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CONTROLLABILITY, OBSERVABILITY AND IDENTIFICATION OF CLASSICAL LINEAR DYNAMIC SYSTEMS

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INTRODUCTION

The area of identification of flexible systems has in recent years achieved increased importance due to the need for obtaining improved response predictions of structural and mechanical systems subjected to various types of loading environments.

One of the first researchers to have worked on the determination of structural properties from dynamic testing data was Berg [1]. Later, various researchers have worked on this problem [e.g. 2-8] attempting both parametric and nonparametric identification of structural systems. Most parametric identification techniques are iterative in nature. Starting with an "initial guess" of the various parameters to be identified, the parameters are updated in a systematic manner so that some error criterion is minimized. However, this minimization process involves a search in parameter space which is primarily localized around the initial guess position. Such iterative methods then leave open the question of uniqueness of the resulting identified parameters. Udwadia and Sharma [9,10] looked at the problem of uniqueness of identification of a general close coupled damped linear dynamic system and showed the relation between the degree of nonuniqueness, and the location at which the response is observed. Beck [11] extended this work and reported on some aspects of the identification of underdamped linear systems having restricted the stiffness matrix to be real, symmetric and positive definite, the mass matrix to be diagonal and the damping matrix to be of Rayleigh form. His work, as also that of references [9] and [10], primarily concentrates on the identification of systems from records obtained during strong ground shaking.

Almost in parallel with the recent interest in the area of identification of large flexible systems, there has been a growing concern in finding ways of actively controlling such systems. Starting in 1972, with the modest aim of controlling tall buildings to achieve adequate human comfort levels in mildly windy environments [12], today, control techniques are being developed for minimizing the response of large building structural systems to large dynamic forces created by intense winds and earthquakes. The recent symposium on structural control [13] shows the wide-spread interest among civil, mechanical and aerospace engineers in this problem area.

In this paper we study some aspects of the controllability, observability [14,15], and identification of general linear, viscously damped dynamic systems which have classical normal modes. The aforementioned three concepts are shown to be intertwined. After establishing the necessary and sufficient conditions for controllability and observability, the results so obtained are then used in developing a sufficient condition for identification of classical linear dynamic systems. The results obtained are all along particularized to systems which are met within common engineering practice, thereby making the conclusions directly available for applications.

THE SYSTEM MODEL

We shall assume the physical system under consideration to be adequately modelled by a lumped parameter viscously damped linear system with the governing equation

$$M\ddot{x} + C\dot{x} + Kx = B\ f\ (t) \tag{1}$$

where,

M is the mass matrix (N X N)

C is the damping matrix (N X N)

K is the stiffness matrix (N X N)

x is the displacement N-vector

f is the forcing function M-vector, and

B is the input matrix (N X M).

The identification problem addresses the determination of the elements of the matrices M, K and C. In general, the extent of our knowledge (of the various elements) of these matrices decreases in the order aforestated - knowledge of the mass matrix being usually greatest (or known in an average sense with the least uncertainty) whereas knowledge of the damping matrix being usually the least (or known with the highest uncertainty).

In this sequel we shall assume that the mass matrix is nonsingular and that the system of equation (1) has normal classical modes, with $\tilde{K} \triangleq M^{-1}K$ capable of being diagonalized in N-space Thus a nonsingular transformation T exists such that $\tilde{K}$ and $\tilde{C} \triangleq M^{-1}C$ can be simultaneously diagonalized [16].

The eigenvalues λ_i, $i = 1,2,\ldots,N$ of the matrix $\tilde{K}$ will be assumed distinct. This assumption while greatly simplifying the mathematical results, is a reasonable one, especially for large systems $(N > 5)$, for there are few practical systems which cannot be approximated with a K which has distinct eigenvalues [17].

SYSTEM CONTROLLABILITY

We begin with establishing the necessary and sufficient condition for the system of equation (1) to be controllable. Roughly speaking, this requires that each normal coordinate defined as a component of the state vector (which in this case has dimension 2N), can be influenced by the input vector f(t).

