Uniqueness in the Identification of Building Structures from Forced Vibration Tests*

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Abstract

This paper investigates the problem of uniqueness in the identification of building structural systems from data gathered during forced-vibration testing. Modelling the structure as an undamped discrete shear beam, it is shown that if the top floor response to a known forcing function which is also applied at the same location is available, then unique identification of the structural system is possible. Several useful results on the ability to uniquely identify some, but not all, of the stiffness constants of the system have also been obtained.

Introduction

The dynamic testing of large structures has become a common practice in the field of structural engineering for the validation and/or updating of structural models. In the area of civil engineering, forced vibration tests are often performed on tall building structures, dams and bridges, to name a few situations [1-4]. Such forced vibration tests are generally carried out by placing one or more "shakers" at one or more locations, and measuring the responses at different locations in the structure [5]. For structures that respond primarily in the linear range, these test procedures were directed, in the past, towards acquiring information about the lower natural frequencies of vibration and the corresponding mode shapes. Nowadays, with the accent on improving our ability to predict structural responses to wide-band excitations, the need for parameter identification for such systems has been widely felt.

Though a large number of identification schemes (which are generally iterative) have been developed, for building structural identification, few investigators have attempted to investigate the uniqueness aspects associated with the inverse problem. It has been shown elsewhere that, from a practical standpoint, this consideration may become a serious one in certain classes of problems such as the identification of structures from their earthquake response [6]. Convergence to the wrong parameter values could occur if the sensors are not distributed in a judicious manner, leading to large inaccuracies in the evaluation of quantities of engineering importance such as base-shears and base bending moments [7].

*Manuscript received November 29, 1983 and in a revised form April 26, 1984. No. 83-CSME-53, EIC Accession No. 1989.

In this paper we provide some uniqueness results which pertain to the forced-vibration testing of building structural systems. We treat an N-story structural system as an N degree-of-freedom, spring-mass system. The identification problem consists of determining of stiffness constants of the system from knowledge of the forced excitation and the corresponding response of one or more of the floors of the system. The mass distribution is always assumed to be known apriori, e.g., from design drawings. Typically, forced vibration tests in building structures are performed by placing a shaker at the top-most story of the structure and measuring the responses at other floor levels. It is shown that under these conditions, unique identification of the stiffness constants is possible by measuring the response at *one* location in the system. Some additional results on partial identification wherein not all, but a few, of the stiffness constants can be uniquely determined are also provided.

Though this work has been motivated by problems in building structural identification, the results arrived at are applicable to all systems that can be expressed by the same matrix equations, such as LC ladder networks.

The results obtained herein are extensions of those presented in [6] to the situation pertinent to forced vibration testing. Throughout the sequel, the shaker excitation is assumed to be broad-band in character. Such excitations are often produced by rotating-mass shakers during "frequency sweep" tests or by pulse-shakers, which generate pulses of short time duration.

Problem Formulation

Consider an N-story structure modelled as an N degree-of-freedom undamped oscillator (Figure 1). Assuming the masses m_i, $i = 1,2,\dots,N$ are known apriori, the identification problem consists of determining the stiffness distribution k_i from a knowledge of the force-time history applied at the ith story level, and the recorded response at the jth story level.

Denoting the absolute motion of the nth floor by $w_n(t)$ and the force applied at the ith mass level by $f_i(t)$, we have the following set of equations:

$$M\ddot{\mathbf{w}} + A\mathbf{w} = \mathbf{h}(t) \tag{1}$$

where M=diag $(m_1, m_2, \ldots, m_N)$,

$$A = \begin{bmatrix} k_1 & -k_1 & & \\ -k_1 & k_1+k_2 & -k_2 & \\ & & & \\ & & & k_{N-1}+k_N \end{bmatrix} \tag{2}$$

and $\mathbf{h}^T(t) = [0,0,\ldots,f_i(t),\ldots,0]$ where the force $f_i(t)$ is located in the ith position in the vector and is applied to mass m_i.

