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Uniqueness of Damping and Stiffness Distributions in the Identification of Soil and Structural Systems

As the interest in the seismic design of structures has increased considerably over the past few years, accurate predictions of the dynamic responses of soil and structural systems has become necessary. Such predictions require a knowledge of the dynamic properties of the systems under consideration. This paper is concerned with the uniqueness of the results in the identification of such properties. More specifically, the damping and stiffness distributions, which are of importance in the linear range of response, have been investigated. An N-storied structure or an N-layered soil medium is modeled as a coupled, N-degree-of-freedom, lumped system consisting of masses, springs, and dampers. Then, assuming the mass distribution to be known, the problem of identification consists of determining the stiffness and damping distributions from the knowledge of the base excitation and the resulting response at any one mass level. It is shown that if the response of the mass immediately above the base is known, the stiffness and damping distributions can be uniquely determined. Following this, some nonuniqueness problems have been discussed in relation to the commonly used ideas of system reduction in the study of layered soil media. A numerical example is provided to verify some of these concepts and the nature of nonuniqueness of identification is indicated by showing how two very different (yet physically reasonable) systems could yield identical excitation-response pairs. Errors in the calculation of the dynamic forces, due to erroneous identification have also been illustrated thus making the results of the present study useful from the practical standpoint of the safe design of structures to ground shaking.

Introduction

The development of dynamic models is necessary for predicting the vibratory response of soil and structural systems to various time histories of ground shaking. Typically, the construction of such mathematical models, for systems that respond linearly, would require a knowledge of the mass, stiffness, and damping distributions throughout the system. Several investigators [1–3][1] have worked in the past on identifying these dynamic properties of soil and structural systems by testing full scale as well as laboratory scale models. With the recent interest in the aseismic design of structures, more and more structures all over the world are being instrumented with strong motion accelerographs, the aim being to determine the structural properties from records obtained during the high level excitations created by ground shocks, earthquakes, etc. Many structures have been instrumented with two accelerographs, one of which is placed in the basement of the structure while the other is placed at some floor level. It is the problem of identification of structural parameters from such "input-output" records that is addressed in this paper.

Though several researchers have tried to deduce the dynamic models from such input-output data in the past, few [4] have tried to investigate the uniqueness aspects associated with the inverse problem. Studies of this kind have been primarily limited in the past to undamped systems [4].

In this paper we treat an N-story structural system as a damped spring-mass system having N degrees of freedom. The identification problem consists of determining the stiffness and damping distri-

[1] Numbers in brackets designate References at end of paper.

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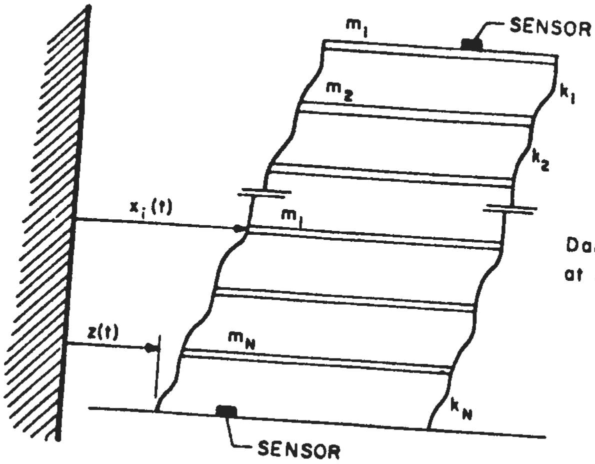
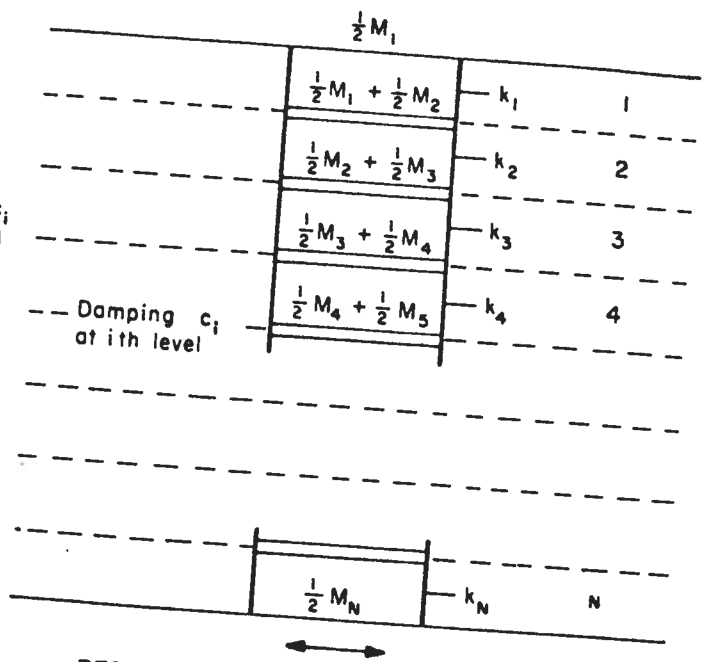


RESPONSE OF N-STORY SHEAR STRUCTURES

Fig. 1(a)



RESPONSE ANALYSIS OF LAYERED SYSTEMS

Fig. 1(b)

butions along the height of the structure from a knowledge of the base excitation and the response of one of the floors (Fig. 1(*a*)). The analysis presented is applicable to, and was motivated in part by, the problem of identification of soil properties in an N-layered medium modeled as an N-degree-of-freedom damped system, where the "base-rock" excitation and the response of one of the layers is known (Fig. 1(*b*)) [5]. We assume that the mass distribution in both the structural and the soil system is known so that the identification problem reduces to determining the damping and the stiffness distributions if we limit ourselves to the linear case.

