Control of Uncertain Nonlinear Multibody Mechanical Systems

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Descriptions of real-life complex multibody mechanical systems are usually uncertain. Two sources of uncertainty are considered in this paper: uncertainties in the knowledge of the physical system and uncertainties in the "given" forces applied to the system. Both types of uncertainty are assumed to be time varying and unknown, yet bounded. In the face of such uncertainties, what is available in hand is therefore just the so-called "nominal system," which is our best assessment and description of the actual real-life situation. A closed-form equation of motion for a general dynamical system that contains a control force is developed. When applied to a real-life uncertain multibody system, it causes the system to track a desired reference trajectory that is prespecified for the nominal system to follow. Thus, the real-life system's motion is required to coincide within prespecified error bounds and mimic the motion desired of the nominal system. Uncertainty is handled by a controller based on a generalization of the concept of a sliding surface, which permits the use of a large class of control laws that can be adapted to specific real-life practical limitations on the control force. A set of closed-form equations of motion is obtained for nonlinear, nonautonomous, uncertain, multibody systems that can track a desired reference trajectory that the nominal system is required to follow within prespecified error bounds and thereby satisfy the constraints placed on the nominal system. An example of a simple mechanical system demonstrates the efficacy and ease of implementation of the control methodology. [DOI: 10.1115/1.4025399]

1 Introduction

All real-life physical systems are known only to within some bounds of uncertainty that may depend on the various levels of their description. The question of how to model the dynamics of such uncertain complex multibody systems to follow prescribed reference trajectories has become a topic of great interest during the past few years. References [1–7] give a brief sampling of some of the researchers who have made substantial contributions, yet several questions remain unanswered at the present time. The uncertainties that arise in complex mechanical systems stem from two main sources: (i) uncertainties in our knowledge of the physical system, like uncertainties in the stiffness and mass distribution, the nature of damping, etc., and (ii) uncertainties in our knowledge of the externally applied forces acting on the system.

Several examples where such uncertainties arise can be found in the areas of precision robotic control, as, for example, required in targeted eye surgery. Another area where such uncertainties arise is in the precision control of low earth satellites subjected to uncertain air drag and in the control of unmanned space vehicles, whose mass and moments of inertia may change over time while simultaneously being subjected to uncertain forces, like solar wind and gravity perturbations. The two sources of uncertainty described above are simultaneously considered in this paper, and in what follows, all these uncertainties are included in what we call the "real-life mechanical system" or the "actual system," whose description is known only imprecisely. While not known precisely, it is assumed, however, that we have estimates of the bounds on the uncertainties involved. Our best assessment of a given actual system will be referred to as the "nominal multibody system" or the "nominal system" for short. This term naturally includes the best assessment of our characterization of the physical system and of the nature of the "given" forces acting on it.

In this paper, the tracking control problem is reformulated in the framework of constrained motion, and we view the control requirements as constraints on the nonlinear dynamical system. Using analytical dynamics instead of control theory, we then obtain closed-form generalized control forces to satisfy these requirements. The aim of this paper is to develop a closed-form equation of motion for a general dynamical system, which, when applied to an actual system, causes this system to follow the trajectory that is prespecified (by the constraints imposed) on the corresponding nominal system and thereby to satisfy the constraints of the nominal system. In what follows, we shall therefore use the terms "requirements" and "constraints" interchangeably, as well as the terms "control forces" and "constraint forces" and the terms "controlled system" and "constrained system." The methodology to obtain the closed-form equation is developed in a two-step process. The first step uses the concept of the fundamental equation in analytical dynamics to provide the closed-form control force needed to satisfy the constraints imposed on the nominal system model, where, as stated before, the nominal model is the model adduced from the best assessment of our characterization of the actual multibody system. Upon specification of the nominal system model, no linearizations/approximations are made in the description of its dynamics and the nonlinear control force that exactly satisfies the desired constraints is obtained in closed form [8–10]. In the next step of the methodology, this nonlinear control force is augmented by an additional additive control force based on a generalization of the notion of a sliding surface. This then provides a general approach to the dynamics of nonlinear uncertain mechanical systems, leading to closed-form nonlinear equations of motion that can guarantee that these systems satisfactorily mimic (within required error bounds) the motions desired of their nominal counterparts.

More specifically, this paper distinguishes itself from previous work on the precision control of multibody mechanical systems in the following four key ways: (i) The methods developed herein include systems whose inertia properties (mass matrices and/or moment of inertia matrices) may be uncertain and time varying, as happens when a spacecraft uses an uncertain amount of fuel in a maneuver. Most of the work done to date in the tracking control of uncertain systems utilizes dynamical models in terms of (first order) differential equations (with the generalized position and velocity as the state vector), in which the coefficients of the derivative terms are usually taken to be unity. Such descriptions

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originate from dynamical systems concepts and can be difficult to apply when dealing with uncertain mechanical systems. For, by using such formulations, one can obtain the necessary control accelerations; however, when dealing with mechanical systems, in order to obtain the necessary control forces to be applied, these control accelerations would yet need to be applied to systems whose masses (and/or moments of inertia) are still uncertain! (ii) We do not use full-state control here, as is often done in the control literature for nonlinear, nonautonomous systems, and application of control directly to the velocity of a mechanical system is generally difficult. (iii) A generalized concept of a sliding surface is developed, thereby making available a set of continuous controllers that can accommodate the practical requirements imposed on the control effort while also eliminating the presence of chattering—a phenomenon commonly found in standard sliding mode control. (iv) We use a closed-form expression for the "exact" control of the nonlinear, nonautonomous, nominal system. This is achieved by visualizing the tracking control problem from a different perspective—reformulating the control objective as a constraint on the system. Hence, the control so obtained relies on recent advances in analytical dynamics rather than on control theory. Having been inspired from analytical dynamics, it is especially suited in its applicability to general multibody mechanical systems. It is capable of minimizing the control cost at *each* instant of time and not just minimizing the integral of the control cost over the duration of the control, as is usually the case when using control theoretic methods. In the control literature, methods such as state-dependent Riccati equations (SDRE) handle the problem through pseudolinearization [11], and methods like backstepping rely on the system having a specific structure [12]. Further, the tracking control objective having been exactly met here for the nominal system now allows the additional additive controller to be more efficacious and fine-tunable in taking care of the uncertainties in the actual system's description.

To demonstrate the effectiveness of the proposed closed-form equation, we consider an example of a triple pendulum, whose masses are taken to be imprecisely known. The control forces, which need to be applied to this actual system so that it follows given, prescribed constraints assigned to the corresponding nominal system and therefore mimics the behavior of the nominal system, are found. Numerical results are provided showing the simplicity, accuracy, and ease of implementation of the control methodology.

2 On the Dynamics of Nominal Multibody Systems

2.1 System Description of the Nominal System. We begin by introducing the description of the nominal system, by which we mean our best assessment of the actual system, whose description is known only imprecisely. It is useful to conceptualize the description of such a nominal multibody system in a three-step procedure [13–16]. We do this in the following way.

First, we describe the uncontrolled system in which the coordinates are all assumed independent of each other. The equation of motion of this system is given, using Lagrange's equation, by

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) \tag{2.1}$$

with the initial conditions

$$q(t=0) = q_0, \quad \dot{q}(t=0) = \dot{q}_0 \tag{2.2}$$

where q is the generalized coordinate n-vector; $M > 0$ is the n by n mass matrix, which is a function of q and t; and Q is an n-vector, called the given force, which is a known function of q, $\dot{q}$, and t.

From Eq. (2.1), we find the acceleration of the uncontrolled system given by

$$a := M^{-1}(q,t)Q(q,\dot{q},t) \tag{2.3}$$

Second, we impose a set of control requirements as constraints on this uncontrolled system. We suppose that the uncontrolled system is now subjected to the m sufficiently smooth control requirements given by [15]

$$\varphi_i(q,\dot{q},t) = 0, \quad i = 1,2,...,m \tag{2.4}$$

where $r \le m$ equations in the equation set of Eq. (2.4) are functionally independent. The control constraints described by Eq. (2.4) include all the usual varieties of holonomic and/or nonholonomic constraints and then some. The presence of the control requirements does not permit all the components of the n-vectors q_0 and $\dot{q}_0$ to be independently assigned. We shall assume that the initial conditions in Eq. (2.2) satisfy the m control requirements. (If not, the control constraints can be expressed in an alternative form so that they are asymptotically satisfied [17] (see Sec. 2.2).)

Differentiating the control requirements in Eq. (2.4) with respect to time t, we obtain the relation [18]

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t) \tag{2.5}$$

where A is an m by n matrix, whose rank is r, and b is an m-vector. We note that each row of A arises by appropriately differentiating one of the m control requirements in the set given in Eq. (2.4).

In the third step, the equation of motion of the "controlled nominal system" or the nominal system is given by

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q^c(q,\dot{q},t) \tag{2.6}$$

where Q^c is the control force n-vector that arises to ensure that the control requirements in Eq. (2.5) are satisfied. The explicit equation of motion of the nominal system is given by the *fundamental equation* [10,17],

$$M\ddot{q} = Q + A^T(AM^{-1}A^T)^+(b - Aa) \tag{2.7}$$

wherein the various quantities have been defined in the previous two steps and the superscript "+" denotes the Moore–Penrose inverse of a matrix. In the above equation and in what follows, we shall suppress the arguments of the various quantities unless required for clarity.