Premultiplying equation (1) by M^{-1} and expressing the displacement vector x by the relation, x = Ty, we get

$$\ddot{y} + \left[\diagdown \zeta_i \diagdown\right] \dot{y} + \left[\diagdown \lambda_i \diagdown\right] y = \tilde{T}\, B\, f(t) \qquad (2)$$

where $T^{-1}\tilde{C}\,T = \left[\diagdown \zeta_i \diagdown\right]$, $T^{-1}\tilde{K}\,T = \left[\diagdown \lambda_i \diagdown\right]$ and $\tilde{T} = (MT)^{-1}$. Further denoting $v_i = \{y_i\ \dot{y}_i\}^T$ we create the 2N-vector $w = \{v_i\}$ so that equation (2) reduces to

$$\dot{w} + A_1\, w = \tilde{T}^*\, B^*\, f(t) \qquad (3a)$$

where,

$$A_1 = \begin{bmatrix} 0 & -1 & & & & & \\ \lambda_1 & \zeta_1 & & & & & \\ & & 0 & -1 & & & \\ & & \lambda_2 & \zeta_2 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & \lambda_N & \zeta_N \end{bmatrix}, \qquad (3b)$$

$$\left.\begin{aligned}
\tilde{T}^* &= \begin{bmatrix} \tilde{T}_{11} & 0 & \tilde{T}_{12} & 0 & \cdots & \tilde{T}_{1N} & 0 \\ 0 & \tilde{T}_{11} & 0 & \tilde{T}_{12} & \cdots & 0 & \tilde{T}_{1N} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \tilde{T}_{N1} & 0 & \tilde{T}_{N2} & 0 & \cdots & \tilde{T}_{NN} & 0 \\ 0 & \tilde{T}_{N1} & 0 & \tilde{T}_{N2} & \cdots & 0 & \tilde{T}_{NN} \end{bmatrix}, \text{ and} \\
B^* &= \begin{bmatrix} 0 \\ b_1^T \\ 0 \\ b_2^T \\ \vdots \\ 0 \\ b_N^T \end{bmatrix},
\end{aligned}\right\} \quad (3b)$$

with the rows of B denoted by b_i^T, $i = 1,2,\ldots,N$. The 2N eigenvalues of A_1 are given by

$$\alpha_i^{\pm} = \frac{\zeta_i}{2} \pm \sqrt{\left(\frac{\zeta_i}{2}\right)^2 - \lambda_i}\ , \quad i = 1,2,\ldots,N \qquad (4)$$

where α_i^+ denotes the eigenvalue with the positive sign before the radical, and α_i^- denotes that with the negative sign before the radical. We now assume that $\alpha_i^{\pm} \neq \alpha_i^{\mp} \neq \alpha_j^{\pm} \neq \alpha_j^{\mp}$ for all $i \neq j$; $i, j \in (1,N)$. Thus the system has no multiple frequencies, and has no critically damped modes. In practical applications, this is again a reasonable assumption for one can, in most cases, approximate the damping matrix by one for which this condition holds [17]. We shall often refer to the eigenvalues also as the set α_i, $i = 1,2, \ldots 2N$, whenever the distinction between α_i^+ and α_i^- is not consequential to the reasoning.

The similarity transformation that diagonalizes A_1 can then be written as

$$S = \begin{bmatrix} \diagdown a_i \diagdown \end{bmatrix} , \text{ where } a_i = \begin{bmatrix} 1 & 1 \\ -\alpha_i^+ & -\alpha_i^- \end{bmatrix} . \tag{5}$$

The inverse of S is obtained as

$$S^- = \begin{bmatrix} \diagdown a_i^{-1} \diagdown \end{bmatrix} \quad \text{with } a_i^{-1} = \frac{1}{\alpha_i^+ - \alpha_i^-} \begin{bmatrix} -\alpha_i^- & -1 \\ \alpha_i^+ & 1 \end{bmatrix} . \tag{6}$$

Finally, denoting w = S u, we obtain

$$\dot{u} + Au = S^{-1} \tilde{T}^* B^* f(t) \quad , \tag{7}$$

where $$A = \begin{bmatrix} \diagdown \alpha_i^+ & \\ & \alpha_i^- \diagdown \end{bmatrix} .$$

Noting that S^{-1} has the form of a block diagonal matrix and that each alternate row of the matrix $\tilde{T}^* B^*$ is zero starting with the first row, we find that if rows (2i -1) and 2i of the matrix $\tilde{T}^* B^*$ are both identically zero, the ith mode of the system would be uncontrollable. But this simply means that $\tilde{T}$ B must have no zero rows for the system described by equation (1) to be controllable. We summarize our result in the following theorem.