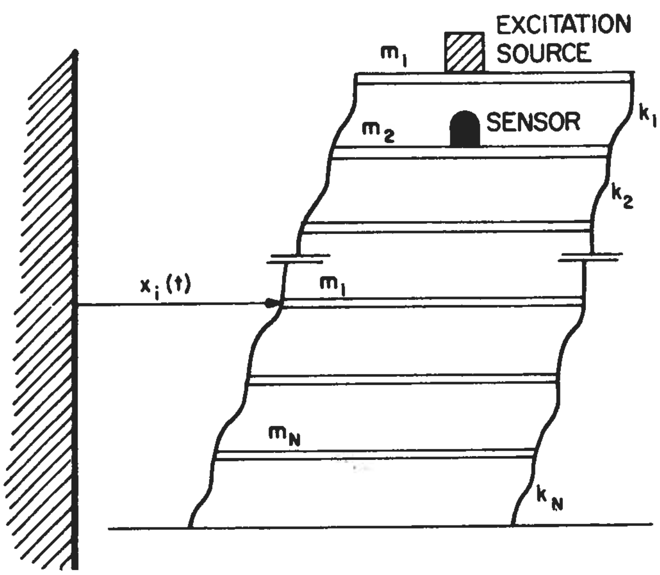

Figure 1. Response of N-Story Shear Structures

Since m_i's and k_i's are real and positive for passive physical systems, we can reduce the system equation to

$$\ddot{\mathbf{y}} + K\mathbf{y} = \mathbf{g}f_i(t) \tag{3}$$

where $\mathbf{g}^T = [0,0,\ldots,1/\sqrt{m_i},0,0,\ldots]$, and

$$K = \begin{bmatrix} b_1 & -a_1 & & \\ -a_1 & b_2 & -a_2 & \\ & & & -a_{N-1} \\ & & -a_{N-1} & b_N \end{bmatrix}$$

in which

$$a_i = \frac{k_i}{\sqrt{m_i m_{i+1}}}, \; 1 \le i \le N-1, \text{ and}$$

$$b_i = \frac{k_{i-1}+k_i}{m_i}, \; 1 \le i \le N \tag{4}$$

with $k_0 = 0$

Taking Laplace Transforms, and replacing the transform variable by $i\sqrt{\lambda}$ we get

$$(K - \lambda I)\mathbf{Y} = \mathbf{g}F_i(\lambda) \tag{5}$$

where $Y(\lambda)$ and $F_i(\lambda)$ represent transformed quantities. Solving (5) for $Y_j(\lambda)$ we have

$$Y_j(\lambda) = \frac{\Delta_j}{\Delta}F_i(\lambda) \tag{6}$$

where $Y_j(\lambda)$ is the transform of $y_j(t)$, Δ = det $(K - \lambda I)$, and, Δ_j is the determinant of the matrix obtained from $(K - \lambda I)$ by replacing its jth column by $\mathbf{g}$. For convenience, let us denote by $Q_i(\lambda)$, the determinant of lower right $(N-i+1) \times (N-i+1)$ submatrix of $(K - \lambda I)$. In this way, $Q_1(\lambda)$ is simply det $(K - \lambda I)$. Also, let $P_i(\lambda)$ be the determinant of the upper left ixi submatrix of $(K - \lambda I)$. This allows us to express (6), for $i = j$, as

$$W_j(\lambda) = \frac{P_{j-1}(\lambda)Q_{j+1}(\lambda)}{Q_1(\lambda)m_j}F_j(\lambda) \tag{7}$$

where $P_0(\lambda) = 1$ and $Q_{N+1}(\lambda) = 1$.

In the following, we present some useful properties of the polynomials $P_i(\lambda)$ and $Q_i(\lambda)$.

Lemma 1

(a) The functions $P_i(\lambda)$ and $Q_i(\lambda)$ defined in the previous section satisfy the recursion relations

$$P_i(\lambda) = (b_i - \lambda)P_{i-1}(\lambda) - a_{i-1}^2 P_{i-2}(\lambda), \quad 2 \le i \le N \text{ with}$$

$$P_1(\lambda) = (b_1 - \lambda), \text{ and,}$$

$$Q_i(\lambda) = (b_i - \lambda)Q_{i+1}(\lambda) - a_i^2 Q_{i+2}(\lambda), \quad 1 \le i \le N \text{ with}$$

$$Q_{N+1}(\lambda) = 1, Q_{N+2} = 0.$$

(b) Each $P_i(\lambda)$ is a polynomial of degree i with $(-1)^i\lambda^i$ as the leading term, i.e.

$$\lim_{\lambda\to\infty} \frac{P_i(\lambda)}{\lambda^i} = (-1)^i.$$

Each $Q_i(\lambda)$ is a polynomial of degree $(N-i+1)$ with $(-1)^{N-i+1}\lambda^{(N-i+1)}$ as the leading term, i.e.

$$\lim_{\lambda\to\infty} \frac{Q_i(\lambda)}{\lambda^{N-i+1}} = (-1)^{N-i+1}$$

Proof:

Part (a) follows directly from the definitions of the P's and Q's. Part (b) follows from (a) by induction. □

Lemma 2

(a) For $1 \le i \le N$, $Q_i(\lambda)$ and $Q_{i+1}(\lambda)$ do not have any common zero if $a_j \neq 0$, $1 \le j \le N-1$.