The investigation points out that though the inverse problem is ill-posed, thus requiring great care in the interpretation of the dynamic models so obtained, unique identification is however possible if the second sensor is properly located. These results for damped systems form a logical extension of the results presented elsewhere for undamped systems [4]. Several extensions of the results in [4] have also been carried out so as to provide physical insight into the nature of the nonuniqueness problem. Limitations of identification carried out using input-output records obtained from building systems and layered soil columns have also been presented. Numerical examples have been provided to illustrate the nature and extent of nonuniqueness.

Problem Statement

Consider an N-layered soil stratum, or an N-story structure modeled as an N-degree-of-freedom lumped mass system (Fig. 1). The masses are represented by m_i, $i = 1, 2, \ldots, N$, the stiffness by k_i, $i = 1, 2, \ldots, N$, and the intermass dampings by c_i, $i = 1, 2, \ldots, N$. From a knowledge of the base motion $z(t)$ and the recorded response at a particular floor level, the stiffnesses k_i and damping values c_i are to be determined. Denoting by $x_n(t)$ the absolute motion of the nth mass, due to the base excitation $z(t)$, we have

$$M\ddot{\mathbf{x}} + C\dot{\mathbf{x}} + K\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ c_N\dot{z} + k_N z \end{bmatrix}$$

where $M = \mathrm{diag}\,(m_1, m_2, \ldots, m_N)$

$$K = \begin{bmatrix} k_1 & -k_1 & & & \\ -k_1 & k_1 + k_2 & -k_2 & & \\ & & \ddots & & \\ & & & & -k_{N-1} \\ & & & -k_{N-1} & k_{N-1} + k_N \end{bmatrix}$$

and

$$C = \begin{bmatrix} c_1 & & & & \\ -c_1 & c_1 + c_2 & -c_2 & & \\ & & \ddots & & \\ & & & & -c_{N-1} \\ & & & & c_{N-1} + c_N \end{bmatrix} \tag{1}$$

Taking the Laplace transform of both sides of equation (1) and denoting the transform variable by s we have

$$(s^2M + sC + K)\mathbf{X}(s) = (0, 0, \ldots, 0, 0, c_Ns + k_N)^T Z(s) \tag{2}$$

where $\mathbf{X}(s)$ and $Z(s)$ are the transforms of $\mathbf{x}(t)$ and $z(t)$, respectively.

Equation (2) can be further expressed as

$$A\mathbf{X}(s) = \mathbf{f}(s)Z(s) \tag{3}$$

where

$$A = \begin{bmatrix} a_1(s) & -b_1(s) & & \\ -b_1(s) & a_2(s) & & \\ & & \ddots & \\ & & & -b_{N-1}(s) \\ & & -b_{N-1}(s) & a_N(s) \end{bmatrix}$$

and $\mathbf{f}(s) = (0, 0, \ldots, b_N(s))^T$ with

$$\left.\begin{aligned} b_i(s) &= c_is + k_i, & 0 \le i \le N \\ a_i(s) &= m_is^2 + b_{i-1}(s) + b_i(s), & 1 \le i \le N \\ \text{and} \quad k_0 &= c_0 = 0 & \end{aligned}\right\} \tag{4}$$

Denoting the determinant of the upper left $i \times i$ submatrix of A by $P_i(s)$ we can solve equation (3) for the transform of ith floor response $X_i(s)$ as

$$X_i(s) = \frac{\Delta_i(s)}{P_N(s)} Z(s) \tag{5}$$

where $\Delta_i(s)$ is the determinant of the matrix obtained from A by replacing its ith column by the vector $\mathbf{f}$.

The identification problem we are studying here can be restated as follows: Given the masses m_i, $1 \le i \le N$, and the base input $Z(s)$ we want to find a j between 1 and N such that given $X_j(s)$, the stiffness k_i and the dampings c_i for $1 \le i \le N$ can be uniquely determined.

In order to do this we first present some useful properties of the polynominals $P_i(s)$ and those of a set of auxilliary functions $Q_i(s)$ in the following Lemmas.

Lemma 1:

(*a*) The functions $P_i(s)$ defined earlier satisfy the recursion relation

$$P_i(s) = a_i(s)P_{i-1}(s) - b_{i-1}^2(s)P_{i-2}, \quad \text{for} \quad 2 \le i \le N \tag{6}$$

where $P_0(s) = 1$ and $P_1(s) = a_1(s)$.

(*b*) Each $P_i(s)$ is a polynomial of degree $2i$ such that

$$\lim_{s\to\infty} \frac{P_i(s)}{s^{2i}} = m_1 m_2 m_3 \ldots m_i, \quad \text{for} \quad 1 \le i \le \text{N}, \tag{7}$$

and

$$(c) \quad \lim_{s\to\infty} \frac{P_i(s)}{s^2 P_{i-1}(s)} = m_i \quad \text{and} \quad \lim_{s\to\infty} \frac{b_i(s)}{s} = c_i, \quad \text{for} \quad 1 \le i \le N \tag{8}$$

Proof:

(*a*) Follows directly from the definitions of the P_i's.