Theorem 1 : Given the system defined by equation (1), which

(1) has a nonsingular mass matrix
(2) has normal classical modes
(3) has λ_i, i = 1,2,...,N all distinct
(4) has $\alpha_i^{\pm} \neq \alpha_i^{\mp} \neq \alpha_j^{\pm} \neq \alpha_j^{\mp}$ for all i, j ε(1,N), (8)

the system is controllable if and only if $\tilde{T}$ B has no rows which are zero.

□

Corollary 1 : Consider the case when the matrix B is a column vector, b, and the forcing function, f(t), a scalar. The system (1) with the above four conditions, is then controllable if the vector $M^{-1}b$ has nonzero components along each of the eigenvectors of $\tilde{K}$.

Proof : By theorem 1, the system would be controllable if and only if $T^{-1} M^{-1}b$ is a vector all of whose components are nonzero. A necessary and sufficient condition for this to be true is that

$$M^{-1}b = Td \quad , \tag{9}$$

where d has no component which is zero. If (9) is valid, $T^{-1} M^{-1}b = d$ and vice versa. □

Corollary 2 : If any row of $\tilde{T}$ B is zero, then both the displacement and velocity related normal coordinates are uncontrollable.

Proof : Noting the block diagonal form of S^{-1}, as given by relation (6), the result follows. □

Corollary 3 : If the matrix $\tilde{K}$ be restricted so that:

it does not have any constant nonzero eigenvector, (10)

then the system is controllable for $b = M.\underline{1}$, where $\underline{1}$ stands for the vector each of whose elements is unity.

Proof : The proof follows directly from Corollary 1. □

We emphasize this case because, often, for systems in which rigid body translations cannot occur, this condition is satisfied by the matrix $\tilde{K}$. Furthermore, it is easy to show that a necessary and sufficient condition that $\tilde{K}$ has a nonzero constant eigenvector is simply that

$$\sum_{j=1}^{N} \tilde{K}_{ij} = c_o \ , \quad \text{for all } i\varepsilon(1,N) \ , \tag{11}$$

where c_o is a constant. This condition can be used as a quick check to see if the vector $b = M.\underline{1}$ would yield the entire system controllable with a single input.

Corollary 4 : If M is a diagonal matrix, and the system (1) satisfies the restrictions (8) and (10), the system is controllable by a single "base input acceleration".

Proof : If a base acceleration $\ddot{a}(t)$ is applied to the system, $B = b = -M\underline{1}$, and $f(t) = \ddot{a}(t)$. By corollary 3, the result follows. □

The above corollary has special applications to the dynamic response of structures to earthquake ground shaking, when, generally speaking, both restrictions (8) and (10) are satisfied. Thus the use of a suitable base input acceleration renders the system controllable.

SYSTEM OBSERVABILITY

Let us denote $r_i = [x_i \ \dot{x}_i]^T$, and create a 2N vector $q = \{r_i\}$ of the system state. Let the observation Q-vector z be obtained by the relation

$$z = G\, q \ , \tag{12}$$

where G is the Q × 2N observation matrix.

Noting the various transformations, z can be expressed as

$$z = GT^*Su \ \underline{\underline{\Delta}}\ \tilde{G}\, u \tag{13}$$

where T* has the form shown in (3b). The system would be observable, if no column of the matrix $\tilde{G}$ is zero. Thus each normal coordinate would be detectable in the output z. It is instructive to expand the product T*S to understand the meaning of the observability criterion. Using relation (5) this yields,