(b) For $1 \le i \le N$, $P_i(\lambda)$ and $P_{i+1}(\lambda)$ do not have any common zero, if $a_j \neq 0$, $1 \le j \le N-1$.

Proof:

(a) The proof is by induction.

Since $Q_N(\lambda) = (b_N - \lambda)$, and $Q_{N-1}(\lambda) = (b_N - \lambda)(b_{N-1} - \lambda) - a_{N-1}^2$, the only common zero they could have is $\lambda = b_N$. But this implies $a_{N-1} = 0$, a contradiction. Thus Q_N and Q_{N-1} have no zero in common. Now assume that Q_i and Q_{i+1} have no common zeros, $i \ge n$, and let

$$Q_{n-1}(\alpha) = Q_n(\alpha) = 0.$$

Then the recursion relation

$$Q_{n-1}(\alpha) = (b_n - \lambda)Q_n(\lambda) - a_n^2 Q_{n+1}(\lambda)$$

implies $Q_{n+1}(\alpha) = 0$, a contradiction since Q_n and Q_{n+1} do not have a common zero; hence the result.

(b) The proof follows along the lines of part (a). □

Lemma 3

Zeros of all $P_i(\lambda)$ and $Q_i(\lambda)$ are simple.

Proof:

Let α be the pth order zero of $Q_i(\lambda)$; then it is an eigenvalue of the lower $(N-i+1)\times(N-i+1)$ submatrix K called K_i. This matrix K_i being a real symmetric matrix can be diagonalized to Λ, a diagonal matrix of eigenvalues, by an orthogonal matrix T i.e.

$$T^T K_i T = \Lambda$$

where p of the diagonal elements of Λ are α. Thus

$$\text{rank } [K_i - \alpha I] = N - i + 1 - p \ ,$$

since T is nonsingular. Thus, $p > 1$ would imply $Q_{i+1}(\alpha) = 0$, a contradiction by Lemma 2, hence $p = 1$.

The proor for all the P_i's is similar. □

Uniqueness Results

(a) Assuming, as is often done in practice, that a shaker is located at the top-most story of the building structure we first show that unique identification of the stiffness constants is guaranteed if the structural response is also measured at the top-most story. The knowledge of the shaker force time history and the response time history at the top level are required.

Theorem 1:

If there exists a set of k_i's $1 \le i \le N$, corresponding to the given functions $W_1(\lambda)$ and $F_1(\lambda)$, and m_i, $1 \le i \le N$, then it is unique.

Proof:

By Equation (7) we have

$$\frac{1}{m_1}\frac{F_1(\lambda)}{W_1(\lambda)} = \frac{Q_1(\lambda)}{Q_2(\lambda)} = (b_1 - \lambda) - a_1^2\frac{Q_3(\lambda)}{Q_2(\lambda)} \ ,$$

which when rearranged yields

$$[\frac{1}{m_1}\frac{F_1(\lambda)}{W_1(\lambda)} + \lambda] = b_1 - \frac{a_1^2}{\lambda}\frac{Q_3(\lambda)\lambda}{Q_2(\lambda)} .$$

Taking the limits on both sides of (8) as $\lambda \to \infty$ gives,

$$b_1 = \underset{\lambda\to\infty}{Lim}[\frac{1}{m_1}\frac{F_1(\lambda)}{W_1(\lambda)} + \lambda] . \tag{8}$$

Thus knowing k_1, the first equation of the set (5) yields

$$a_1\frac{W_2(\lambda)}{W_1(\lambda)} = [(b_1 - \lambda)m_1 - \frac{F_1(\lambda)}{W_1(\lambda)}]\frac{1}{(m_1 m_2)^{1/2}} . \tag{9}$$