(*b*) The proof follows by induction: For $i = 1$,

$$P_1(s) = m_1 s^2 + b_1(s)$$

Hence

$$\lim_{s\to\infty} \frac{P_1(s)}{s^2} = m_1$$

Let us assume that

$$\lim_{s\to\infty} \frac{P_{i-1}(s)}{s^{2i-2}} = m_1 \cdot m_2 \cdot m_3 \ldots m_{i-1}$$

By equation (6) we have

$$\frac{P_i(s)}{s^{2i}} = \frac{a_i(s)}{s^2} \cdot \frac{P_{i-1}(s)}{s^{2i-2}} - \left[\frac{b_{i-1}^2(s)}{s^2} \cdot \frac{P_{i-2}(s)}{s^{2i-4}}\right] \frac{1}{s^2}$$

Using equation (4) we get

$$\lim_{s\to\infty} \frac{P_i(s)}{s^{2i}} = m_i \cdot \lim_{s\to\infty} \frac{P_{i-1}(s)}{s^{2i-2}} = m_1 \cdot m_2 \ldots m_i.$$

(*iii*) Part (*c*) can be proved by expressing $P_i(s)/s^2 P_{i-1}(s)$ as

$$\frac{P_i(s)}{s^{2i}} \cdot \frac{s^{2i-2}}{P_{i-1}(s)} \text{ and using Part } (b).$$

The other result follows directly from the definition of $b_i(s)$ as given in equation (4).

Lemma 2: The functions $Q_i(s)$ defined by the relation

$$Q_i(s) = \frac{b_i(s)}{s}\left[1 - b_i(s)\frac{P_{i-1}(s)}{P_i(s)}\right], \quad \text{for} \quad 1 \le i \le N \tag{9}$$

satisfy the recursion relation

$$Q_i = \frac{b_i}{s} \cdot \frac{m_i s + Q_{i-1}}{m_i s + \dfrac{b_i}{s} + Q_{i-1}} \tag{10}$$

Proof: By definition,

$$Q_i = \frac{b_i(s)}{s}\left[1 - b_i(s)\frac{P_{i-1}(s)}{P_i(s)}\right]$$

Using equation (6), this can be expressed as

$$Q_i = \frac{b_i}{s} \frac{(a_i - b_i)P_{i-1} - b_{i-1}^2 P_{i-2}}{a_i P_{i-1} - b_{i-1}^2 P_{i-2}}$$

which can be reduced to

$$Q_i = \frac{b_i}{s} \cdot \frac{m_i s^2 + b_{i-1} - (b_{i-1}^2 P_{i-2}/P_{i-1})}{m_i s^2 + b_i + b_{i-1} - (b_{i-1}^2 P_{i-2}/P_{i-1})}$$

$$= \frac{b_i}{s} \cdot \frac{m_i s + Q_{i-1}(s)}{m_i s + \dfrac{b_i}{s} + Q_{i-1}(s)}$$

Lemma 3:

$$\lim_{s\to\infty} Q_i(s) = c_i \quad \text{for} \quad 1 \le i \le N \tag{11}$$

Proof: Using equations (4) and (9) we have the relation

$$Q_i(s) = \frac{(c_i s + k_i)}{s}\left[1 - b_i(s) \cdot \frac{P_{i-1}(s)}{P_i(s)}\right],$$

which can be rearranged to yield

$$Q_i(s) = \left[c_i + \frac{k_i}{s}\right]\left[1 - \frac{b_i(s)}{s^2} \cdot \frac{P_{i-1}(s)}{P_i(s)} \cdot \frac{s^{2i}}{s^{2i-2}}\right] \tag{12}$$

Taking limits on both sides of equation (12) and using Lemma 1, we obtain the desired result.

Lemma 4:

$$\lim_{s\to 0} \frac{Q_i(s)}{s} = m_1 + m_2 + \ldots m_i \quad \text{for} \quad 1 \le i \le N \tag{13}$$

Proof:

$$\frac{Q_1(s)}{s} = \frac{(c_1 s + k_1)}{s^2}\left[1 - \frac{c_1 s + k_1}{m_1 s^2 + c_1 s + k_1}\right]$$

$$= (c_1 s + k_1)\left[\frac{m_1}{m_1 s^2 + c_1 s + k_1}\right]$$

Taking the limit as $s \to 0$,

$$\lim_{s\to 0} \frac{Q_1(s)}{s} = m_1$$

The proof now follows by induction. Assuming that the result is valid for $1 \le n \le i$, we have

$$\frac{Q_i(s)}{s} = \frac{m_i + Q_{i-1}(s)/s}{\dfrac{m_i s^2}{b_i} + 1 + \dfrac{Q_{i-1}(s)}{s} \cdot \dfrac{s^2}{b_i}},$$

from which we obtain

$$\lim_{s\to 0} \frac{Q_i(s)}{s} = m_i + \lim_{s\to 0} \frac{Q_{i-1}(s)}{s}$$

$$= m_i + m_{i-1} + \ldots + m_2 + m_1$$

Lemma 5:

$$\lim_{s\to\infty} (Q_i(s) - c_i)s = k_i - \frac{c_i^2}{m_i} \tag{14}$$

Proof: By definition.

$$Q_i(s) = \frac{b_i}{s}\left[1 - \frac{b_i P_{i-1}(s)}{P_i(s)}\right].$$

Using equation (4), and rearranging we have

$$Q_i(s) - c_i = \frac{k_i}{s} - \frac{(c_i s + k_i)^2}{s} \frac{P_{i-1}(s)}{P_i(s)}.$$

Multiplying by s, taking the limit as $s \to \infty$, and using Lemma 1, the result follows.

