Unified Approach to Modeling and Control of Rigid Multibody Systems

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A unified approach is developed to model complex multibody mechanical systems and design controls for them. The characterization of such complex systems often requires the use of more coordinates than the minimum number to describe their configurations and/or the use of modeling constraints to capture their proper physical descriptions. When required to satisfy prescribed control requirements, it becomes necessary that the generalized control forces they are subjected to exactly satisfy these modeling constraints so that their physical descriptions are correctly preserved. The control requirements imposed can always be interpreted as a set of additional control constraints, and they may or may not be consistent with the modeling constraints that describe the physical system. This paper considers both the cases when the control constraints are consistent with the modeling constraints and when they are inconsistent. Such inconsistencies can arise when dealing with underactuated systems. A user-prescribed control cost is minimized at each instant of time in both cases. No linearizations/approximations of the nonlinear mechanical systems are made throughout. Insights into the control methodology are afforded through its geometric interpretation. Numerical examples with full-state control and underactuated control are considered, demonstrating the simplicity of the approach, its ease of implementation, and its effectiveness.

I. Introduction

RECENT advances in the control of nonlinear, nonautonomous mechanical systems have led to the development of a new perspective in which control requirements are viewed as constraints that are imposed on these systems. Instead of the use of conventional control theory, this approach is grounded in recent results from the field of analytical dynamics through the use of the fundamental equation of mechanics [1,2]. The salient advantage of this perspective is its simplicity, effectiveness, and ease of implementation. The (generalized) control forces required to enforce these control requirements (constraints) are obtained in closed form without the need for any linearizations/approximations of the dynamical systems involved or the need for imposing any a priori structure on the nature of the controller [3–6]. These (generalized) control forces are readily computable, making real-time control of complex nonlinear, nonautonomous dynamical systems possible. For nonlinear, nonautonomous systems for which the dynamical models are assumed to be known and that permit full-state control, this approach ensures that the control requirements are exactly satisfied while simultaneously minimizing a user-specified quadratic control cost at each instant of time [3–7]. For example, the tracking control of a set of slave gyroscopes that exhibit chaotic motions so that each slave exactly tracks the chaotic motions of a master gyroscope (or gyro), while simultaneously minimizing a quadratic control cost, was demonstrated in [8]. The same approach is further used to obtain the closed-form tracking control of a cluster of nonlinearly coupled slave gyros so they exactly track the chaotic motions of a master gyro [9]. Applications to the control of systems with complex and highly nonlinear dynamics such as the formation flight of spacecraft in nonuniform gravity fields illustrate the simplicity and effectiveness of the closed-form approach [10–12]. Current extensions to dynamical systems subjected to (generalized) forces that are only imprecisely known and/or to systems whose description is only imprecisely known can be found in [13–16]. Also, stable full-state control of a general nonlinear, nonautonomous mechanical system has been achieved by casting the objective of realizing asymptotically stable control as a Lyapunov constraint on the system [5,6,15,16]. More recently, Pappalardo used the approach to develop a method for the kinematic and dynamic analysis of rigid multibody systems [17].

Despite the variety of problems that the approach has been able to handle, its application to large-scale complex dynamical systems requires that it be able to simultaneously handle both modeling constraints and control requirements in an effective manner. Most large-scale complex nonlinear systems are modeled through the use of modeling constraints because their use significantly simplifies the task of the modeler [18]. Here, the term "large-scale" refers to nonlinear nonautonomous systems with a large number of degrees of freedom. The important aspect here is that the modeling constraints are quintessential for providing a proper mathematical model of the physical mechanical system at hand and they must be satisfied at all times in order to maintain the integrity of system's physical description. When such systems are further subjected to control requirements, such as when required to follow specific trajectories (as in trajectory tracking), these control requirements, as stated previously, can also be interpreted as constraints on the mechanical system, but with a difference. For, were the control constraints not to be exactly satisfied, this would naturally lead to an inadequate satisfaction of the control requirements, resulting in poor control. But, if the modeling constraints are dissatisfied, the integrity of the physical description of the mechanical system is jeopardized because, now, one is not dealing with the proper mathematical model of the physical system. Moreover, the (quadratic) costs associated with the modeling constraints and the control constraints can be different because the cost function to be minimized for the modeling constraints comes from analytical dynamics, whereas that for the control constraint is specified by the control engineer.