$$T^*S = \begin{bmatrix} T_{11} & T_{11} & T_{12} & T_{12} & \cdots & T_{1N} & T_{1N} \\ -\alpha_1^+T_{11} & -\alpha^-T_{11} & -\alpha_2^+T_{12} & -\alpha_2^-T_{12} & \cdots & -\alpha_N^+T_{1N} & -\alpha_N^-T_{1N} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ T_{i1} & T_{i1} & T_{i2} & T_{i2} & \cdots & T_{iN} & T_{iN} \\ -\alpha_1^+T_{i1} & -\alpha_1^-T_{i1} & -\alpha_2^+T_{i2} & -\alpha_2^-T_{i2} & \cdots & -\alpha_N^+T_{iN} & -\alpha_N^-T_{iN} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\alpha_1^+T_{N1} & -\alpha_1^-T_{N1} & -\alpha_2^+T_{N2} & -\alpha_2^-T_{N2} & \cdots & -\alpha_N^+T_{NN} & -\alpha_N^-T_{NN} \end{bmatrix} \quad (14)$$

The displacement measurement at node j of the system, for instance, would require the observation matrix, G, to be of the form

$$G = [0\ 0\ 0\ 0\ \ldots\ 0\ 1\ 0\ 0\ 0] ,$$

where the (2j-1)st element is unity. The product GT*S would then simply be the (2j-1)st row of the matrix T*S. If this row has any element (say the 2ith element) equal to zero, then the system would be unobservable as far as the 2ith normal coordinate is concerned. We note in passing that if the 2ith element is zero then so is the (2i-1)st element of that row. Thus the displacement measurement at a node of a particular mode of vibration would make that mode unobservable from the displacement output obtained at that point.
A similar argument can be made when the velocity of the jth node is observed, with G = [0 0 0 0 .. 0 1 0 .. 0 0] where the 2jth element is unity. However, we observe that the measurement of displacement

is in general, from the observability point of view, superior to the measurement of velocity because (equation (14)) the velocity measurements cannot observe the mode corresponding to a zero eigenvalue ($\alpha_i = 0$, $i \in (1,2N)$). We then have the following results.

Theorem 2 : The system defined by equation (1) under the restrictions (8), is observable if the product GT*S has no columns which are zero.

Corollary 5 : If under the restrictions (8), the measurement vector consists only of the nodal velocities (i.e. $\dot{x}_j$, $j \in (1,N)$), then the mode corresponding to the zero eigenvalue, α_i, is unobservable.

Proof : The proof follows directly from the structure of equation (14). □

Corollary 6 : If the displacement of each node is measured, the system is observable; also if the system has no zero eigenvalue, the measurement of each nodal velocity renders the system observable.

Proof : Using (13) and (14) and noting that T is nonsingular, the result follows. □

Whereas Corollary 6 yields a sufficient condition for the system to be observable, it is not a necessary condition. If some apriori information on the matrices M and K is available, one can often render the system observable without the measurement of the displacement of each node (i.e. each component of $\underset{\sim}{x}$). For instance, in many applications K is known to be banded. Perhaps the commonest occurance of this arises in structural and mechanical systems when M is often diagonal and K is a symmetric banded matrix. In an attempt to answer the questions (a) how many displacement (or velocity) measurements are sufficient to render such systems observable, and (b) where should those measurements be made, we present the following result.

Theorem 3 : Consider the system of equation (1) with the restrictions (8). If the matrix $\tilde{K}$ is banded and has a structure given by

$$\tilde{K} = \begin{bmatrix} \tilde{k}_{1,1} & \tilde{k}_{1,2} & \cdots & \tilde{k}_{1,\delta} & & & \\ \tilde{k}_{2,1} & \tilde{k}_{2,2} & \cdots & \cdots & \tilde{k}_{2,\delta} & & \\ \vdots & & & & \ddots & & \\ \tilde{k}_{\delta,1} & \tilde{k}_{\delta,2} & \cdot\ \tilde{k}_{\delta\delta} & \cdots & \cdots & \tilde{k}_{\delta,2\delta-1} & \\ & \ddots & & \ddots & & & \tilde{k}_{N-\delta+1,N} \\ & & \ddots & & \ddots & & \vdots \\ & & & \tilde{k}_{N,N-\delta+1} & \cdots & \cdots & \tilde{k}_{N,N} \end{bmatrix}$$

with no elements of $\tilde{K}$ which lie within the band-width equal to zero, then the system (1) is observable through the use of $(\delta-1)$ displacement measurements (or $(\delta-1)$ velocity measurements if $\tilde{K}$ has no zero eigenvalue). Furthermore, these measurements would involve the first $(\delta-1)$ or the last $(\delta-1)$ components of the vector x (or $\dot{x}$).