We shall now show that all the k_i's and c_i's can be uniquely determined if the input $Z(s)$ and the response $X_N(s)$ are known.

Theorem 1: If the masses m_i are known for $1 \le i \le N$, $z(t)$ is known and $x_j(t)$ is known for $j = N$, then the stiffnesses k_i and dampings c_i, $1 \le i \le N$, can be uniquely determined.

Proof: From equation (5) we have

$$\frac{X_N(s)}{Z(s)} = \frac{b_N P_{N-1}(s)}{P_N(s)}. \tag{15}$$

Using Lemmas 1 and 2 together with some simplifications we obtain

$$\frac{X_N(s)}{Z(s)} = \frac{b_N}{s}\left[\frac{1}{m_N s + \frac{b_N}{s} + Q_{N-1}(s)}\right] \tag{16}$$

which with the help of equation (4) implies

$$\frac{X_N(s)}{Z(s) - X_N(s)} = \frac{(sc_N + k_N)}{s}\left[\frac{1}{m_N s + Q_{N-1}(s)}\right] \tag{17}$$

Multiplying both sides of equation (17) by s and taking limits as $s \to \infty$,

$$\lim_{s\to\infty}\left[\frac{sX_N(s)}{Z(s) - X_N(s)}\right] = \lim_{s\to\infty}\left[\frac{c_N}{m_N + \frac{Q_{N-1}(s)}{s}} + \frac{k_N}{m_N s + Q_{N-1}(s)}\right]$$

Using Lemma 3,

$$\lim_{s\to\infty}\left[\frac{sX_N(s)}{Z(s) - X_N(s)}\right] = \frac{c_N}{m_N} \tag{18}$$

Again multiplying both sides of equation (17) by s^2 and taking limits as $s \to 0$ we get

$$\lim_{s\to 0}\frac{s^2X_N(s)}{Z(s) - X_N(s)} = \lim_{s\to 0}\left[\frac{c_N s}{m_N + \frac{Q_{N-1}(s)}{s}} + \frac{k_N}{m_N + \frac{Q_{N-1}(s)}{s}}\right]$$

which by Lemma 4 becomes

$$\lim_{s\to 0}\left[\frac{s^2X_N(s)}{Z(s) - X_N(s)}\right] = \frac{k_N}{\Sigma m_N} \tag{19}$$

Equations (18) and (19) give the values of c_N and k_N since the mass m_N is assumed to be known. Using these values in equation (10), $Q_{N-1}(s)$ can be evaluated.

We now show that if $Q_i(s)$ is known, the c_n, k_n for $1 \le n \le i$ can be found using the following three steps:

(1) By Lemma 3, $c_i = \lim_{s\to\infty} Q_i(s)$

(2) By Lemma 5

$$k_i = \lim_{s\to\infty}(Q_i(s) - c_i)s + \frac{c_i^2}{m_i}$$

Since c_i is known from step (1), k_i can be determined.

(3) Using k_i and c_i from the foregoing, b_i can be evaluated, and from the recursion relation (Lemma 3) Q_{i-1} is obtained as

$$Q_{i-1}(s) = \frac{m_i b_i - \left[m_i s + \frac{b_i}{s}\right]Q_i(s)}{\frac{b_i}{s} + Q_i} \tag{20}$$

Once $Q_{i-1}(s)$ is determined, we go back to step 1.

The results developed here can be particularized to the case of an undamped shear beam by assuming all the c_i's, $1 \le i \le N$, to be known and equal to zero. In that case, as has been previously shown [4], a knowledge of $X_N(s)$, $Z(s)$ and m_i for $1 \le i \le N$ leads to unique identification.

We next present the following result:

Theorem 2: If $x_i(t)$ and $x_{i+1}(t)$ the responses of the ith and $(i+1)$th masses are known for some $1 \le i \le N$ then the stiffness k_n and c_n can be uniquely determined for $1 \le n \le i$ (Fig. 1).

Proof: We observe that equation (5) can be expressed as

$$X_i(s) = b_N \cdot b_{N-1} \ldots b_i \frac{P_{i-1}(s)}{P_N(s)} Z(s) \tag{21}$$

Hence

$$\frac{X_i(s)}{X_{i+1}(s)} = b_i \frac{P_{i-1}(s)}{P_i(s)} \tag{22}$$

and it follows then by Theorem 1 that all the c_n's and k_n's for $1 \le n \le i$ can be uniquely determined.

The result here indicates that if the motion of the ith mass and $(i+1)$th mass are monitored, then all the stiffness k_n and dampings c_n above the *lower* sensor location can be uniquely determined. We note here that the previous result does not require the knowledge of N, the number of degrees-of-freedom of the system. If N is not known, then *no* information regarding the k_n's and c_n's for $i + 1 \le n \le N$ can be extracted from a knowledge of $x_i(t)$ and $x_{i+1}(t)$ since the order N of the polynomial $P_N(s)$ in equation (21) is unknown.