Traditionally, when modeling constraints are present in addition to control requirements, control has been designed under simplified assumptions by simplifying/approximating the dynamical system and/or imposing structure on the controller. For example, in [19], the constrained motion of a biped robot was tackled by linearizing the system about an operating point and assuming the modeling constraints to be holonomic. Under these simplified assumptions, linear control was obtained for the nonlinear system by pole placement techniques. In [20], a geometric approach inspired by [21,22] was presented for the analysis and control of constrained mechanical systems. A different approach using feasible trajectories

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that satisfied the modeling constraints in real time was presented in [23].

In contrast to the aforementioned methods, the current paper takes the analytical dynamics-based view in which the control requirements are seen as control constraints. No simplifying assumptions are made about the dynamics of the system, nor is any structure imposed on the controller. A unified approach for modeling and control is developed so that when the control constraints (requirements) are consistent with the modeling constraints–constraints are said to be consistent if there is at least one solution that satisfies all the constraints, otherwise they are inconsistent–both sets of constraints can be (exactly) satisfied simultaneously and the physical system will precisely meet the control requirements while simultaneously minimizing a user-prescribed control cost at each instant of time. This situation (where the constraints are consistent) arises when full-state control is used and the control requirements are feasible. However, when the constraints become inconsistent, satisfaction of the modeling constraints is always enforced (since these constraints pertain to the proper description of the physical system) at the expense of not exactly satisfying the control constraints (requirements) while still minimizing (in the L_2-norm sense) a user-desired control cost. This situation can arise when dealing with underactuated systems. In what follows, the terms "control requirements" and "constraints" will be used synonymously. The set of constraints may contain holonomic and/or nonholonomic constraints.

The task of providing such a unified approach was first introduced in a landmark paper by Schutte [24]. Motivated by the general result for constrained systems that did not satisfy d'Alembert's principle [25,26], Schutte developed an approach for modeling and controlling mechanical systems with general holonomic and nonholonomic constraints [24]. To the authors' knowledge, this is the first and only paper that addresses this important and practical problem. The paper considers control constraints that are, in general, not consistent with the modeling constraints. First, (generalized) control forces are obtained that enforce the control constraints. Next, these control forces are projected into the space of forces that produce accelerations consistent with the modeling constraints. These control forces are referred to in [24] as permissible forces because their addition does not violate the modeling constraints. As stated in the paper, the method does not guarantee that the control requirements will be ultimately met.

The current work is different from [24] in the following aspects:

1) It conceptualizes the control problem in a different manner by considering the constraint force needed to satisfy the modeling constraints to be a function of the control force needed to satisfy the control requirements.

2) A user-specified norm of the (generalized) control force is minimized at each instant of time while ensuring that the modeling constraints are exactly satisfied at each instant of time.

3) The control constraints are satisfied in the least-square sense at all times; when the control constraints are consistent with the modeling constraints, both sets of constraints are exactly satisfied.

4) The control force is obtained, irrespective of the consistency of the constraints, so there is no need to check for consistency to choose whether to use the fundamental equation of motion [3,4] or the approach in [24].

5) The control cost function could be the same as the Gaussian [27] or different, as prescribed by the user.

The organization of the paper is as follows. Section II provides the description of the constraints, introduces the notation that will be used in the sequel, and poses the control problem in this notation. In the next section, a brief review of the fundamental equation of motion is provided along with some further notation. In Sec. IV, the general case in which the modeling constraints [Eq. (13)] and the control constraints [Eq. (14)] need not be consistent with each other is treated. In Sec. V, the special case in which the constraints are consistent is considered. Two possibilities are considered here, depending on whether or not the control cost function $J_c(t)$ is the same as the Gaussian $G(t)$. Both cases are investigated in detail and the simplifications that emerge are considered. The expression for the control force in Sec. V can be viewed as a simplified version of that obtained in the general case dealt with in Sec. IV. Section VI provides a geometric understanding of the current approach. Three numerical examples that demonstrate the approach and its effectiveness are provided in Sec. VII. Conclusions are given in Sec. VIII.