Knowing the right-hand side of the expression above, and noting that

$$a_1\frac{W_1(\lambda)}{W_2(\lambda)}(\frac{m_1}{m_2})^{1/2} = \frac{Q_2}{Q_3} = (b_2 - \lambda) - a_2^2\frac{Q_4(\lambda)}{Q_3(\lambda)}$$

the use of Lemma 1, then gives

$$\underset{\lambda\to\infty}{Lim}[\lambda + a_1\frac{W_1(\lambda)}{W_2(\lambda)}(\frac{m_1}{m_2})^{1/2}] = b_2 . \tag{10}$$

Using the second relation of the set (5) we now can find $W_3(\lambda)$ and the recursion continues. We now show that if $W_i(\lambda)$, k_i, are assumed to be known for $1 \le i \le n$, then $W_{n+1}(\lambda)$ and k_{i+1} can be determined. We have,

$$a_n\frac{W_{n+1}(\lambda)}{W_n(\lambda)} = [(b_n - \lambda)m_n^{1/2} - a_{n-1}\frac{W_{n-1}(\lambda)}{W_n(\lambda)}m_{n-1}^{1/2}]\frac{1}{m_{n+1}^{1/2}} \tag{11}$$

where the right hand is known by assumption. Also,

$$(\frac{m_n}{m_{n+1}})^{1/2}\cdot a_n \cdot \frac{W_n(\lambda)}{W_{n+1}(\lambda)} = \frac{Q_{n+1}}{Q_{n+2}} = (b_{n+1} - \lambda) - a_{n+1}^2\frac{Q_{n+3}}{Q_{n+2}} \tag{12}$$

so that

$$\underset{\lambda\to\infty}{Lt}[a_n(\frac{m_n}{m_{n+1}})^{1/2}\frac{W_n}{W_{n+1}} + \lambda] = b_{n+1} . \tag{13}$$

Thus k_{n+1} is determined since k_i, $1 \le i \le n$ and m_i, $i \epsilon (1,N)$ are all known. This proves the theorem. □

Corollary:

Consider two systems $\{k_i, m_i\}$ and $\{\tilde{k}_i, m_i\}$ subjected to the set of forcing functions $F_1^j(t)$ applied to mass level m_1. Let the masses m_i, $1 \le i \le N$, be known.

Then if $w_1^j(t) = \tilde{w}_1^j(t)$, for all forcing functions j, $k_i = \tilde{k}_i$, $1 \le i \le N$.

Proof:

The proof directly follows from the theorem. □

(b) We next present some useful results on partial identification when two sensors are used to measure the response at two story levels, the shaker force time history being unknown.

Theorem 2:

Consider the system governed by Equation (1) subjected to an unknown forcing function $f_i(t)$. If the responses of the system $w_n(t)$ and $w_{n+1}(t)$ are measured, where $n \ge i$, then k_ℓ, $n \le \ell \le N$ can be uniquely determined provided m_ℓ, $n+1 \le \ell \le N$ are known.

Proof:

$$\frac{W_{n+1}(\lambda)}{W_n(\lambda)}(\frac{m_{n+1}}{m_n})^{1/2} = a_n\frac{Q_{n+1}}{Q_{n+1}} . \tag{14}$$

Noting Lemma 1, we have

$$k_n = \underset{\lambda\to\infty}{Lim}[\lambda\cdot m_{n+1}\frac{W_{n+1}(\lambda)}{W_n(\lambda)}] . \tag{15}$$

Thus we also obtain,

$$\frac{Q_{n+2}}{Q_{n+1}} = \frac{m_{n+1}}{k_n}\frac{W_{n+1}(\lambda)}{W_n(\lambda)} .$$

Let us assume that $Q_{\ell+1}/Q_{\ell+2}$ and k_ℓ are known for $n < \ell < r - 1$. Then using the recursion relation we obtain

$$\frac{Q_r}{Q_{r+1}} = (b_r - \lambda) - a_r^2 \frac{Q_{r+2}}{Q_{r+1}}.$$

Thus

$$b_r = \underset{\lambda \to \infty}{Lt} [\lambda + \frac{Q_r}{Q_{r+1}}]. \quad (16)$$

But

$$a_{r-1} = \frac{k_{r-1}}{\sqrt{m_r m_{r-1}}} \text{ and thus}$$

$$a_r = b_r (\frac{m_r}{m_{r+1}})^{1/2} - \frac{k_{r-1}}{\sqrt{m_r m_{r+1}}}. \quad (17)$$