If, however, the degree N of the system is known, information about k_n and c_n for $i < n \le N$ can be extracted although a unique identification of these parameters is not possible. To illustrate this, we now prove a theorem related to undamped structural systems regarding the uniqueness of identification from the responses of the ith and $(i+1)$th masses of an N-mass system.

Theorem 3: Given an undamped system $\{k_j, m_j\}$, $m_j, k_j > 0$ whose total number of degrees of freedom, N, are known, there exist $(N-i)!$ different systems, all of which yield identical ith and $(i+1)$th mass response pairs $[x_i^n(t), x_{i+1}^n(t)]$ when subjected to an ensemble, $z^n(t)$, $n = 1, 2, \ldots$, of base motions.

Proof: From equation (21) with $c_i = 0$, we have as before

$$X_{i+1}(s) = k_N \cdot k_{N-1} \ldots k_{i+1} \frac{P_i(s)}{P_N(s)} Z(s) \tag{24}$$

If we assume that in addition to the system $\{k_j, m_j\}$ there exists another system $\{\tilde{k}_j, m_j\}$ such that both systems yield the same $X_i^n(s)$ and $X_{i+1}^n(s)$ when subjected to the same though unknown inputs $Z^n(s)$, $n = 1, 2, \ldots$, then it follows that

$$k_N \cdot k_{N-1} \ldots k_{i+1} \frac{P_i(s)}{P_N(s)} = \tilde{k}_N \tilde{k}_{N-1} \ldots \tilde{k}_{i+1} \frac{\tilde{P}_i(s)}{\tilde{P}_N(s)} \tag{25}$$

where the tildas denote quantities related to the $\{\tilde{k}_j, m_j\}$ system. Further, from Theorem 2 we have

$$\tilde{k}_n = k_n \text{ for } 1 \le n \le i \tag{26}$$

so that $\tilde{k}_n$, $i \le n \le N$ need only to be determined from relation (25). However, from the definition of $P_i(s)$, we have $\tilde{P}_i(s) = P_i(s)$.

Cross multiplying equation (25) and equating powers of s^{2N} we have,

$$k_{i+1} \cdot k_{i+2} \ldots k_N = \tilde{k}_{i+1} \cdot \tilde{k}_{i+2} \ldots \tilde{k}_N \tag{27}$$

and

$$\tilde{P}_N(s) = P_N(s)$$

In order to determine the $\tilde{k}$'s we equate the coefficients of various powers of s on both sides of equation (27). This leads to N nonlinear algebraic equations in the k's which have the following form:

$$\sum_{i_1=1}^{N} \alpha_{1i_1}\tilde{k}_{i_1} = \sum_{i_1=1}^{N} \alpha_{1i_1}k_{i_1} = a_1 \tag{28-1}$$

$$\sum_{i_2>i_1}\sum_{i_1=1}^{N} \alpha_{2i_1i_2}\tilde{k}_{i_1}\tilde{k}_{i_2} = \sum_{i_2>i_1}\sum_{i_1=1}^{N} \alpha_{2i_1i_2}k_{i_1}k_{i_2} = a_2 \tag{28-2}$$

$$\sum_{i_n>i_{n-1}} \cdots \sum_{i_2>i_1}\sum_{i_1=1}^{N} \alpha_{ni_1i_2\ldots i_n}\tilde{k}_{i_1}\tilde{k}_{i_2}\ldots\tilde{k}_{i_n}$$

$$= \sum_{i_n>i_{n-1}} \cdots \sum_{i_2>i_1}\sum_{i_1=1} \alpha_{ni_1i_2\ldots i_n}k_{i_1}k_{i_2}\ldots k_{i_n} = a_n \tag{28-n}$$

$$\tilde{k}_1\tilde{k}_2\ldots\tilde{k}_N = k_1k_2\ldots k_N = a_N \tag{28-N}$$

In this set, the a_i's are all known from the left-hand side of equation

(27) and can be expressed in terms of the roots, λ_i, of the equation $P_N(\lambda) = 0$ as follows:

$$
\begin{aligned}
a_1 &= \sum_{i=1}^{N} \lambda_i \\
a_2 &= \frac{1}{2} \sum_{\substack{i=1 \\ j=1}}^{N} \lambda_i \lambda_j \\
&\cdots\cdots\cdots \\
a_N &= \lambda_1 \lambda_2 \ldots \lambda_N
\end{aligned}
\qquad (29)
$$

It can be shown that the a_i, $1 \le i \le N$, are all greater than zero and that the α's are known and positive since they only involve the m_i [4]. Using equation (26) we note that the nth equation for $1 \le n \le N - i$ of the set (28) is of degree n. The remaining i equations are all of degree $N - i$. Using the first $(N - i)$ equations to solve for the $(N - i)$ unknowns k_n, $i \le n \le N$, we can use Bezout's theorem to establish the number of solution sets.