We note that Eq. (2.7) is valid (i) whether or not the control requirements are holonomic or nonholonomic, (ii) whether or not they are nonlinear functions of their arguments, and (iii) whether or not they are functionally dependent. The control force that the uncontrolled system is subjected to, because of the presence of the control requirements in Eq. (2.4), can be explicitly expressed as

$$Q^c(t) := Q^c(q(t),\dot{q}(t),t) = A^T(AM^{-1}A^T)^+(b - Aa) \tag{2.8}$$

The control force given in Eq. (2.8) is optimal in the sense that it minimizes the control cost ${Q^c}^T M^{-1} Q^c$ at *each* instant of time [17,18].

We refer to the system described by Eq. (2.7) as the nominal system, implying that (1) it includes our best assessment of the information we have regarding the system's parameters and structure and the nature of the given force n-vector Q that the system is subjected to and (2) it exactly satisfies the control requirements placed on it. Premultiplying both sides of Eq. (2.7) with M^{-1}, the acceleration of the nominal system that satisfies the constraint in Eq. (2.4) can be expressed as

$$\ddot{q} = a + M^{-1}A^T(AM^{-1}A^T)^+(b - Aa) := a + M^{-1}Q^c(t) \tag{2.9}$$

a relation which we shall require later on.

2.2 Example. To demonstrate the applicability of the proposed methodology, we introduce an example of a simple multibody system. We will continue this example all the way through this paper. It is straightforward to extend this example to more general situations.

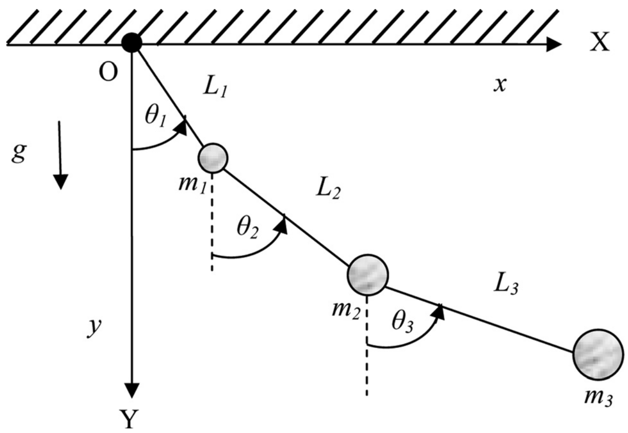

Fig. 1 Triple pendulum with the datum at the origin O

Consider a planar pendulum consisting of three masses: m_1, m_2, and m_3 suspended from massless rods of lengths L_1, L_2, and L_3 moving in the XY-plane (see Fig. 1). An inertial frame of reference is fixed at the point of suspension, O, of the triple pendulum, and the X-axis is taken as the datum for computing the potential energy of the system. Though simple, the system can exhibit complex dynamics.

The masses are constrained to move so that the total energy, $E(t)$, of the system is required to equal the sum of the energies (kinetic and potential) of only the two masses m_2 and m_3 (i.e., $E(t) = E_2(t) + E_3(t)$, where we have denoted $E_i(t)$ as the total energy of mass m_i.

The three-step approach described in Sec. 2.1 is now illustrated. We begin by writing the equation of the uncontrolled system (corresponding to Eq. (2.1)) using the generalized coordinate 3-vector $q(t) = [\theta_1(t), \theta_2(t), \theta_3(t)]^T$, whose components, in the absence of the above-mentioned energy constraint, are independent of one another. Lagrange's equations then yield the relation

$$M(q; m_1, m_2, m_3)\ddot{q} = Q(q, \dot{q}; m_1, m_2, m_3) \tag{2.10}$$

where the elements of the 3 by 3 symmetric matrix M are given by

$$\begin{aligned} &M_{11} = (m_1 + m_2 + m_3)L_1^2; \quad M_{12} = (m_2 + m_3)L_1 L_2 \cos(\theta_{12}); \\ &M_{13} = m_3 L_1 L_3 \cos(\theta_{13}); \quad M_{22} = (m_2 + m_3)L_2^2; \\ &M_{23} = m_3 L_2 L_3 \cos(\theta_{23}); \quad M_{33} = m_3 L_3^2 \end{aligned} \tag{2.11}$$

and the elements of the 3-vector Q are given by

$$\begin{aligned} Q_1 &= -(m_2 + m_3)L_1 L_2 \dot{\theta}_2^2 \sin(\theta_{12}) - m_3 L_1 L_3 \dot{\theta}_3^2 \sin(\theta_{13}) \\ &\quad - (m_1 + m_2 + m_3) g L_1 \sin\theta_1 \\ Q_2 &= (m_2 + m_3)L_1 L_2 \dot{\theta}_1^2 \sin(\theta_{12}) - m_3 L_2 L_3 \dot{\theta}_3^2 \sin(\theta_{23}) \\ &\quad - (m_2 + m_3) g L_2 \sin\theta_2 \\ Q_3 &= m_3 L_1 L_3 \dot{\theta}_1^2 \sin(\theta_{13}) + m_3 L_2 L_3 \dot{\theta}_2^2 \sin(\theta_{23}) \\ &\quad - m_3 g L_3 \sin\theta_3 \end{aligned} \tag{2.12}$$

In the above, we have denoted $\theta_{ij}(t) = \theta_i(t) - \theta_j(t)$, and we explicitly show in Eq. (2.10) the parameters m_1, m_2, and m_3, which we will later on consider to be known only imprecisely.

Using the X-axis as the datum (see Fig. 1), in the second step, we describe the energy constraint $E(t) = E_2(t) + E_3(t)$, which is equivalent to the relation

$$E_1(t) = 0 \tag{2.13}$$

where the energy E_1 of mass m_1 is given by

$$E_1 = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 - m_1 g L_1 \cos\theta_1 \tag{2.14}$$

Since the system may not initially (at time $t = 0$) satisfy this constraint, we modify the constraint in Eq. (2.13) using the trajectory stabilization relation [17],

$$\dot{E}_1 + \alpha E_1 = 0 \tag{2.15}$$

where $\alpha(t) > 0$ is a positive function. By Eqs. (2.14) and (2.15), we obtain the constraint equation

$$A\ddot{q} := \left[L_1^2 \dot{\theta}_1 \;\; 0 \;\; 0\right]\ddot{q} = -g L_1 \sin\theta_1 \dot{\theta}_1 - \alpha\left(\frac{1}{2} L_1^2 \dot{\theta}_1^2 - g L_1 \cos\theta_1\right) := b \tag{2.16}$$

For the final step to obtain the equations of motion of the (controlled) nominal system, we use the information from Eqs. (2.10)–(2.12) and (2.16) in Eq. (2.7). Premultiplying both sides of the equation by M^{-1}, we obtain the constrained acceleration of the (controlled) nominal system as (see Eq. (2.9))

$$\ddot{q} = a + M^{-1} A^T (A M^{-1} A^T)^+ (b - Aa) \tag{2.17}$$

2.3 Numerical Results and Simulations of the Control Problem. In what follows, we shall assume that the real-life triple pendulum described above has masses whose values are imprecisely known and that our best assessment of their values is $m_1 = 1$ kg, $m_2 = 2$ kg, and $m_3 = 3$ kg. Thus, these are the values of the three masses of our nominal system.

The lengths of the massless rods are $L_1 = 1$ m, $L_2 = 1.5$ m, and $L_3 = 2$ m. At $t = 0$, the masses are located with the angles of $\theta_1(0) = 1$ rad, $\theta_2(0) = 0$ rad, and $\theta_3(0) = 0$ rad with respect to the vertical Y-axis (see Fig. 1). The initial velocities of the three bobs are taken to be $\dot{\theta}_1(0) = 0.01$ rad/s, $\dot{\theta}_2(0) = 0$ rad/s, and $\dot{\theta}_3(0) = 0$ rad/s. We note that these initial conditions do not satisfy the constraint $E_1 = 0$. Thus, the parameter α in Eq. (2.15) is chosen to be $0.1\|A\|_2^2$, where $\|A\|_2$ is the L^2 norm of the matrix A in Eq. (2.16). The acceleration due to gravity is downwards and of magnitude $g = 9.81\text{m/s}^2$. Numerical integration throughout this paper is done in the MATLAB environment, using a variable time step integrator with a relative error tolerance of 10^{-8} and an absolute error tolerance of 10^{-12}.

Figure 2 plots the trajectory of mass m_3 of the triple pendulum in the XY-plane for a period of 10 seconds. The start of the trajectory is marked by a circle, and its end is marked by a square. From here on throughout this paper, the start and the end of all trajectories are indicated likewise. The energies of the three masses are shown in Fig. 3. We see that the total energy (E) is the sum of

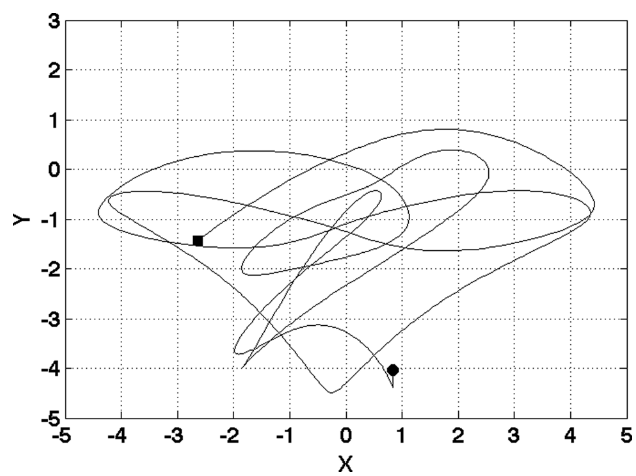

Fig. 2 Trajectory of mass m_3 in the XY-plane (meter) of the triple pendulum shown for a duration of 10 s. The trajectory starts at the circle and ends at the square.