To continue the recursion, we determine

$$\frac{Q_{r+1}}{Q_{r+2}} = a_r^2 [b_r - \lambda - \frac{Q_r}{Q_{r+1}}]^{-1}. \ \square$$

Thus knowledge of $w_n(t)$ and $w_{n+1}(t)$ yield sufficient information to identify k_ℓ, $\ell \geq n$, uniquely. We note that explicit knowledge of the forcing function time-history $f_i(t)$ is not required; only the fact that $w_n(t)$ and $w_{n+1}(t)$ are responses to the function $f_i(t)$, and the forcing function is applied at or above the upper of the two measurement locations, is required. Furthermore, unique identification of all stiffness constants is possible if $w_1(t)$ and $w_2(t)$, the responses at the two top-most stories are available, when the shaker is placed at the top story.

Corollary:

If $w_1(t)$ and $w_2(t)$ are known for a forcing function $f_1(t)$, and m_ℓ, $1 \leq \ell \leq N$, are known, then the k_i's, $1 \leq i \leq N$, can be uniquely determined.

Proof:

Using Theorem 2 for $i = n = 1$, the result follows. □

Corollary:

If $f_1(t)$ and $w_2(t)$ are known, then given m_ℓ, $1 \leq \ell \leq N$, the k_i's, $1 \leq i \leq N$ can be uniquely determined.

Proof:

We have

$$\frac{W_2(\lambda)}{F_1(\lambda)} = \frac{\sqrt{m_2}}{\sqrt{m_1}} a_1 \frac{Q_3}{Q_1}.$$

Thus

$$a_1 = \underset{\lambda \to \infty}{Lt} [\frac{\sqrt{m_1}}{\sqrt{m_2}} \frac{\lambda^2 W_2(\lambda)}{F_1(\lambda)}],$$

and the value of k_1 is obtained using (4). Knowing k_1, $w_2(t)$ and $f_1(t)$ and using the first of the equation set (1), $w_1(t)$ can be found. Application of the previous corollary then guarantees unique identification of k_i, $1 \leq i \leq N$. □

We now show that if the excitation source which forces the motion in the structure is placed at the level of mass m_{i+1}, then the knowledge of $w_n(t)$ and $w_{n+1}(t)$ where $n \leq i$ will yield knowledge of k_ℓ, $1 \leq \ell \leq n$.

Theorem 3:

If the forcing function $f_{i+1}(t)$ yields the known responses w_n and w_{n+1} when $n \leq i$, then k_ℓ, $1 \leq \ell \leq n$ can be uniquely determined, if the m_i's, $1 \leq i \leq n$ are known. Knowledge of the force-time history $f_{i+1}(t)$ is not required.

Proof:

We have

$$\frac{W_n(\lambda)}{W_{n+1}(\lambda)} = \frac{k_n}{m_n} \frac{P_{n-1}}{P_n}. \quad (18)$$

Using Lemma 1, we get

$$\frac{k_n}{m_n} = -\underset{\lambda \to \infty}{Lim} \frac{\lambda W_n(\lambda)}{W_{n+1}(\lambda)}. \quad (19)$$

Thus,

$$\frac{P_{n-1}}{P_n} = \frac{m_n}{k_n} \frac{w_n(\lambda)}{W_{n+1}(\lambda)}. \quad (20)$$

We now assume that we know $P_{\ell-1}(\lambda)/P_\ell(\lambda)$ for $i \leq \ell \leq n$.

Using the recursion relation of Lemma 1, we get

$$\frac{P_{i+1}(\lambda)}{P_i(\lambda)} = (b_{i+1} - \lambda) - a_i^2 \frac{P_{i-1}}{P_i}.$$

Taking limits as $\lambda \to \infty$, we get

$$b_{i+1} = \underset{\lambda \to \infty}{Lim} [\lambda + \frac{P_{i+1}}{P_i}]. \quad (21)$$

This allows us to find k_i from

$$k_i = m_{i+1} b_{i+1} - k_{i+1}.$$

We continue the recursion by noting that

$$\frac{P_i}{P_{i-1}} = a_i^2 [b_{i+1} - \lambda - \frac{P_{i+1}(\lambda)}{P_i(\lambda)}]^{-1}. \ \square$$

We have thus shown that the measurement of the response at two consecutive mass levels when the forcing function is applied at a location which is at or below the location of the lower recording level, allows unique identification of the stiffness distribution above the lower positioned sensor.