Bezout's theorem states that [6] if $f_1, f_2, \ldots, f_m$ be hypersurfaces in m-dimensional projective space which intersect in a finite set $\{M_j\}$ of points, and if d_i be the degree of f_i, there may then be assigned multiplicities σ_j to the M_j independent of the coordinate system, such that counted with these multiplicities the number of intersections is $d = d_1 \cdot d_2 \ldots d_m$. Thus, if the number of solutions is finite, then there are $(N - i)(N - i - 1)(N - 1 - 2) \ldots 3.2.1$, i.e., $(N - i)!$ solution sets counted with their multiplicities, of which one set is given by $\tilde{k}_i = k_i$, $1 \le i \le N$.

We have thus shown that knowledge of the ith and $(i + 1)$th floor responses of an N-degree-of-freedom system lead to solution set vectors of the form $\{k_1, k_2, k_3 \ldots k_i; k_{i+1} \ldots k_n\}^T$ in which the first i elements are uniquely determinable. The degree of nonuniqueness in the determination of the remaining components of the vector is $(N - i)!$, if N, the number of degrees-of-freedom of the system, is known.

Applications of Previous Results

The preceding section illustrates that with a knowledge of the base motions and the motions at the first floor level in a structure modeled as a shear beam, unique identification of the complete stiffness and damping distribution can be obtained. As has been shown before [3, 4] the measurement of response at any other floor level would lead to nonunique identification.

Several researchers in the past have concentrated on building structural models that yield "close fits" between the measured response at a particular floor and the model response at the same location for a specific base input. The establishment of a model from such history matching may be grossly in error because (1) the model may not be able to predict the response to other base inputs and (2) the forces in the various elements could be incorrect even for the specific input for which the identification through history matching was done. This will be illustrated in the numerical example which follows. The foregoing arguments apply to soil columns as well, with two possible significant differences: (1) In the structural identification problem, when nonuniqueness occurs, the number of possible solution sets can be reduced through a knowledge of available structural data. For instance, one can make use of the fact that in many building structures, the stiffness is a monotonically decreasing function of height. Such constraints on the nature of the distributions as well as on the parameters themselves are less reliable for soil columns. (2) Except for a few downhole measurements, most earthquake response studies are carried out by the placement of the sensor right on the surface of the ground. The bed-rock input though generally unknown, is often assumed. Identification of soil properties from such "bed-rock input-ground surface response" type studies for an N-layered soil medium is an extremely ill-posed problem with $N!$ possible solution sets. Also, as mentioned earlier, the constraints on the nature of the stiffness distribution in soil columns are less reliable than for structural identification, further aggravating the nonuniqueness situation.

In general, the higher up from the base the sensor is located, the more ill-posed the inverse problem gets. Structural response measured at the roof would lead, in general, to $N!$ possible solution sets of stiffnesses for undamped systems [4]. Nor is this problem completely solved by the placement of an additional sensor at an intermediate floor between the base and the roof. Though this may reduce the ill-posedness dramatically, it will still not lead in general to unique identification. As shown in [4], the placement of a sensor at the nth floor level leads to S_1 solution sets ($S_1 = (N - n)!(n - 1)!(n - 1)$). Also, a knowledge of the base and roof motions leads to S_2 solution sets ($S_2 = N!$). Further, a knowledge of the nth floor and roof response leads to S_3 solution sets ($S_3 = (N - n + 1)!$). Thus a knowledge of the nth floor response in conjunction with the base input as well as the roof motions would then lead to S solution sets that would comprise the intersection of the aforementioned three sets. We note that though the number of solution sets S will be less than S_2, uniqueness is not in general guaranteed.

Theorems 2 and 3 are especially applicable to columns in layered soil media where the properties are assumed uniform throughout each layer. If the number of "layers" of soil strata below the lower sensor location is unknown, no information can be extracted about the soil properties below the lower sensor. This is obvious, since a knowledge of the number of layers is a necessary prerequisite for their description. If, however, the number of layers from the surface to bed-rock are known (or arbitrarily fixed), information about the layer properties below the lowest sensor location can be extracted although once again, the problem becomes ill-posed.

Numerical Example

To highlight the nature of the nonuniqueness problem, we consider a two layered soil medium represented by a two-degree-of-freedom damped oscillator defined by the parameter set $\{m_1, m_2, k_1, k_2, c_1, c_2\}$. The surface response $x_1(t)$ and the base rock input can then be shown to be related through the following equation:

$$
-s^2 \left[\frac{X_1(s)}{X_1(s) - Z_1(s)} \right] = \frac{s^2 c_1 c_2 + s(k_1 c_2 + k_2 c_1) + k_1 k_2}{s^2 m_1 m_2 + s[(c_1 + c_2)m_1 + c_1 m_2] + (k_1 + k_2)m_1 + k_1 m_2} \qquad (30)
$$

where s is the Laplace transform variable.

