrices and applications to multi-body dynamics*, Proceedings of the Royal Society of London A, **462** (2006), 2097–2117.

[18] F. E. Udwadia and R. E. Kalaba, *A new perspective on constrained motion*, Proceedings of the Royal Society of London A, **439** (1992), 407–410.

[19] F. E. Udwadia and R. E. Kalaba, *An alternative proof for Greville's formula*, Journal of Optimization Theory and Applications, **94** (1997), 23–28.

[20] F. E. Udwadia and R. E. Kalaba, "Analytical Dynamics: A New Approach," Cambridge University Press, 1996.

[21] F. E. Udwadia and R. E. Kalaba, *Explicit equations of motion for mechanical systems with non-ideal constraints*, ASME J. Appl. Mech., **68** (2001), 462–467.

[22] F. E. Udwadia and R. E. Kalaba, *On the foundations of analytical dynamics*, Int. J. Nonlin. Mech., **37** (2002), 1079–1090.

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