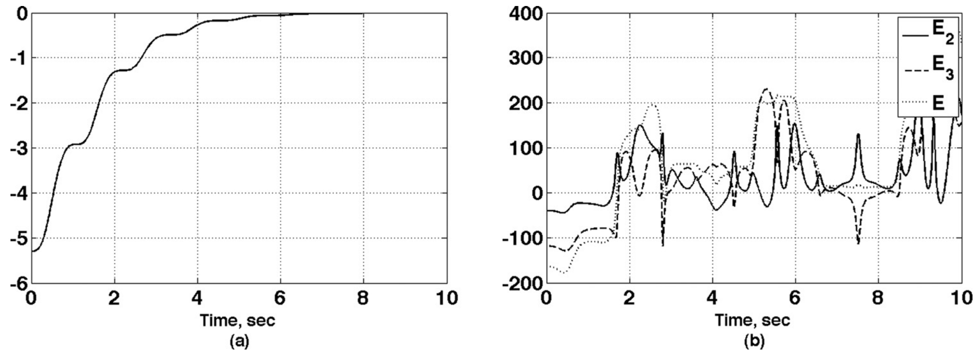

Fig. 3 Energies in N-m: (*a*) E_1; (*b*) $E = E_2 + E_3$

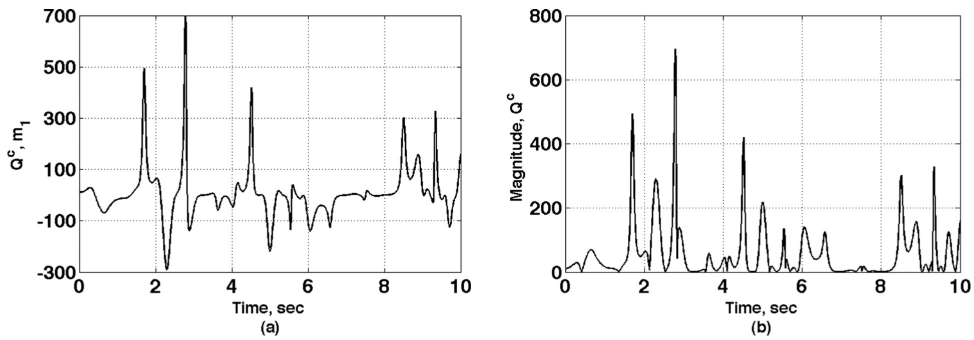

Fig. 4 (*a*) Control force applied to mass m_1 of the nominal system to satisfy $E = E_2 + E_3$; (*b*) magnitude of the control force. The control forces on masses m_2 and m_3 are zero.

the energies of mass $m_2(E_2)$ and mass $m_3(E_3)$ (i.e., $E = E_2 + E_3$). Figure 3(*a*) also shows the extent of error in satisfying this constraint, $E_1 = 0$. In Fig. 4, we show the control force Q^c in Eq. (2.8) on the nominal system in order to follow the desired constraint $E_1 = 0$. Since only the first element of the matrix A in Eq. (2.16) is nonzero, the control forces on masses m_2 and m_3 are zero, since the right-hand side of Eq. (2.8) is the product of A^T with a scalar. Figure 4(*a*) shows the control force required to be applied to mass m_1 to satisfy the constraint given in Eq. (2.15), and Fig. 4(*b*) shows its magnitude.

3 On the Dynamics of Actual Uncertain Systems

As mentioned before, there are always uncertainties in the description of any real-life dynamical systems. These uncertainties arise due to our lack of precise knowledge of the system and/or of the given forces acting on it. With the conceptualization of the nominal system given in Sec. 2, these uncertainties are now assumed to be encapsulated in the elements of the n by n matrix M and/or the n-vector Q (see Eq. (2.1)) of a dynamical system.

3.1 Description of the Actual System. We assume that the mass matrix of the uncertain real-life system, which we do not know exactly, is $M_a := M + \delta M > 0$, where $M > 0$ is the n by n nominal mass matrix—our best estimate of the mass matrix of the actual system—and δM is the n by n matrix that characterizes our uncertainty in the mass matrix of the actual system. The subscript "a" denotes the *actual*, real-life system whose knowledge is uncertain. Similarly, the given force n-vector acting on the real-life system is taken to be $Q_a := Q + \delta Q$, where the n-vector Q denotes the given force on the nominal system and δQ denotes the n-vector of uncertainty in Q.

The equation of motion of the actual unconstrained (uncontrolled) system, whose description is known only imprecisely, is then given by

$$M_a(\tilde{q}, t)\ddot{\tilde{q}} = Q_a(\tilde{q}, \dot{\tilde{q}}, t) \tag{3.1}$$

where $\tilde{q}$ is the generalized coordinate n-vector of the actual system; the n by n matrix $M_a > 0$ is the mass matrix of the actual system, which is a function of $\tilde{q}$ and t; and the n-vector Q_a is the given force acting on the actual system, which is a function of $\tilde{q}$, $\dot{\tilde{q}}$, and t. Equation (3.1) is then the description of the actual system, which is known only imprecisely, since $\delta M(\tilde{q}, t)$ and $\delta Q(\tilde{q}, \dot{\tilde{q}}, t)$ are, in general, unknown.

Our aim is to control this actual system so that it mimics the motion of the nominal system (within given error tolerances) and thereby satisfies the control requirements (constraints) in Eq. (2.4) (or equivalently Eq. (2.5)) imposed on the nominal system. With no exact knowledge of δM and δQ, the only control force that we have at hand to satisfy the control requirement in Eq. (2.4) is the one we have obtained for the nominal system—our best estimate of the actual system. We then attempt to track the trajectory of the nominal system and control the actual system so that it satisfies the trajectory requirements given by the set in Eq. (2.4) by using this control force Q^c, which is explicitly obtained in Eq. (2.8). Thus, the equation of motion of the actual system, so controlled, becomes

$$M_a\ddot{\tilde{q}} := Q_a(\tilde{q}, \dot{\tilde{q}}, t) + Q^c(t) \tag{3.2}$$

Premultiplying both sides of Eq. (3.2) by M_a^{-1}, the acceleration of the actual system is given by

$$\ddot{\tilde{q}} := M_a^{-1}Q_a(\tilde{q}, \dot{\tilde{q}}, t) + M_a^{-1}Q^c(t) \tag{3.3}$$

We note that Eq. (3.2) involves (i) the description of the actual system given by Eq. (3.1), whose parameters are only known imperfectly, and (ii) the control force $Q^c(t)$ given by Eq. (2.8), which is obtained on the basis of our best estimate of this actual system, namely, on the basis of the corresponding nominal system.

By applying this control force to the actual system described by Eq. (3.1), one obtains a different state $(\tilde{q},\dot{\tilde{q}})$ from that obtained for the (controlled) nominal system $(q,\dot{q})$. This causes the trajectories of the actual system and the nominal system to differ, with a corresponding error in satisfaction of our desired trajectory requirements in Eq. (2.4).

We note that, even if we apply the correct control force to the actual system by assuming that we have somehow gained precise knowledge of our uncertain system so that

$$M_a\ddot{q}_a = Q_a(q_a,\dot{q}_a,t) + A_a^T(A_aM_a^{-1}A_a^T)^+(b_a - A_aa_a) \tag{3.4}$$

the actual system's response $(q_a,\dot{q}_a)$ will not track the trajectory of the nominal system $(q,\dot{q})$.

We note that, in Eq. (3.4), q_a denotes the generalized coordinate n-vector of the actual system, which is obtained by using the correct control force that the actual system is required to be subjected to so that it satisfies the constraint in Eq. (2.5), namely, $A_a(q_a,\dot{q}_a,t)\ddot{q} = b_a(q_a,\dot{q}_a,t)$. In Eq. (3.4), since M_a and Q_a are assumed to be known, $a_a := M_a^{-1}Q_a$.

Premultiplying both sides of Eq. (3.4) by M_a^{-1}, the acceleration of the actual system can be expressed as

$$\ddot{q}_a = a_a + M_a^{-1}A_a^T(A_aM_a^{-1}A_a^T)^+(b_a - A_aa_a) \tag{3.5}$$

We illustrate this by continuing our example of the triple pendulum system considered in Sec. 2, with uncertainties in the masses m_1, m_2, and m_3. We assume that each mass has a random uncertainty of up to ±10% of our best estimate of it (i.e., of its nominal value). For illustrative purposes, however, we pick a specific system with $\delta m_1 = 0.1$, $\delta m_2 = -0.2$, and $\delta m_3 = 0.3$ and perform a simulation using Eq. (3.5), with all other parameter values the same as those prescribed in Sec. 2.3. We note that the elements of the 3 by 3 symmetric matrix M_a and of the 3-vector Q_a are given in a manner similar to Eqs. (2.11) and (2.12), respectively. In this case, we have replaced m_i in Eqs. (2.11) and (2.12) with $m_i = m_i + \delta m_i, i = 1,2,3$. We note that $A_a = A$ and $b_a = b$, since our constraint in Eq. (2.16) does not involve any of the masses m_i. The response of mass m_3 over a duration of 10 seconds is shown in Fig. 5 for illustration. We observe that it is vastly different from that of the corresponding nominal system shown in Fig. 2 over the same duration of time, though both systems satisfy the energy constraint in Eq. (2.15), pointing out that the actual system does not track the trajectory of the nominal system.