If another system, defined by the parameters $\{m_1, m_2, \tilde{k}_1, \tilde{k}_2, \tilde{c}_1, \tilde{c}_2\}$ exists such that it yields exactly the same surface response $x_1(t)$, for any arbitrary but known input $z(t)$, then we must have

$$
\frac{s^2 c_1 c_2 + s(k_1 c_2 + k_2 c_1) + k_1 k_2}{s^2 + s[(c_1 + c_2)m_1 + c_1 m_2] + [(k_1 + k_2)m_1 + k_1 m_2]} = \frac{s^2 \tilde{c}_1 \tilde{c}_2 + s(\tilde{k}_1 \tilde{c}_2 + \tilde{c}_1 \tilde{k}_2) + \tilde{k}_1 \tilde{k}_2}{s^2 + s[(\tilde{c}_1 + \tilde{c}_2)m_1 + \tilde{c}_1 m_2] + [(\tilde{k}_1 + \tilde{k}_2)m_1 + \tilde{k}_1 m_2]} \qquad (31)
$$

Noting that the quantities on each side of the equation are rational functions of polynomials in s, for equation (31) to be true for all s, we must have

$$
\begin{aligned}
c_1 c_2 &= \tilde{c}_1 \tilde{c}_2; \quad (c_1 + c_2)m_1 + c_1 m_2 = (\tilde{c}_1 + \tilde{c}_2)m_1 + \tilde{c}_1 m_2 \\
k_1 k_2 &= \tilde{k}_1 \tilde{k}_2; \quad (k_1 + k_2)m_1 + k_1 m_2 = (\tilde{k}_1 + \tilde{k}_2)m_1 + \tilde{k}_1 m_2
\end{aligned}
\qquad (32)
$$

and

$$
k_1 c_2 + k_2 c_1 = \tilde{k}_1 \tilde{c}_2 + \tilde{k}_1 \tilde{c}_1.
$$

The previous five nonlinear algebraic equations have the two solution sets

$$
\tilde{k}_1 = k_1, \quad \tilde{c}_1 = c_1, \quad \tilde{k}_2 = k_1, \quad \tilde{c}_2 = c_1 \qquad (33a)
$$

and

$$
\tilde{k}_1 = k_2 m_R, \quad \tilde{c}_1 = c_2 m_R, \quad \tilde{k}_2 = k_1 / m_R, \quad \tilde{c}_2 = c_1 / m_R \qquad (33b)
$$

when m_R is a mass ratio and equals $m_1/(m_1 + m_2)$.

For most soil columns modeled by n-degree-of-freedom systems, the estimates of the model parameters from earthquake data are