Proof : Noting the structure of T*S we need only show that the matrix $\tilde{K}$ of the form given by equation (15) has no nontrivial eigenvector whose first $(\delta-1)$ components (or last $(\delta-1)$ components) are identically zero.

Assume that $\tilde{K}$ has an eigenvector, e of the form $e = [0 \ . \ 0 \ .. \ 0 \ e_\delta \ e_{\delta+1} \ . \ . \ e_N]^T$, and that the corresponding eigenvalue is μ.

Then the use of the relation

$$\tilde{K}\, e = \mu e \tag{16}$$

renders $e_\delta = 0$ by use of the first equation of the set of simultaneous equations represented by (16). Moving to the succeeding equation of the set (16), the component $e_{\delta+1}$ is proved zero. In this manner we prove that the vector e is zero. Thus if e is nontrivial, its first $(\delta-1)$ components cannot all be zero. One can similarly show that the last $(\delta-1)$ components are also nonzero.

The result on velocity measurements is obvious from the above and Corollary 6. We mention in passing that $\tilde{K}$ need not be symmetric. □

Corollary 7 : If the Q-vector renders the system (1) observable, then so does any other Q-vector z_1 which is related to z by the relation

$$z_1 = Fz \quad ,$$

where F has rank Q.

Proof : Given the measurement vector z_1, the vector z can always be obtained as $z = F^{-1}z_1$. The rank of F guarantees the existance of its inverse. □

SYSTEM IDENTIFICATION

Having established the conditions for observability and controllability, we prove the following result.

Theorem 4 : If the system (1) under the restrictions (8) is controllable, and the complete state vector is observed, $\tilde{K}$ and $\tilde{C}$ can be uniquely identified.

Proof : Consider the single input case. Recasting equation (1), we have

$$\dot{p} = Dp + b^* f(t) \tag{17}$$

where f(t) is the single input forcing function,

$$p = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} , \qquad b^* = \begin{Bmatrix} 0 \\ b \end{Bmatrix}$$

and,

$$D = \begin{bmatrix} 0 & I \\ -\tilde{K} & -\tilde{C} \end{bmatrix} . \qquad (18)$$

As the system is controllable, there exists a vector b such that [14] the matrix

$$[b^* \; Db^* \; \dots \; D^{2N-1}b^*]$$

has rank 2N.

If $f(t) = \delta(t)$, assuming the system starts from rest,

$$p(t) = e^{Dt}b^* ,$$

so that

$$\left.\begin{aligned} p(o) &= b^* \\ \dot{p}(o) &= Dp(o) = Db^* \\ \ddot{p}(o) &= D\dot{p}(o) = D^2b^* = D.Db^* \\ &\dots\dots\dots\dots \\ &\dots\dots\dots\dots \\ p^{2N}(o) &= Dp^{2N-1}(o) = D^{2N}b^* = DD^{2N-1}b^* \end{aligned}\right\} \qquad (19)$$

Thus

$$[\dot{p}(o) \; \ddot{p}(o) \; \dots \; p^{2N}(o)] = D[p(o) \; \dot{p}(o) \; \dots \; p^{2N-1}(o)] ,$$

from which

$$D = [\dot{p}(o) \; \ddot{p}(o) \; \dots \; p^{2N}(o)][p(o) \; \dot{p}(o) \; \dots \; p^{2N-1}(o)]^{-1} . \qquad (20)$$

The inverse matrix in relation (20) exists because

$$[p(o)\ \dot{p}(o)\ \dots\ p^{2N-1}(o)] = [b^*\ Db^*\ \dots\ D^{2N-1}b^*]$$

has rank 2N. □

Theorem 5 : If the system (1) under restriction (8) is controllable, observation of the complete displacement vector x, for all time, is sufficient to enable a unique identification of the matrices $\tilde{K}$ and $\tilde{C}$.