Corollary:

If the location of the sensors is fixed and made immovable at two consecutive story levels, then the stiffness constants k_ℓ, $1 \leq \ell \leq N$, can all be uniqueley determined by obtaining the responses of the system for at most two different shaker locations.

Proof:

Using Theorems 2 and 3, the result follows. □

(c) We next present a result for the situation in which the response measurement sensors are fixed at two consecutive levels i.e. w_n and w_{n+1} are available with $n > 1$. If two different types of excitation are used unique identification of all the stiffness constants is possible.

Theorem 4:

If (a) the responses $w_i(t)$ and $w_{i+1}(t)$ to the forcing function $f_j(t)$ are known, for some $1 < i \leq N$, and, $j \leq i$, and (b) the responses $v_i(t)$ and $v_{i+1}(t)$ to the base motion $W_{N+1}(t)$ are known for the same two locations, then, the stiffness k_n can be uniquely determined for $1 \leq n \leq N$, if $1 \leq m_i \leq N$ are known.

Proof:

In view of condition (a), and Theorem 2, knowledge of $w_i(t)$ and $w_{i+1}(t)$ implies knowledge of k_j, $i \leq j \leq N$. Also in view of condition (b), and the results proved in [7], knowledge of $v_i(t)$

and $v_{i+1}(t)$ to the base motion $w_{N+1}(t)$ implies knowledge of k_j, $1 \leq j \leq i$. Thus, the result. □

We have shown that if the response of *any* two adjacent mass levels (stories) can be obtained for (a) base excitations of the structure, and (b) forced vibrations caused by shaking the top most mass (roof) of the structure, then sufficient data is available to completely identify the stiffness distribution in the structure uniquely.

Some Illustrative Examples

To ellucidate some of the results obtained, let us present here, two illustrative examples. Consider a three degree of freedom system subjected to a forcing function $f_1(t)$ at mass level m_1 (Figure 1). The governing equations of motion can be described by

$$\begin{bmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{bmatrix} \begin{matrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{matrix} + \begin{bmatrix} k_1 & -k_1 & \\ -k_1 & k_1+k_2 & -k_2 \\ & -k_2 & k_2+k_3 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = \begin{bmatrix} f_1(t) \\ 0 \\ 0 \end{bmatrix} . \quad (22)$$

Let us assume that the values of m_1, m_2 and m_3 are known apriori (say from design drawings) and that we are interested in estimating the values of k_1, k_2 and k_3 by measuring the response of the system to our excitation function $f_1(t)$. This, incidentally, is a common situation for it is generally much more difficult to estimate the stiffness distribution in a system from design drawings than it is to estimate its mass distribution.

A relevant question one might pose then is as follows: Given that such a forced vibration test is being undertaken, would it be more advantageous, in so far as estimating the correct parameter values k_1, k_2 and k_3 is concerned from test data, to measure the response $x_1(t)$ of mass m_1 or $x_2(t)$ of mass m_2 or $x_3(t)$ of mass m_3? Theorem 1 provides an answer to this question and prescribes that unique identification would be possible if the time history of response $x_1(t)$ is measured at the top-most mass level.

Example 1:

To be specific, consider the system with the following parameter values:

$$m_1 = 1, m_2 = 2, m_3 = 1 ;$$
$$k_1 = 10, k_2 = 20, k_3 = 30 .$$

At this point, the parameters k_i, $i = 1,2,3$ are assumed to be unknown. While one could rerun the proof of Theorem 1 putting in the numerical values for the various algebraic quantities, greater insight will be acquired by following a slightly different line of approach.

Taking Fourier transforms on both sides of Equation (22) we have

$$\begin{bmatrix} k_1-\lambda & -k_1 & \\ -k_1 & k_1+k_2-\lambda & -k_2 \\ & -k_2 & k_2+k_3-\lambda \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} F_i \\ 0 \\ 0 \end{bmatrix} \quad (23)$$

where the capital letters indicate the transform quantities.