3.2 Description of the Controlled Actual Systems. To compensate for the uncertainty, the control force given by the second member on the right-hand side of Eq. (3.2), $Q^c(t)$, needs to be modified, since it was calculated on the basis of the nominal system and is now instead being applied to the actual unknown system. We do this by adding another control force Q^u from a compensating controller, resulting in a new state $(q_c,\dot{q}_c)$ (see Fig. 6). We define the difference between $q_c(t)$ and $q(t)$ as the tracking error $e(t)$ (see Fig. 6). In this paper, we develop this additive controller based on a generalization of the notion of a sliding surface, which is discussed in Sec. 3.2.2. A broad introduction to sliding mode control may be found in Refs. [19] and [20].

The equation of motion of the controlled actual system thus becomes

$$M_a(q_c,t)\ddot{q}_c = Q_a(q_c,\dot{q}_c,t) + Q^c(t) + Q^u \tag{3.6}$$

where q_c is the generalized coordinate n-vector of the controlled actual system, $Q^c(t)$ is the control force that is obtained from the corresponding nominal system and that causes the nominal system to satisfy the constraint in Eq. (2.5), and Q^u is the additional control force n-vector, which we shall develop in closed form. We now refer to Eq. (3.6) as the description of the "controlled actual system" or "controlled system" for short. Premultiplying both sides of Eq. (3.6) by M_a^{-1}, the acceleration of this controlled system can then be expressed as

$$\ddot{q}_c = a_a + M_a^{-1}Q^c(t) + M_a^{-1}M\ddot{u} \tag{3.7}$$

Here, $a_a := M_a^{-1}Q_a$ and $Q^u := M\ddot{u}$, where $\ddot{u}$ is the additional generalized acceleration provided by the additional control forces Q^u to compensate for uncertainties in our knowledge of the actual system.

It is important to note that the mass matrix M in Eq. (3.7) is that of the nominal system—the only mass matrix we have in hand, since the mass matrix M_a of the actual system is unknown. Hence, after we obtain a compensating control acceleration $\ddot{u}$, in order to obtain the control force, we need to multiply it with this mass matrix M, so that $Q^u = M\ddot{u}$. However, the generalized acceleration of the controlled actual system due to the compensating control acceleration $\ddot{u}$ is $M_a^{-1}Q^u := M_a^{-1}M\ddot{u}$, which is shown in the last term of Eq. (3.7). And so we observe that this term $M_a^{-1}Q^u$ still contains the mass matrix M_a of the actual system, which is uncertain! However, as shown later in the proof of Lyapunov stability in the Appendix, our control approach will take care of this uncertainty as well. Before embarking on the determination of Q^u, we consider the uncertainties in the dynamics of the mechanical system next.

3.2.1 Uncertainties in the Dynamics of Mechanical Systems. Defining the tracking error as

$$e(t) = q_c(t) - q(t) \tag{3.8}$$

and differentiating Eq. (3.8) twice with respect to time, we get

$$\ddot{e} = \ddot{q}_c - \ddot{q} \tag{3.9}$$

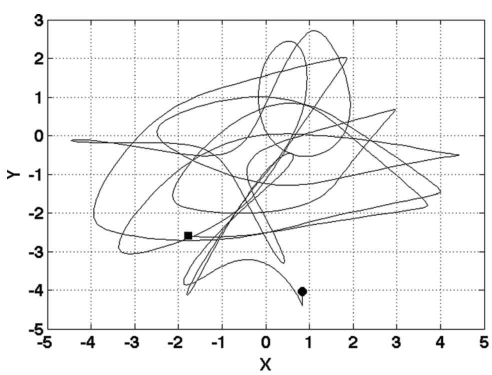

Fig. 5 Trajectory of mass m_3 of the actual system over a period of 10 s. The masses are $m_1 = 1.1$ kg ($\delta m_1 = 0.1$), $m_2 = 1.8$ kg ($\delta m_2 = -0.2$), and $m_3 = 3.3$ kg ($\delta m_3 = 0.3$). The system satisfies the energy constraint in Eq. (2.15).

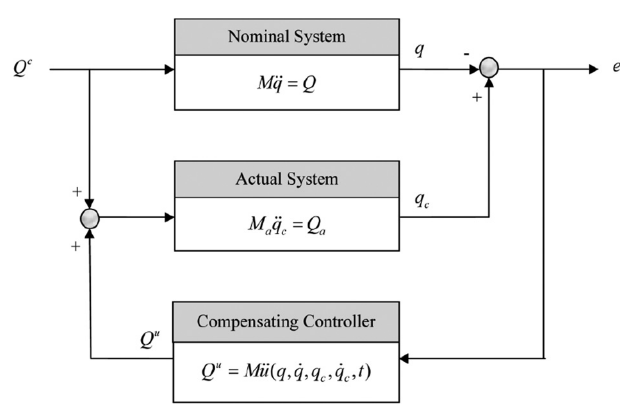

Fig. 6 The block diagram of the controlled actual system. Note that the compensating controller uses the mass matrix of the nominal system.

which, upon use of Eqs. (2.9) and (3.7), yields

$$\ddot{e} = [a_a(q_c,\dot{q}_c,t) - a(q,\dot{q},t)] + \left[M_a^{-1}(q_c,t) - M^{-1}(q,t)\right]Q^c(t) + M_a^{-1}M\ddot{u} := \delta\ddot{q} + M_a^{-1}M\ddot{u} = \delta\ddot{q} + [I - (I - M_a^{-1}M)]\ddot{u} := \delta\ddot{q} + \ddot{u} - \bar{M}\ddot{u} \quad (3.10)$$

In the above equation, we have defined

$$\bar{M} = I - M_a^{-1}(q_c,t)M(q,t) = I - (M(q_c,t) + \delta M(q_c,t))^{-1}M(q,t) = I - \left(M^{-1}(q,t)M(q_c,t) + M^{-1}(q,t)\delta M(q_c,t)\right)^{-1} \quad (3.11)$$

and denoted the acceleration $\delta\ddot{q}$ as

$$\delta\ddot{q}(q,\dot{q},q_c,\dot{q}_c,t) = [a_a(q_c,\dot{q}_c,t) - a(q,\dot{q},t)] + \left[M_a^{-1}(q_c,t) - M^{-1}(q,t)\right]Q^c(t) \quad (3.12)$$

where $a_a := M_a^{-1}Q_a$, with $M_a := M(q_c,t) + \delta M(q_c,t)$ and $Q_a := Q(q_c,\dot{q}_c,t) + \delta Q(q_c,\dot{q}_c,t)$.

The aim in this section is to find a suitable bound on $\delta\ddot{q}$, which we shall use in Sec. 3.2.2 to develop a set of additive control forces to compensate for the uncertainties involved in our knowledge of the actual multibody system.

Using Taylor's expansion, Eq. (3.12) can be expanded as

$$\begin{aligned}\delta\ddot{q}(q,\dot{q},q_c,\dot{q}_c,t) &= M_a^{-1}(q,t)Q_a(q,\dot{q},t) - M^{-1}(q,t)Q(q,\dot{q},t)\\ &+ M_a^{-1}(q,t)\left[\sum_{j=1}^{n}\frac{\partial Q_{a,i}}{\partial q_{c,j}}\Big|_{q,\dot{q},t}(q_{c,j}-q_j) + \sum_{j=1}^{n}\frac{\partial Q_{a,i}}{\partial \dot{q}_{c,j}}\Big|_{q,\dot{q},t}(\dot{q}_{c,j}-\dot{q}_j)\right]\\ &+ \left[\sum_{j=1}^{n}\frac{\partial M_{a,ik}^{-1}}{\partial q_{c,j}}\Big|_{q,t}(q_{c,j}-q_j)\right]\left[Q_a(q,\dot{q},t) + \sum_{j=1}^{n}\frac{\partial Q_{a,i}}{\partial q_{c,j}}\Big|_{q,\dot{q},t}(q_{c,j}-q_j) + \sum_{j=1}^{n}\frac{\partial Q_{a,i}}{\partial \dot{q}_{c,j}}\Big|_{q,\dot{q},t}(\dot{q}_{c,j}-\dot{q}_j)\right]\\ &+ \left\{M_a^{-1}(q,t) + \left[\sum_{j=1}^{n}\frac{\partial M_{a,ik}^{-1}}{\partial q_{c,j}}\Big|_{q,t}(q_{c,j}-q_j)\right] - M^{-1}(q,t)\right\}Q^c(t)\\ &+ H.O.T.,\ \text{for } i = 1,\dots,n \text{ and } k = 1,\dots,n \end{aligned} \quad (3.13)$$

where $H.O.T.$ denotes the higher-order terms in $(q_c - q)$ and $(\dot{q}_c - \dot{q})$.