II. Description of Constraints and Notation

Consider a system for which the unconstrained motion can be described by the equations

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) \tag{1}$$

where M is a symmetric positive definite mass matrix, and Q is the generalized impressed (given) force vector. The n-dimensional column vector that represents the configuration of the system is $q \in R^n$. A dot on top of a variable represents the derivative with respect to time, and two dots represent the second derivative with respect to time.

Let us assume that n_m constraints need to be imposed on the unconstrained system described by Eq. (1) so that the constrained system now provides a proper description of the physical system under consideration. Let these constraints be of the form

$$\phi_m(q,\dot{q},t) = 0 \tag{2}$$

In the preceding equation, $\phi_m \in R^{n_m}$ is a column vector. These constraints can be holonomic and/or nonholonomic.

It is important to note that these modeling constraints are essential in providing a proper mathematical description of the physical system, and they must be satisfied at all times if the mathematical model is to represent the physical system with probity. Furthermore, these constraints must be consistent; else, once again, the mathematical modeling of system will be incorrect.

We assume that the constraints are smooth enough to be differentiated a sufficient number of times to get equations of the form [1]

$$A_m(q,\dot{q},t)\ddot{q} = b_m(q,\dot{q},t) \tag{3}$$

where A_m is an $n_m \times n$ matrix. Nature applies a force $Q_m(q,\dot{q},t)$ (called the "constraint force") on the system so that these modeling constraints are enforced. According to Gauss's principle, nature appears to apply the constraint force in such a way that the Gaussian $G(t)$ defined as

$$G(t) = Q_m^T(q,\dot{q},t)M^{-1}(q,t)Q_m(q,\dot{q},t) \tag{4}$$

is minimized at each instant of time [1]. Thus, nature appears to minimize a quadratic cost $G(t)$ with the specific weighting matrix M^{-1}. The consequent equation of motion of the constrained mechanical system is then

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q_m(q,\dot{q},t) \tag{5}$$

When control requirements (objectives) are placed on this mechanical system, these requirements can be viewed as a set of additional constraints. Thus, the system is subjected to control requirements of the form

$$\phi_c(q,\dot{q},t) = 0 \tag{6}$$

where ϕ_c is a column vector of n_c dimensions.

We assume that these constraints are smooth enough to be differentiated and expressed in the form

$$A_c(q,\dot{q},t)\ddot{q} = b_c(q,\dot{q},t) \tag{7}$$

where A_c is an $n_c \times n$ matrix. To satisfy the control objectives, one needs to apply a (generalized) control force $Q_c(q,\dot{q},t)$ to the system so that the system also satisfies the set of control constraints given in

Eq. (7). This control force Q_c is obtained by minimizing the control cost

$$J_c(q,\dot q,t) = Q_c^T(q,\dot q,t)W(q,\dot q,t)Q_c(q,\dot q,t) \tag{8}$$

where $W(q,\dot q,t)$ is a user-prescribed symmetric, positive definite matrix.

The two sets of constraints given in Eqs. (3) and (7) are very different in character. Although the modeling constraints given in Eq. (3) must be exactly satisfied for the mathematical model to represent the physical system with integrity, the control constraints given by Eq. (7) may or may not. When the control constraints are exactly satisfied, the control objectives for (controlling) the physical system are met; when they are not, then we may have inadequate control of the physical system.

The goal is to find the (generalized) force n vectors Q_m and Q_c such that the (controlled) dynamical system described by the equation

$$M(q,t)\ddot q = Q(q,\dot q,t) + Q_m(q,\dot q,t) + Q_c(q,\dot q,t) \tag{9}$$

satisfies the following conditions.

1) Q_m is found such that a) the dynamical system always satisfies the modeling constraints [Eq. (3)] and b) the Gaussian $G(t)$ shown in Eq. (4) is minimized at each instant of time.

2) Q_c is found such that a) the dynamical system satisfies the control constraints [Eq. (7)] as best possible (L_2 norm), and b) the user-specified control cost $J_c(t)$ shown in Eq. (8) is minimized at each instant of time.