Solving for X_i, we get

$$X_i = F_1 \frac{\Delta i}{\Delta} , \quad (24)$$

where

$$\left.\begin{aligned} \Delta_1 &= 2\lambda^2 - \lambda(k_1 + 3k_2 + 2k_3) + k_1k_2 + k_2k_3 + k_3k_1 \\ \Delta_2 &= k_1(k_2 + k_3 - \lambda) \\ \Delta_3 &= k_1k_2 \\ -\Delta &= 2\lambda^3 - \lambda^2(3k_1 + 3k_2 + 2k_3) \\ &\quad + \lambda(4k_1k_2 + 3k_1k_3 + k_2k_3) - k_1k_2k_3 \end{aligned}\right\} \quad (25)$$

a) If $x_1(t)$ is measured, then $X_1(\lambda)$ is known and therefore the ratio

$$-\frac{X_1(\lambda)}{F_1(\lambda)} = \frac{\Delta_1}{\Delta} , \quad (27)$$

is known. Let us ask the question, how many sets of values $\{k_1,k_2,k_3\}$ yield identical values of $X_1(\lambda)/F_1(\lambda)$? If this ratio is the same for all λ, for the sets $\{k_1,k_2,k_3\}$, we then require using the parameter values for the system,

$$\left.\begin{aligned} &3k_1 + 3k_2 + 2k_3 = 150 \\ &4k_1k_2 + 3k_1k_3 + k_2k_3 = 2300 \\ &k_1k_2k_3 = 6000 \\ &k_1 + 3k_2 + 2k_3 = 130 \\ &k_1k_2 + k_2k_3 + k_3k_1 = 1100 . \end{aligned}\right\} \quad (28)$$

The Equations (26) result because the numerator and denominator of (27) do not have any pole-zero cancellation as guaranteed by Lemma 2. It turns out, that Equations (28) have one unique solution k_1=10, k_2=20, k_3=30.

Thus, if two systems defined by two different sets of values $\{k_1,k_2,k_3\}$, have identical 'top mass response, forcing function' time-history pairs then the two systems must be identical. One thus has sufficient information to uniquely identify the system stiffness parameters from the top-mass response. This result is guaranteed by Theorem 1.

We note, in passing, that for the set (28) to be valid, all we need is a segment of data over a continuous, finite interval of λ. Forcing functions that do that are amply available.

b) To find if we could uniquely identify k_1,k_2,k_3 from a knowledge of $x_2(t)$ and $f_1(t)$, we write

$$-\frac{X_2(\lambda)}{F_1(\lambda)} = \frac{\Delta_2}{\Delta} . \quad (29)$$

Under the proviso that in the numerator and denominator of (29), no pole-zero cancellation occurs, we have

$$\left.\begin{aligned} &k_1 = 10 \\ &k_2 + k_3 = 50 \\ &k_1k_2k_3 = 6000 \\ &3k_1 + 3k_2 + 2k_3 = 150 \\ &4k_1k_2 + 3k_1k_3 + k_3k_2 = 2300 . \end{aligned}\right\} \quad (30)$$

In fact even if a pole-zero cancellation occurs in relation (29) the data available, $x_2(t)$, will have sufficient information to identify the stiffness distribution uniquely. In the following example we shall illustrate this.

c) The expression for $X_3(\lambda)$ yields

$$-\frac{X_3(\lambda)}{F_1(\lambda)} = \frac{k_1k_2}{\Delta} . \quad (31)$$

This gives the set of equations

$$\begin{aligned} &k_1k_2 = 200 \\ &4k_1k_2 + 3k_3k_1 + k_2k_3 = 2300 \end{aligned} \quad (32)$$

$$3k_1 + 3k_2 + 2k_3 = 150 \tag{32}$$
$$k_1 k_2 k_3 = 6000$$

which for this example again yields an unique set of parameters k_1, k_2, k_3. It turns out that for a three-degree of freedom system, knowledge of $x_3(t)$ uniquely defines the stiffness distribution. For systems with larger number of degrees of freedom, this may not be true in general. In a sense, therefore, our three-degree of freedom example is slightly restrictive in exhibiting the structure of the results.

Example 2:

Consider the system: $m_1 = 1$, $m_2 = 2$, $m_3 = 1$, $k_1 = 50$, $k_2 = 20$, $k_3 = 30$.

Should we measure the response $x_2(t)$ of such a system to a given $f_1(t)$, we get

$$-\frac{X_2(\lambda)}{F_1(\lambda)} = \frac{k_1}{2\lambda^2 - (3k_1 + k_2)\lambda + k_1 k_2 - k_2^2}$$

due to the pole-zero cancellation that occurs. Thus two systems that have identical "$X_2(\lambda), F_1(\lambda)$" pairs for all λ values, simply require

$$k_1 = 50$$
$$3k_1 + k_2 = 170$$
$$k_1 k_2 - k_2 = 600$$
$$k_1 = k_2 + k_3 .$$

This yields $k_1 = 50$, $k_2 = 20$ and $k_3 = 30$ as per the corollary following Theorem 2.