We note that, in Eq. (3.13), $Q_{a,i}$, $q_{c,j}$, and q_j denote the corresponding ith and jth components of the n-vectors Q_a, q_c, and q, respectively. Also, $M_{a,ik}^{-1}$ represents the (i,k) element of the n by n matrix M_a^{-1}.

The aim is to develop a controller $\ddot{u}$ such that the motion of the controlled actual system closely tracks the motion of the nominal system. We assume for the moment that the compensating control acceleration $\ddot{u}$ is capable of this and causes the trajectory of the controlled actual system $(q_c,\dot{q}_c)$ to sufficiently approximate that of the nominal system so that $(q_c,\dot{q}_c) \approx (q,\dot{q})$. Under this assumption, we take the lowest-order terms in Eq. (3.13) and approximate $\delta\ddot{q}$ as

$$\delta\ddot{q}(q,\dot{q},t) \approx \left[M_a^{-1}(q,t)Q_a(q,\dot{q},t) - M^{-1}(q,t)Q(q,\dot{q},t)\right] + \left[M_a^{-1}(q,t) - M^{-1}(q,t)\right]Q^c(t) \quad (3.14)$$

and similarly approximate $\bar{M}$ as (see Eq. (3.11))

$$\bar{M} \approx I - \left(I + M^{-1}(q,t)\delta M(q,t)\right)^{-1} \quad (3.15)$$

Since [21]

$$M_a^{-1}(q,t) = [M(q,t) + \delta M(q,t)]^{-1} = M^{-1} - M^{-1}(I + \delta M\, M^{-1})^{-1}\delta M\, M^{-1} \quad (3.16)$$

expanding Eq. (3.14) and utilizing Eq. (3.16), we obtain

$$\delta\ddot{q}(t) \approx -(M+\delta M)^{-1}\delta M\, M^{-1}(Q+Q^c) + (M+\delta M)^{-1}\delta Q \quad (3.17)$$

which includes the combined effect of the uncertainties δM and δQ. By taking the norm of the relation in Eq. (3.17), one can obtain an estimate of the bound, $\Gamma(t)$, on $\|\delta\ddot{q}\|$ as

$$\|\delta\ddot{q}(t)\| \approx \left\|-(M+\delta M)^{-1}\delta M\, M^{-1}(Q+Q^c) + (M+\delta M)^{-1}\delta Q\right\| \le \Gamma(t) \quad (3.18)$$

where $\Gamma(t)$ is a positive function of time. This bound depends on δM and δQ, which in turn depends on the state of our knowledge (or ignorance) about the actual system.

A further simplification of the right-hand side of the relation in Eq. (3.18) using knowledge of only the bounds on the uncertainties in the mass matrix, $\|\delta M\|$, and the given force, $\|\delta Q\|$, under the assumption that $\|M^{-1}\delta M\| << 1$, yields

$$\|\delta\ddot{q}(t)\| \le \left(1 + \|M^{-1}\|\|\delta M\|\right)\|M^{-1}\|\left(\|M^{-1}\|\|Q+Q^c\|\|\delta M\| + \|\delta Q\|\right) \quad (3.19)$$

3.2.2 Generalized Sliding Surface Control. Having obtained an estimate of the bound $\|\delta\ddot{q}\| \le \Gamma(t)$, our aim in this section is to develop a set of compensating control forces that can guarantee the tracking of the nominal system's trajectory (to within desired error bounds), despite our uncertain knowledge of the actual system. To do this, we use a generalization of the concept of a sliding surface [19,20,22,23]. The formulation permits the use of a large class of control laws that can be adapted to the practical limitations of the specific compensating control force being used and the extent to which we want to compensate for the uncertainties.

Noting Eq. (3.10), the tracking error signal in acceleration can be expressed as

$$\ddot{e} = \delta\ddot{q} + M_a^{-1}M\ddot{u} := \delta\ddot{q} + \ddot{u} - \bar{M}\ddot{u} \quad (3.20)$$

where $\bar{M}$ can be approximated as $\bar{M} \approx I - (I + M^{-1}(q,t)\,\delta M(q,t))^{-1}$ (see Eq. (3.15)).

We note that $\|\delta\ddot{q}\| \le \Gamma(t)$. Here, we have used the bound $\Gamma(t)$ that is related to the uncertainties involved in the actual system and that is obtained from the relation in Eq. (3.18) (or Eq. (3.19)). In what follows, we shall denote $\|\cdot\|$ to mean the infinity norm.

We now define a sliding surface

$$s(t) = ke(t) + \dot{e}(t) \tag{3.21}$$

where $k > 0$ is an arbitrary small positive number and s is an n-vector. Our aim is to maneuver the system to the sliding surface $s \in \Omega_\varepsilon$, whereupon by Eq. (3.21), ideally speaking, when the size of the surface Ω_ε is zero, we obtain the relation $\dot{e} = -ke$, whose solution $e(t) = e_0 \exp(-kt)$ shows that the tracking error $e(t)$ exponentially reduces to zero along this lower-dimensional surface in phase space.

To ensure that the controlled actual system in Eq. (3.7) is restricted to the sliding surface $s \in \Omega_\varepsilon$, we apply an additional compensating control force $Q^u := M\ddot{u}$, where $\ddot{u}$ is explicitly given as (see Appendix)

$$\ddot{u} = -[k\dot{e}(t) + \sigma\beta(t)f(s)] \tag{3.22}$$

with $k > 0$. The function $\beta(t)$ is considered such that

$$\beta(t) \geq \frac{n(\Gamma(t) + \beta_0)}{\alpha_0} > 0 \tag{3.23}$$

where

$$\beta_0 > k\|\bar{M}(t)\|\|\dot{e}(t)\| \text{ and } 0 < \alpha_0 < 1 - n\sigma\|\bar{M}(t)\| \tag{3.24}$$

are any arbitrary positive constants over the time duration over which the control is applied. As noted below, the function $f(s)$ in Eq. (3.22) belongs to the set of continuously differentiable functions.

The positive constant σ is chosen such that

$$\gamma \leq \sigma \leq 1 \tag{3.25}$$

where

$$\gamma := \frac{\|s\|\|f(s)\|}{s^T f(s)} \leq 1 \tag{3.26}$$

We note that, since $\gamma \leq 1$, the choice $\sigma = 1$ would suffice in Eq. (3.23) when choosing α_0.

The ith component, $f_i(s)$, of the n-vector $f(s)$ is defined as

$$f_i(s) = g(s_i/\varepsilon), \quad i = 1, ..., n \tag{3.27}$$

where s_i is the ith component of the n-vector s, ε is defined as any (small) positive number, and the function $g(s_i/\varepsilon)$ is *any* arbitrary monotonically increasing odd continuously differentiable function of s_i on the interval $(-\infty, +\infty)$ that satisfies

$$\|f_i(s)\| = \|g(s_i/\varepsilon)\| \geq \frac{\Gamma(t) + k\|\bar{M}(t)\|\|\dot{e}(t)\|}{\Gamma(t) + \beta_0}, \quad \text{if } s_i \text{ is outside the surface } \Omega_\varepsilon(t) \tag{3.28}$$

where $\Omega_\varepsilon(t)$ is defined as the surface of the n-dimensional cube around the point $s = 0$, each of whose sides has a computable length (as shown below). We note that the right-hand side of the relation in Eq. (3.28) is always less than unity, since $\beta_0 > k\|\bar{M}(t)\|\|\dot{e}(t)\|$, and hence the relation in Eq. (3.28) will always be satisfied when $\|f(s)\| \geq 1$.

The control force $Q^u := M\ddot{u}$, where $\ddot{u}$ is defined as in Eq. (3.22), ensures that the controlled actual system is restricted to a region (which could be made as close to the surface $s = 0$ as we desire) around the sliding surface. The proof is given in the Appendix, where we also show that the asymptotic bounds on the errors in tracking the displacement and velocity of the nominal system are respectively given by

$$\|e(t)\| \leq \frac{L_\varepsilon}{2k} \text{ and } \|\dot{e}(t)\| \leq L_\varepsilon, \quad \text{as } t \to \infty \tag{3.29}$$

where

$$L_\varepsilon(t) <\approx 2\varepsilon g^{-1}[(\Gamma(t) + k)/(\Gamma(t) + \beta_0)] \tag{3.30}$$

For ease of implementation, one could *choose* the function $\Gamma(t)$ to be a constant by taking it to be the upper bound, Γ_m, so that $\|\delta\ddot{q}(t)\| \leq \Gamma_m$ for $t \in [0, T]$, where $[0, T]$ is the interval over which the control is applied. The relation in Eq. (3.30) then becomes a constant

$$L_\varepsilon <\approx 2\varepsilon g^{-1}[(\Gamma_m + k)/(\Gamma_m + \beta_0)] \tag{3.31}$$

Main result. The closed-from generalized sliding surface control described above for the uncertain system,

$$\begin{aligned} M_a\ddot{q}_c &= Q_a + Q^c(t) + M\ddot{u} \\ &= Q_a + Q^c(t) - M\left[k\dot{e} + n\sigma\left(\frac{\Gamma(t) + \beta_0}{\alpha_0}\right)f(s)\right] \end{aligned} \tag{3.32}$$

where:

(i) the control force $Q^c(t)$ is given by Eq. (2.8) and is obtained on the basis of the nominal system
(ii) $k > 0$ is an arbitrary small positive number
(iii) σ can be chosen to be unity, and for the function $f(s)$, *any* arbitrary monotonically increasing odd continuous function of s on the interval $(-\infty, +\infty)$, as described in Eq. (3.27), with $\|f(s)\| \geq 1$ outside Ω_ε would be sufficient
(iv) $\|\delta\ddot{q}(t)\| \leq \Gamma(t)$, where $\Gamma(t)$ is chosen based on the estimate of $\|\delta\ddot{q}(t)\|$ from Eq. (3.18)
(v) α_0 is a small positive number that satisfies

$$0 < \alpha_0 < 1 - n\sigma\|\bar{M}(t)\| \tag{3.33}$$

over the time duration over which the control is done

(vi) Under the proviso and the expectation that $\|\bar{M}\|\|\dot{e}\| << 1$, β_0 is chosen such that

$$\beta_0 = k \tag{3.34}$$

will cause the actual system to track the trajectory of the nominal system within the estimated error bounds given by Eq. (3.29)

Proof. Using Eq. (3.9) in Eq. (3.20), we have

$$\ddot{e} = \ddot{q}_c - \ddot{q} = \delta\ddot{q} + M_a^{-1}M\ddot{u} \tag{3.35}$$

so that

$$\ddot{q}_c = \ddot{q} + \delta\ddot{q} + M_a^{-1}M\ddot{u} \tag{3.36}$$

Consider Eq. (3.12),

$$\begin{aligned} \delta\ddot{q} &= (a_a - a) + \left(M_a^{-1} - M^{-1}\right)Q^c(t) \\ &= \left(a_a + M_a^{-1}Q^c(t)\right) - \left(a + M^{-1}Q^c(t)\right) \\ &= a_a + M_a^{-1}Q^c(t) - \ddot{q} \end{aligned} \tag{3.37}$$

In the last equality above, we have used Eq. (2.9). Substituting Eq. (3.37) in Eq. (3.36), we then get

$$\ddot{q}_c = a_a + M_a^{-1}Q^c(t) + M_a^{-1}M\ddot{u} \tag{3.38}$$

Premultiplying both sides of Eq. (3.38) by M_a, we obtain

$$M_a \ddot{q}_c = Q_a + Q^c(t) + M\ddot{u} \tag{3.39}$$

Finally, using Eqs. (3.22) and (3.29), the main result follows. ■

4 Numerical Results and Simulations

In this section, we continue to illustrate the methodology in the presence of uncertainties by considering the same example of the triple pendulum. The approach is straightforward to apply to other systems. While our nominal system has $m_1 = 1, m_2 = 2$, and $m_3 = 3$, there is an uncertainty of up to $\pm 10\%$ in each of these values when describing the actual system.

With imperfect knowledge of the parameters in the system, in order to control the actual system's motion so that it tracks the motion of the controlled nominal system and thereby satisfies the constraints imposed on the nominal system, we would have to use Eq. (3.32), which contains the additional control force to compensate for our uncertainty in the knowledge of the actual system.

We next select the structure and parameters for the controller $\ddot{u}$ given by Eq. (3.22). We choose

$$f_i(s) = \alpha_c (s_i/\varepsilon)^3 \tag{4.1}$$

where $\alpha_c, \varepsilon > 0$ and ε is a suitable small number. We then obtain in closed form the additional controller needed to compensate for uncertainties in the actual system as

$$\ddot{u}_i(t) = -k\dot{e}_i - n\sigma \left(\frac{\Gamma(t) + \beta_0}{\alpha_0} \right) \alpha_c (s_i/\varepsilon)^3 \tag{4.2}$$

We note that, with this choice of $f_i(s) = \alpha_c (s_i/\varepsilon)^3$, the region outside the surface Ω_ε is the region outside of the n-dimensional cube around $s = 0$, each of whose sides has length $L_\varepsilon <\approx 2\varepsilon((\Gamma_m + k)/\alpha_c(\Gamma_m + \beta_0))^{1/3}$(see Eq. (3.31)). In this region, Eq. (A.15) (see Appendix) assures us that the control given by Eq. (4.2) will cause $s(t)$ to strictly decrease until it reaches the boundary $s \in \Omega_\varepsilon$ and remains on or inside this n-box thereafter.

Premultiplying both sides of Eq. (3.32) by M_a^{-1} and using the additional controller Eq. (4.2), we obtain the closed-form equation of motion of the controlled actual system as

$$\ddot{q}_c = a_a + M_a^{-1} Q^c(t) - M_a^{-1} M \left[k\dot{e} + n\sigma \left(\frac{\Gamma(t) + \beta_0}{\alpha_0} \right) \alpha_c (s/\varepsilon)^3 \right] \tag{4.3}$$

which will cause the actual system to track the trajectory of the nominal system, thereby compensating for the uncertainty in our knowledge of the actual system.

However, while we have no knowledge of the actual parameters, in order to affect a compensating controller, a suitable bound on the uncertainty in $\delta\ddot{q}$ is required. We next estimate Γ_m and $\Gamma(t)$. We note that $\Gamma_m \geq \Gamma(t)$, where $\Gamma(t)$ is the bound on $\delta\ddot{q}$ (see Eq. (3.18)) in the presence of the ± 10 percent uncertainties in each of the masses m_1, m_2, and m_3, as described in Sec. 3.1. In order to estimate Γ_m and $\Gamma(t)$, we use Eq. (3.18) and perform a Monte Carlo simulation using 1014 uniformly distributed, independent samples of the uncertain masses δm_1, δm_2, and δm_3. The location of the actual masses for each sample is shown in Fig. 7, and the probability density function of $\|\delta\ddot{q}\|$ (at each instant of time t) that is obtained is shown in Fig. 8.

The mass properties of our actual system, though unknown, lie somewhere inside the box shown in Fig. 7. In order to illustrate the efficacy of our control force in compensating for our lack of exact knowledge of the actual system, we pick the set $\delta m_1 = 0.1$, $\delta m_2 = -0.2$, and $\delta m_3 = 0.3$, which is assumed to represent our actual system. To check the performance of our controller, we

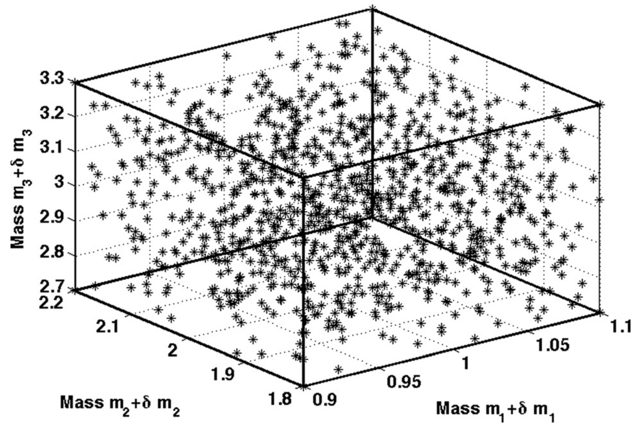

Fig. 7 The three masses $m_i \pm \delta m_i$, $i = 1, 2, 3$, of the actual system lie somewhere in the box shown. The figure shows 1014 uniformly distributed random points generated from a Monte Carlo simulation.

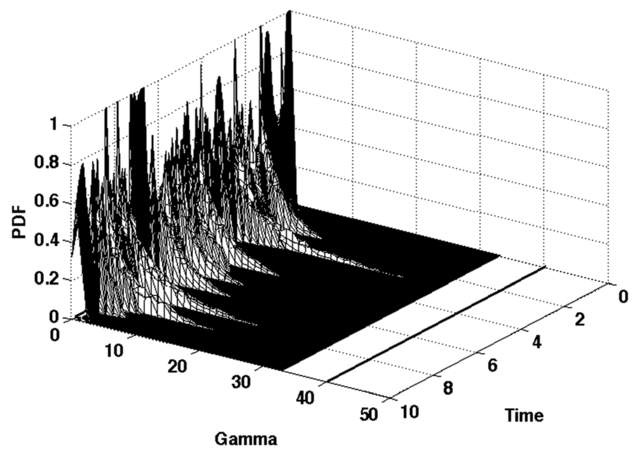

Fig. 8 Probability density function of $\|\delta\ddot{q}\|$ at each time t using Eq. (3.18) for the 1014 simulation points in which the masses have $\pm 10\%$ uncertainties

perform a simulation using Eq. (4.3) by choosing $\Gamma_m = \Gamma(t) = 40$, as shown in Fig. 8 (solid line), and using the parameters $n = 3$, $k = 10, \beta_0 = k, \alpha_c = 2, \alpha_0 = 0.01, \sigma = 1$, and $\varepsilon = 10^{-2}$ to specify our controller. All the other parameter values are the same as those prescribed in Sec. 2.3. We note that the chosen set of deviations from the nominal values ($m_1 + \delta m_1 = 1.1$, $m_2 + \delta m_2 = 1.8$, and $m_3 + \delta m_3 = 3.3$) represents simply one possibility out of the random triples shown in Fig. 7.

Though the value of Γ_m has been obtained here through Monte Carlo simulation, in general, such a simulation may be unfeasible for complex multibody systems, and an estimate of Γ_m would need to be made based on experience, on the level of perceived uncertainty in the system, and available data. However, experimentation with the value of the bound Γ_m shows that the magnitude of the additional control force Q^u is insensitive to its overestimation and identical results are obtained when using $\Gamma_m = 100$ and $\Gamma_m = 200$. Hence, only a rough estimate of Γ_m is required and conservatively overestimating Γ_m causes negligible change in Q^u.