Proof : Knowledge of $x_i(t)$, i = 1,2,...,N for all time yields complete information on the higher derivatives of $x_i(t)$ which are involved in the various vectors $p^r(o)$, $r\ \varepsilon(1,2N)$, of equation (20). Using (20), D can be determined. We note that the observation of the complete vector x guarantees system observability. □

Theorem 6 : If the system (1) is controllable and restriction (8) is satisfied, measurement of the complete velocity vector, yields $\tilde{K}$ and $\tilde{C}$ uniquely, if for any i, $\alpha_i \neq o$, $i\ \varepsilon(1,2N)$.

Proof : By relation (19),

$$[\ddot{p}(o)\ \dddot{p}(o)\ \dots\ p^{2N+1}(o)] = D[\dot{p}(o)\ \ddot{p}(o)\ \dots\ p^{2N}(o)] \quad . \tag{21}$$

But

$$[\dot{p}(o)\ \ddot{p}(o)\ \dots\ p^{2N}(o)] = D[b^*Db^*\dots\ D^{2N}b^*] \quad . \tag{22}$$

If D has no zero eigenvalue, then the rank of the matrix on the right is 2N and its inverse exists. Thus

$$D = [\ddot{p}(o)\ \dddot{p}(o)\ \dots\ p^{2N+1}(o)][\dot{p}(o)\ \ddot{p}(o)\ \dots\ p^{2N}(o)]^{-1} \tag{23}$$

Clearly if $\dot{x}_i(t)$, i = 1,...,N are all measured, the higher derivatives $p^r(o)$, $r > 1$ can be found. The right hand side of (23) can be determined and uniquely ascertained. □

Partial identification of the system of equation (1) can be thought of as the determination of all the eigenvalues $\alpha_i^{\pm}$, $i = 1,2,\ldots,N$. One is often interested in the determination of these eigenvalues α_i. To this end we prove the following theorem.

Theorem 7 : Consider the system of equation (1) under the restriction (8). If the system is controllable and observable, the eigenvalues $\alpha_i^{\pm}$, $i = 1,2,\ldots,N$ can be uniquely determined.

Proof : Let z be the observation Q-vector which makes the system observable. Then

$$z = G\, q = G\, T^*\, S\, u \quad . \tag{24}$$

Assuming $q(o) = o = u(o)$, and taking Laplace transforms, we have, using relation (7),

$$Z(s) = GT^*S\ [Is+A]^{-1}\ S^{-1}\ \tilde{T}^*\ B^*\ f(s) \tag{25}$$

where s is the transform variable. Since the system is controllable, the matrix $H \stackrel{\Delta}{=} S^{-1}\ \tilde{T}^*\ B^*$ has no zero rows; also, since the system is observable the matrix $L \stackrel{\Delta}{=} GT^*S$ has no zero columns. Let h_i^T denote the ith row of H and ℓ_i denote the ith column of L. Then

$$Z(s) = \sum_{i=1}^{N} \left[\frac{\ell_{2i-1} h^T_{2i-1}}{s - \alpha_i^+} + \frac{\ell_{2i} h^T_{2i}}{s - \alpha_i^-} \right] f(s) \tag{26}$$

where $\ell_i h_i^T$ is the $(Q \times M)$ vector outer product. Since the elements of ℓ_i and h_i for $i \,\varepsilon (1,2N)$ cannot all be zero, each of the matrices $\ell_i h_i^T$, $i \,\varepsilon (1,2N)$, has at least one element which is nonzero. Thus there exists at least one component of the observation vector z which has a contribution from any given eigenvalue α_i, $i \,\varepsilon (1,2N)$.

Taking the input forcing function $f(t) = f_0(t)\ \underline{1}$, where $\underline{1}$ is an M dimensional unit vector, we can express each component Z_j of the vector Z by

$$Z_j(s) = \sum_{i=1}^{N} \left[\frac{\beta_{ij}}{s-\alpha_i^+} + \frac{\gamma_{ij}}{s-\alpha_i^-} \right] f_0(s) \tag{27}$$

where β_{ij} and λ_{ij} are suitable constants.