Discussion and Conclusions

In this paper we have modelled an N-story structure by an N-degree of freedom lumped mass system. The mass distribution is assumed to be known apriori. The structure is taken to be undamped and the forced excitation is assumed to be broadband. Identification of the stiffness constants have been investigated when the forcing function is applied at the top most story. If the force-time history of the excitation is available, unique identification of all the stiffnesses is guaranteed with the measurement of the response at just *one* location, namely the top story. If the force-time history of excitation is unknown, the response measurement at the top two stories in the structure will again yield unique identification of the complete stiffness distribution.

If the measurements of the response are made at *any* two consecutive stories (Figure 1) n and $n+1$, then as long as the unknown forcing function is applied above or at the nth floor level, the stiffness distribution, k_i, $n \le i \le N$ is uniquely identifiable from the measurements. If the unknown forcing function is applied below or at the $(n+1)$th story level, then the stiffness distribution k_i, $1 \le i \le n$ can be uniquely identified from the measured data. Thus, given that the sensors in a structure are located at stories n and $n+1$, unique identifications of k_i, $1 \le i \le N$ can be achieved by taking measurements for at most two shaker locations.

The result of Theorem 4 is an extension of a known result by Levinson [8]. Levinson shows that if the complete frequency spectrum is known for two different sets of boundary conditions, then the stiffness distribution can be uniquely deduced. The result obtained in Theorem 4 shows that for the two different types of boundary conditions prescribed, the response at *any two consecutive* stories is sufficient to uniquely identify the complete stiffness distribution. The particular usefulness of this result arises by virtue of the fact that responses to base excitations in tall structures can be obtained from data available during ground blasts and strong earthquake ground shaking. It is to be noted that the measurement of a single time history will, in general, *not* yield unique identification even when the two different types of excitations of Theorem 4 are used. The minimum number of time histories required for unique identification of the complete stiffness distribution always appears to be two.

Though the structural system in this paper is modelled in a simplistic manner and may be unrealistic for several situations, the model used here is one commonly encountered in engineering analysis and design of tall buildings, nuclear power plants and subassemblages. We note that the results obtained are valid for undamped systems and are therefore, in a sense, restrictive. Also, it should be pointed out that the analysis is pertinent to noise free data, so that even if the identification problem may have a unique solution, in practice, the noise corruption could lead to large variances in the estimates of the identified parameters.

An important application of the results indicated here is to the area of geotechnical engineering, wherein a multi-layered soil stratum is often modelled as a discrete lumped mass system as shown in Figure 2. Assuming that the mass density distribution with depth is known, the result of Theorem 1, namely that, if the surface response to a measured surface source of excitations of SH waves is available, then the stiffness distribution with depth is uniquely identifiable, is of great usefulness for soils and geological exploration.

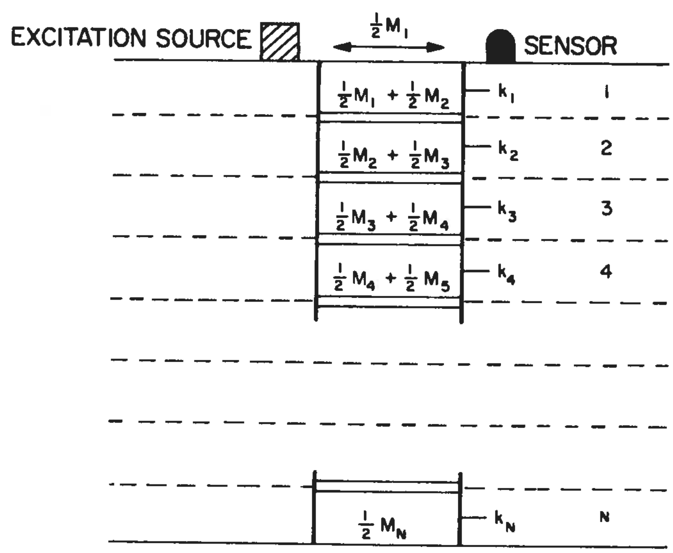

Figure 2. Response Analysis of Layered Systems

Acknowledgements

This work was funded in part by a grant from the National Science Foundation. Additional funding was provided by Design, Analysis and Software Engineering, Inc.

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