The constrained trajectory of mass m_3 in the XY-plane of the controlled actual system is illustrated in Fig. 9. We see that the controlled system (given by Eq. (4.3) and shown in Fig. 9) tracks the nominal system (given by Eq. (2.17) and shown in Fig. 2), while, as shown before, the actual system (given by Eq. (3.5) and shown in Fig. 5) deviates from the desired nominal system. We note that all three systems satisfy the energy constraint in

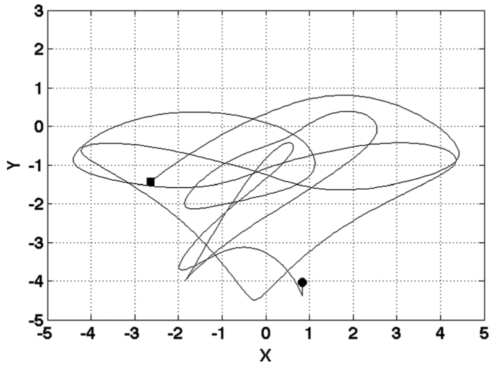

Fig. 9 Trajectory response (meter) of mass m_3 over a period of 10 s of the controlled actual system when the uncertainties in the masses are prescribed as $\delta m_1 = 0.1$ kg, $\delta m_2 = -0.2$ kg, and $\delta m_3 = 0.3$ kg and the uncertainty bound in Eq. (4.3) is chosen to be $\Gamma(t) = 40$

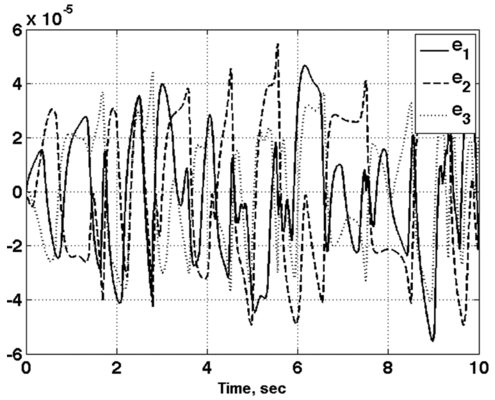

Fig. 10 Tracking errors between the controlled nominal system and the controlled actual system ($e_i(t) := \theta_i(t) - \theta_{c_i}(t)$, $i = 1, 2, 3$) in radians of the masses m_1, m_2, and m_3

Eq. (2.15). This illustrates the performance of the closed-form Eq. (4.3), showing that the controlled actual system tracks the trajectories prespecified by the nominal system in the presence of the $\pm 10\%$ uncertainties in masses of the triple pendulum and the constraint imposed on it, given by Eq. (2.15). Figure 10 shows the displacement errors $(q - q_c)$ between the nominal system in Eq. (2.17) and the controlled actual system in Eq. (4.3). The tracking errors are small, which are seen to be of $O(10^{-5})$. We see that these errors are within the estimated error norms $\|e(t)\| <\approx L_\varepsilon/2k \approx 8 \times 10^{-4}$, as prescribed by Eqs. (3.29) and (3.31), where

$$L_\varepsilon <\approx 2\varepsilon((\Gamma_m + k)/\alpha_c(\Gamma_m + \beta_0))^{1/3} \approx 1.6 \times 10^{-2} \tag{4.4}$$

We note that use of the smooth cubic function $f_i(s)$ given in Eq. (4.1) eliminates chattering.

Premultiplying Eq. (4.3) by M_a, we obtain (see Eq. (3.32))

$$M_a \ddot{q}_c = Q_a + Q^c - M\left(k\dot{e} + n\sigma\left(\frac{\Gamma(t) + \beta_0}{\alpha_0}\right)\alpha_c (s/\varepsilon)^3\right)$$
$$:= Q_a + Q^c + M\ddot{u} := Q_a + Q^c + Q^u \tag{4.5}$$

The total control force applied to the actual system is given by $Q^T = Q^c + Q^u$. Here, Q^c is the control force obtained from the nominal system and Q^u is the force applied by the additional compensating controller to compensate for our inexact knowledge of the actual system. The control forces Q^T and Q^u on the masses m_1, m_2, and m_3 of the actual pendulum are shown in Fig. 11. The magnitude of the additional control forces, Q^u, applied by the compensating controller is seen to be small relative to the magnitude of the total control forces, Q^T.

5 Conclusion

In this paper, a set of closed-form control forces for uncertain nonlinear multibody mechanical systems is developed. These control forces are able to guarantee tracking of a desired reference trajectory within prescribed error bounds, which the nominal system—our best estimate of the actual real-life situation—is

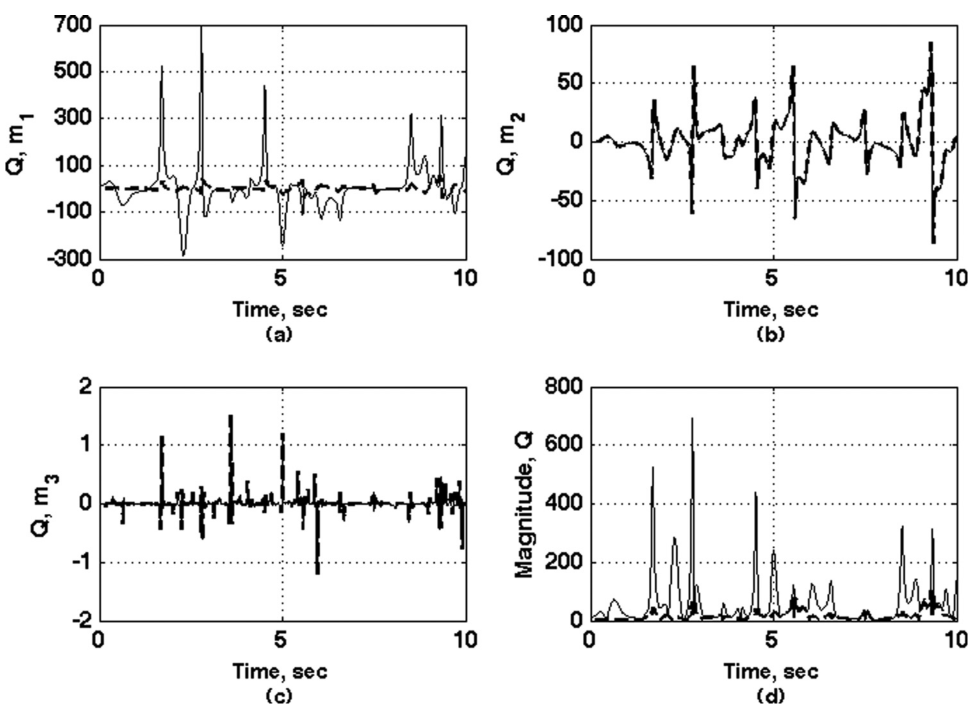

Fig. 11 Control forces (newtons) on the controlled actual system. The solid line shows the total control force, Q^T, and the dashed line shows the additional force, Q^u, needed to compensate for uncertainties in the actual system.

required to follow. The control is carried out in a two-step process. The nominal system's control is done through the use of results obtained in analytical dynamics by recasting the control problem as a problem of constrained motion. The requisite closed-form control minimizes the control cost at each instant of time. The uncertainty is handled through an additional additive controller based on a generalized sliding surface. The approach is simple (when compared with SDRE or backstepping) and yields the control of the nominal system in closed-form while allowing continuously differentiable functions to be used in the design of the additional controller that takes care of the uncertainties. The final closed-form continuous controller provides a new, analytical dynamics–based approach for handling the tracking control problem for uncertain multibody mechanical systems that may be highly nonlinear and nonautonomous.

The main contributions of this paper are:

(i) We obtain the exact closed-form solution to the energy control problem of a multibody system through the use of analytical dynamics. The control force that must be applied to the system because of the presence of the energy constraint imposed on the system is easily obtained. Also, when starting with initial states that do not satisfy this energy requirement, the error in satisfying it converges to zero exponentially.

(ii) The general closed-form equation of motion for uncertain nonlinear multibody systems—the so-called controlled actual system—has been developed. The novelty in the approach developed here is that we first use the fundamental equation to obtain an exact control force of the nominal, nonlinear, nonautonomous, mechanical system. Appeal here is made to results in analytical dynamics rather than control theoretic approaches. This control force, Q^c, ensures that the trajectory constraints are exactly satisfied by the nominal system and that it optimizes the control cost given by $Q^{c^T} M^{-1} Q^c$ at *each* instant of time. More general control costs can also be considered, as in Ref. [17]. Control of the actual system, in which both the mass matrix and the given force vector may be only imprecisely known, is then carried out using the concept of generalized sliding surfaces.

(iii) We have generalized the concept of a sliding surface by including continuously differentiable functions, $f_i(s)$, as opposed to the standardly used discontinuous signum functions and saturation functions [19,20,22,23]. This results in trajectories approaching the sliding surface, and a bound on the distance, within which they remain from the surface, is analytically obtained. The control functions, $f_i(s)$, and the parameters that define the compensating control force can therefore be chosen depending on practical considerations of the control environment and on the extent to which the compensation of the uncertainties is desired. The parameters can be adjusted so that desired error bounds can be guaranteed when the uncertain system is required to track the nominal system. Thus, when dealing with large, complex multibody systems, greater flexibility is afforded. For example, the use of a cubic function may obviate the need for a high-gain controller and would also allow continuous control, thereby preventing chattering.