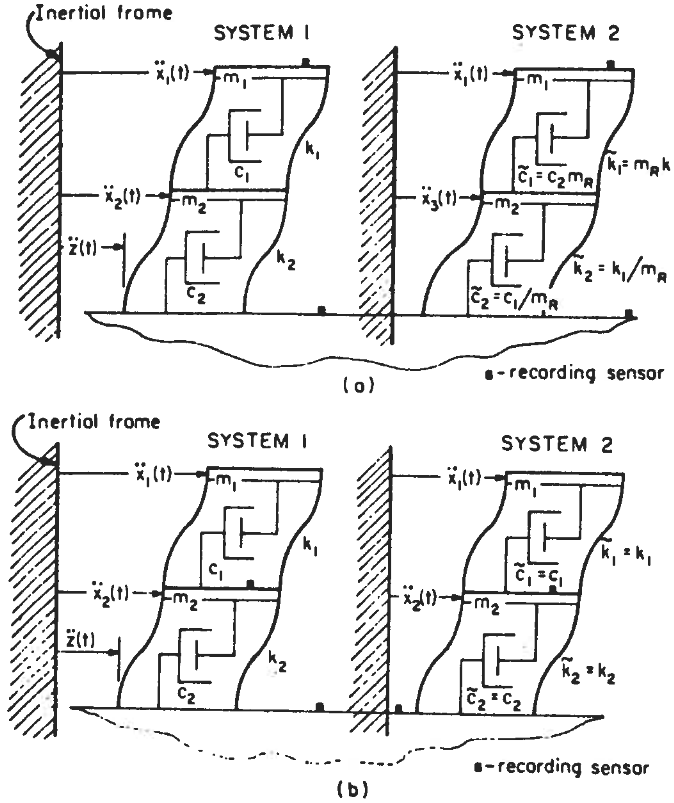


Fig. 2

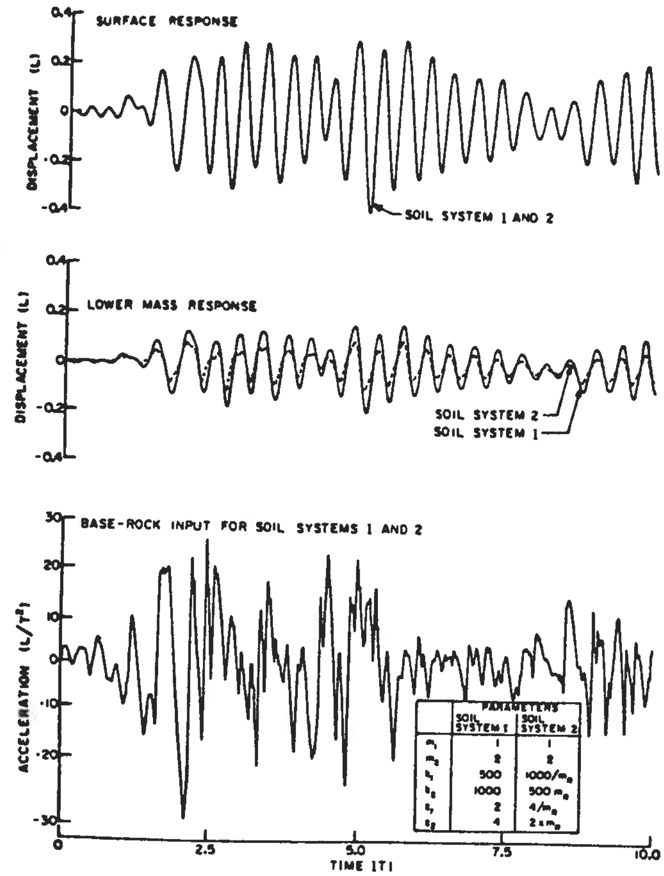


Fig. 3

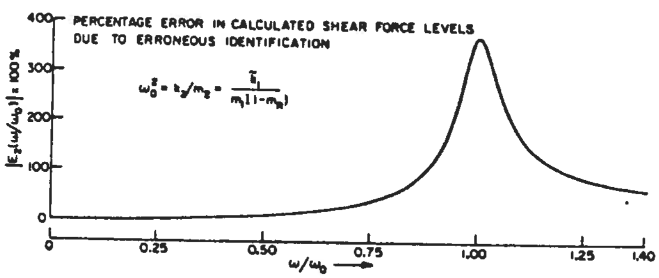


Fig. 4

obtained by (*a*) starting out with an initial guess of the parameters (*b*) adjusting one or more of the parameters so that the "closest-fit" between the calculated surface response and the measured response for the supposedly known base rock motion, is achieved. These adjustments are generally carried out in an iterative manner, and validation of the model is then often justified on the basis of a close history match between the model response and the measured responses. The foregoing analysis indicates that from a practical point of view such an iterative adjustment of the model parameters, may lead to parameter estimates that *do* converge, though to the wrong values (Figs. 2(*a*) and 2(*b*)).

To illustrate this, Fig. 3 shows the model response (solid line) at the surface of the system $\{m_1 = 1, m_2 = 2, k_1 = 500, k_2 = 1000, c_1 = 2, c_2 = 4\}$ for an earthquake ground motion base input. Also shown (dashed line) is the response of its companion system described by equation set (33*b*). The two systems are totally indistinguishable from each other if only a knowledge of the base rock and surface motions is provided. It may be noted that though both models yield identifical surface motions, motions of the lower lumped mass are different in the two cases.

It may be argued that, from the engineering point of view, the two models are both correct in so far as our interest is limited only to the prediction of the surface response. However, the models yield very different intermass shear force values. After some manipulation, it can be shown that the percentage difference in the shear force values between the two systems can be expressed as:

$$E_1(s) = \frac{\tilde{F}_1(s) - F_1(s)}{\tilde{F}_1(s)} = \frac{s(c_1k_2 - k_2c_1)}{(k_1k_2 + sc_1k_2)} \quad (34)$$

$$E_2(s) = \frac{\tilde{F}_2(s) - F_2(s)}{\tilde{F}_2(s)}$$

$$= \frac{m_2s^2[(m_1 + m_2)k_1 - m_1k_2)] + (m_1 + m_2)(c_2k_1 - c_1k_2)s]}{(m_1 + m_2)k_1 \cdot (m_2s^2 + k_2 + c_2s)} \quad (35)$$

where $F_1(s)$ and $F_2(s)$ represent shear forces in the elements with stiffnesses k_1 and k_2. Setting $s = i\omega$, we observe that significant differences in shear force calculations would arise for $\omega_0 = \sqrt{k_2/m_2}$ showing that such nonuniqueness problems may become critical from the structural analysis point of view though they may not affect the response prediction capability of the models. It can be shown from equation (35) that in general

$$|E_2[(\omega/\omega_0) = 1]| \geq \sqrt{\frac{k_2m_2}{c_2}}\left[1 - m_R\frac{k_2}{k_1}\right]$$

the equality occurring when $c_1k_2 = c_2k_1$. Fig. 4 shows a plot of $|E_2(\omega/\omega_0)|$ calculated as a function of the dimensionless frequency $\omega^* = \omega/\omega_0$ for the numerical example considered. We observe that for the system under consideration an error of about 400 percent in the calculation of shear force may occur if the system is identified erroneously.

Discussion and Conclusions

In this paper an N-layered soil system (or an N-stored shear structure) has been modeled as a lumped mass-spring-damper system. It has been shown that identification of the stiffness and damping distributions with height (assuming viscous damping) can be uniquely carried out through a knowledge of the base motions and that of the mass immediately above the base. The paper further points to some of the possible difficulties that may arise in the identification of soil systems when modeling soil columns for which the number of degrees of freedom are not exactly known. Base rock excitation records (even if they can be somehow accurately inferred) and surface ground motion records do not have sufficient information in them to uniquely tie down the stiffness and damping estimates.

It must be pointed out that the model analyzed here is very simplistic. However. such models are often used in practice to idealize structural as well as soil systems. Though this study is restricted to noise-free data and to the linear range of response. the analysis presented here. we feel. is an indication of the types of problems that would arise in the testing and identification of such systems.

Nonuniqueness of system properties. if records are available at mass levels other than the one immediately above the base. is illustrated through a numerical example. The practical aspect of the nonuniqueness problem is emphasized by showing that convergence (most identification schemes are iterative). to the wrong set of parameters may lead to very different forces calculated in the various members.

As observed from the results presented herein. the records at higher and higher mass levels seem to have less and less detailed information about the distribution of dynamic properties throughout the system.

Acknowledgments

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