Assume that two different sets of eigenvalues $\alpha_i^\pm$ and $\tilde{\alpha}_i^\pm$ exist such that z(t) is identical for any $f_0(t)$. Then

$$Z_j(s) = \sum_{i=1}^{N} \left[\frac{\beta'_{ij}}{s-\tilde{\alpha}_i^+} + \frac{\gamma'_{ij}}{s-\tilde{\alpha}_i^-} \right] f_0(s) \ . \tag{28}$$

Subtracting equation (28) from equation (27) we have,

$$0 = \sum_{i=1}^{N} \left[\left(\frac{\beta_{ij}}{s-\alpha_i^+} - \frac{\beta_{ij}}{s-\tilde{\alpha}_i^+} \right) + \left(\frac{\gamma_{ij}}{s-\alpha_j^-} - \frac{\gamma'_{ij}}{s-\tilde{\alpha}_i^-} \right) \right] f_0(s) . \tag{29}$$

Relation (29) would be true for all s only if the set $\alpha_i^\pm$ is identical to the set $\tilde{\alpha}_i^\pm$ (with perhaps a relabelling), for the poles must exactly cancel each other. Moreover, $\beta_{ij} = \beta'_{ij}$ and $\gamma_{ij} = \gamma'_{ij}$. Noting that for each $\alpha_p^+(\alpha_p^-)$ there is at least one Z_j for which $\beta_{pj}(\gamma_{pj})$ is nonzero, the result of the theorem follows. However, the unique determination of the λ_i's and ζ_i's is not always possible from a knowledge of all the eigenvalues $\alpha_i^\pm$, i = 1,2,...,N.

□

Often, in mechanical and structural systems, the mass and stiffness matrices are symmetric and positive definite. The λ_i's then correspond to the undamped natural frequencies of vibration of the system, and the ζ_i's are related to percentages of critical damping in the various modes of vibration. For such syste we have the following result.

Theorem 8 : When $\lambda_j > o$, j = 1,2,...,N, for all the "underdamped" modes of vibration, that is for all j such that

$$\left(\frac{\zeta_j}{2}\right)^2 - \lambda_j < o \ ,$$

the system of equation (1) satisfying restriction (8), if observable and controllable, yields the values of λ_j and ζ_j uniquely.

Proof : Since the system is observable and controllable, by Theorem 7, all the $\alpha_i^{\pm}$, i = 1,2,...,N can be uniquely determined. Also, if for any j

$$\left(\frac{\zeta_j}{2}\right)^2 - \lambda_j < o \quad ,$$

then

$$\alpha_j^{\pm} = \frac{\zeta_j}{2} \pm i\sqrt{\lambda_j - \left(\frac{\zeta_j}{2}\right)^2} \quad . \tag{30}$$

The $\alpha_j^{\pm}$'s are complex conjugates of each other. Knowledge of these complex conjugate pair thus yields ζ_j and λ_j uniquely. It can be easily seen that for those modes, i, which are "overdamped", knowledge of the $\alpha_i^{\pm}$'s does not, in general, lead to a unique determination of the corresponding λ_i's and ζ_i's. □

We note in passing that the results of Theorems 7 and 8 are obviously valid for a single input, as in the case B* = b*. For an underdamped system, the λ_i's and ζ_i's are then both uniquely ascertained by the use of a single control input.

CONCLUSIONS AND DISCUSSION

This paper attempts to bring together the various concepts of controllability and observability of classical linear dynamic systems and attempts to pose the identification problem in terms of these concepts.

The theory is developed for classical linear systems satisfying certain conditions. These conditions are in practice quite reasonable especially where large dynamic systems are concerned.

We show sufficient and necessary conditions for the system to be controllable by a single input. We indicate the fact that a sufficient condition for the observability of the system is the measurement of each nodal displacement. We show that for a controllable system, the measurement of each nodal displacement is sufficient to yield unique identification of the matrices $M^{-1}K$ and $M^{-1}C$. Generally speaking, the matrix M is often estimated with a sufficient degree of accuracy by design drawings and/or experimental data. Thus if M is known aproiri, the matrices K and C can be determined from such displacement data.

Lastly, we prove that for underdamped systems which are both controllable and observable, unique identification of the λ_i and ζ_i is possible by use of a single control input. For systems which have real, symmetric, positive definite stiffness and mass matrices, this corresponds to a complete knowledge of the undamped frequencies of vibration and the percentages of critical damping.

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