(iv) For brevity, we have illustrated through numerical examples uncertainties that are related to the properties of a simple physical system. And here too we have illustrated the effectiveness of the approach when only the mass properties of the system are uncertain. While such uncertainties are often the most pernicious, the formulation of the methodology encompasses both general sources of uncertainties—uncertainties in the description of the physical system and uncertainties in knowledge of the given forces applied to the system. The set of closed-form control forces developed herein is therefore general enough to be applicable to complex dynamical systems in which uncertainties of both these types may arise.

Appendix

Differentiating Eq. (3.21) with respect to time and using Eq. (3.20), we get

$$\dot{s} = k\dot{e} + \ddot{e} = k\dot{e} + \delta\ddot{q} + \ddot{u} - \bar{M}\ddot{u} \tag{A.1}$$

Since $(\dot{q}_c - \dot{q})$ can be measured, to cancel the known term $k\dot{e} = k(\dot{q}_c - \dot{q})$ in Eq. (A.1), we choose the controller $\ddot{u}$ to be of the form

$$\ddot{u} = -k\dot{e}(t) + v(t) \tag{A.2}$$

so that

$$\dot{s} = v(t) + \delta\ddot{q}(t) - \bar{M}(t)[-k\dot{e}(t) + v(t)] \tag{A.3}$$

We note again that $\|\delta\ddot{q}\| \le \Gamma(t)$. Here, we have used the bound $\Gamma(t)$ that is related to the uncertainties involved in the actual system and that is obtained from the relation in Eq. (3.18). In what follows, we shall denote $\|\cdot\|$ to mean the infinity norm.

We now define a control n-vector $v(t)$ so that

$$v(t) := -\sigma\beta(t)f(s) \tag{A.4}$$

where the function $\beta(t)$, the positive constant σ, and the n-vector $f(s)$ are defined as in Eqs. (3.23), (3.25), and (3.27), respectively.

We shall now show that the system in Eq. (3.7) can indeed be maneuvered to the sliding surface $s \in \Omega_\varepsilon$ when Ω_ε is defined as any appropriately small surface around $s = 0$, whose exact description will be shortly discussed.

Result. The control law

$$\ddot{u} = -k\dot{e}(t) + v(t) = -[k\dot{e}(t) + \sigma\beta(t)f(s)] \tag{A.5}$$

with $k > 0$ and $v(t)$ defined in Eq. (A.4) will cause $s(t) \to \Omega_\varepsilon$.

Proof. Consider the Lyapunov function

$$V = \frac{1}{2}s^T s \tag{A.6}$$

Differentiating Eq. (A.6) once with respect to time, we get

$$\dot{V} = s^T \dot{s} \tag{A.7}$$

Substituting Eqs. (A.3) in (A.7), we have

$$\dot{V} = s^T(t)v(t) + s^T(t)\delta\ddot{q}(t) + ks^T(t)\bar{M}(t)\dot{e}(t) - s^T(t)\bar{M}(t)v(t) \tag{A.8}$$

Then, using Eq. (A.4) in Eq. (A.8), we obtain

$$\dot{V} = -\sigma\beta s^T f(s) + s^T\delta\ddot{q} + ks^T\bar{M}\dot{e} + \sigma\beta s^T\bar{M}f(s) \tag{A.9}$$

so that

$$\dot{V} \le -\sigma\beta s^T f(s) + \|s^T\|\|\delta\ddot{q}\| + k\|s^T\|\|\bar{M}\|\|\dot{e}\| + \sigma\beta\|s^T\|\|\bar{M}\|\|f(s)\| \tag{A.10}$$

Then, using the relation $\|\delta\ddot{q}\| \le \Gamma(t)$, we obtain

$$\dot{V} \le -\sigma\beta s^T f(s) + \|s^T\|\Gamma(t) + k\|s^T\|\|\bar{M}\|\|\dot{e}\| + \sigma\beta\|s^T\|\|\bar{M}\|\|f(s)\| \tag{A.11}$$

Since (see Eqs. (3.25) and (3.26))

$$\sigma s^T f(s) \geq \|s\| \|f(s)\| \tag{A.12}$$

the relation in Eq. (A.11) becomes

$$\begin{aligned} \dot{V} &\leq -\|s^T\| \left(\beta \frac{\|s\|}{\|s^T\|} \|f(s)\| - \sigma\beta\|\bar{M}\|\|f(s)\| - \Gamma(t) - k\|\bar{M}\|\|\dot{e}\| \right) \\ &\leq -\|s^T\| \left[\beta\left(\frac{1}{n} - \sigma\|\bar{M}\|\right)\|f(s)\| - \Gamma(t) - k\|\bar{M}\|\|\dot{e}\| \right] \\ &= -\|s^T\| \left[\frac{\beta}{n}(1 - n\sigma\|\bar{M}\|)\|f(s)\| - \Gamma(t) - k\|\bar{M}\|\|\dot{e}\| \right] \end{aligned} \tag{A.13}$$

where the second inequality follows because $\|s\|/\|s^T\| \geq 1/n$.

Using Eq. (3.23) in Eq. (A.13), we then have

$$\begin{aligned} \dot{V} &\leq -\|s^T\| \left[\frac{(\Gamma(t) + \beta_0)}{\alpha_0}(1 - n\sigma\|\bar{M}\|)\|f(s)\| - \Gamma(t) - k\|\bar{M}\|\|\dot{e}\| \right] \\ &\leq -\|s^T\| [(\Gamma(t) + \beta_0)\|f(s)\| - \Gamma(t) - k\|\bar{M}\|\|\dot{e}\|] \end{aligned} \tag{A.14}$$

where the last inequality follows because $(1 - n\sigma\|\bar{M}\|)/\alpha_0 > 1$.

Since, by Eq. (3.28),

$$(\Gamma(t) + \beta_0)\|f(s)\| - \Gamma(t) - k\|\bar{M}(t)\|\|\dot{e}(t)\| := \Delta(t) \geq 0$$

outside the surface $\Omega_\varepsilon(t)$, we have

$$\dot{V} \leq -\|s^T\|\Delta(t), \text{outside the surface } \Omega_\varepsilon(t) \tag{A.15}$$

so that the derivative $\dot{V}$ is negative, and we have convergence to the closed set interior to the region enclosed by the surface Ω_ε.

Thus, for the right-hand side of the relation in Eq. (A.14) to be negative, we require the relation in Eq. (3.28), namely,

$$\|f(s)\| = \|g(s/\varepsilon)\| \geq \frac{\Gamma(t) + k\|\bar{M}(t)\|\|\dot{e}(t)\|}{\Gamma(t) + \beta_0} := \Xi(t) \tag{A.16}$$

where, as noted in Sec. 3.2.2, $\Xi(t) < 1$. The relation in Eq. (A.16) then yields

$$\|s\| \geq \varepsilon g^{-1}[\Xi(t)] \tag{A.17}$$

In the region in which $\|s\|$ satisfies Eq. (A.17), the Lyapunov derivative $\dot{V}$ is negative. This shows us that the controller in Eq. (A.5) will cause $s(t)$ to decrease until it reaches the boundary $s \in \Omega_\varepsilon(t)$. Further, since $\Xi(t) < 1$ and the function g(*) is a monotonically increasing function, $\Omega_\varepsilon(t)$ is enclosed in an n-dimensional cube of constant size around the point $s = 0$, each of whose sides has length

$$L_\varepsilon(t) = 2\varepsilon g^{-1}[\Xi(t)] < 2\varepsilon g^{-1}(1) := \Sigma \tag{A.18}$$

This gives an estimate of the n-dimensional cubical region Ω_ε, each of whose sides is estimated to be of *constant* length Σ, to which trajectories of the controlled actual system will be attracted to.

Noting the fact that $\|s(t)\|$ is bounded by $L_\varepsilon/2$ inside the surface Ω_ε, we now have an estimate of the error bounds given by

$$\|e(t)\| \leq \frac{\Sigma}{2k} \text{ and } \|\dot{e}(t)\| \leq \Sigma, \text{ as } t \to \infty \tag{A.19}$$

Further, under the proviso $\|\bar{M}(t)\|\|\dot{e}(t)\| < 1$ for $t \in [0, T]$, where $[0, T]$ is the interval over which the control is applied, which is something that we expect, we then have

$$L_\varepsilon(t) <\approx 2\varepsilon g^{-1}[(\Gamma(t) + k)/(\Gamma(t) + \beta_0)] \tag{A.20}$$

For ease of implementation, one could *choose* the function $\Gamma(t)$ to be a constant by taking it to be the upper bound, Γ_m, so that $\|\delta\ddot{q}(t)\| \leq \Gamma_m$ for $t \in [0, T]$, where $[0, T]$ is the interval over which the control is applied. Then, the relation in Eq. (A.20) becomes

$$L_\varepsilon <\approx 2\varepsilon g^{-1}[(\Gamma_m + k)/(\Gamma_m + \beta_0)] \tag{A.21}$$

where L_ε is a constant. One can then, accordingly, obtain an estimate of the error bounds by replacing Σ in the expressions in Eq. (A.19) by the expression on the right-hand side of Eq. (A.21).

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