In what follows, instead of the accelerations, we use "scaled" accelerations, which were first introduced in [26]. Scaling consists of premultiplying Eq. (9) by the matrix $M^{-1/2}$. Thus, after scaling, Eq. (9) can be rewritten as

$$M^{1/2}\ddot q = M^{-1/2}Q + M^{-1/2}Q_m + M^{-1/2}Q_c \tag{10}$$

which upon defining the scaled accelerations as

$$\ddot q_s = M^{1/2}\ddot q, \quad a_s = M^{-1/2}Q, \quad \ddot q_s^m = M^{-1/2}Q_m, \quad \ddot q_s^c = M^{-1/2}Q_{c,} \tag{11}$$

simplifies to

$$\ddot q_s = a_s + \ddot q_s^m + \ddot q_s^c \tag{12}$$

The subscript s denotes scaled quantities. We refer to $\ddot q_s$ as the scaled acceleration of the controlled system, a_s as the scaled acceleration of the unconstrained system, $\ddot q_s^m$ as the scaled constraint acceleration, and $\ddot q_s^c$ as the scaled control acceleration. However, from here on, scaled acceleration vectors will simply be referred to as accelerations for brevity, unless required for clarity.

The modeling constraint can now be written in scaled form as

$$B_m\ddot q_s = b_m \quad \text{with } B_m := A_m M^{-1/2} \tag{13}$$

and the control constraint as

$$B_c\ddot q_s = b_c \quad \text{with } B_c := A_c M^{-1/2} \tag{14}$$

With the two sets of constraints given in Eqs. (13) and (14), and the two different cost functions given in Eqs. (4) and (8) to be minimized while enforcing these constraints, several different situations are possible.

III. Fundamental Equation of Motion

Before dealing with the general case involving both control and modeling constraints, let us first ignore the control constraints and look at the system with only the modeling constraints. Consider the motion of the constrained system for which the unconstrained motion is described by Eq. (1) along with the modeling constraints described by Eq. (2) [or, alternatively, by Eq. (13)]. The equation of motion of the constrained system is given as

$$M(q,t)\ddot q = Q(q,\dot q,t) + Q_m(q,\dot q,t) \tag{15}$$

which in terms of scaled accelerations can be rewritten as

$$\ddot q_s = a_s + \ddot q_s^m \tag{16}$$

The constraint acceleration $\ddot q_s^m$ of the constrained system is explicitly found using the fundamental equation of motion as [2]

$$\ddot q_s^m = B_m^+(b_m - B_m a_s) \tag{17}$$

where X^+ is the Moore–Penrose inverse of the matrix X [28]. Thus, the equation of the motion of the constrained system, when the only constraints to be satisfied are the modeling constraints, can be written simply as

$$\ddot q_s = a_s + B_m^+(b_m - B_m a_s) \tag{18}$$

or, alternatively, as

$$\ddot q_s = (I_n - B_m^+B_m)a_s + B_m^+b_m \tag{19}$$

In the preceding, I_n denotes an $n\times n$ identity matrix. We denote the scaled acceleration of the physical system subjected only to the modeling constraints and, as yet, not subjected to any control constraints by $\ddot u_s$ (referred to as scaled acceleration of the uncontrolled system) so that

$$\ddot u_s := (I_n - B_m^+B_m)a_s + B_m^+b_m \tag{20}$$

Two important properties of the matrix $(I_n - B_m^+B_m)$ in Eqs. (19) and (20) are noteworthy.

1) The matrix $(I_n - B_m^+B_m)$ is an orthogonal projection matrix.

2) It projects any scaled acceleration vector into the null space of B_m, thus ensuring that the modeling constraint [Eq. (13)] is always satisfied.

The first property can be quickly shown as follows:

$$\begin{aligned}(I_n - B_m^+B_m)^2 &= I_n - 2B_m^+B_m + B_m^+B_mB_m^+B_m \\ &= I_n - 2B_m^+B_m + B_m^+B_m = I_n - B_m^+B_m\end{aligned} \tag{21}$$

In the preceding, we have made use of the Moore–Penrose condition $B_m^+B_mB_m^+ = B_m^+$. Furthermore, the matrix $(I_n - B_m^+B_m)$ is symmetric because $(B_m^+B_m)^T = B_m^+B_m$.

To verify the second property, consider any n vector v. Its projection is $(I_n - B_m^+B_m)v$. Then, we have

$$B_m(I_n - B_m^+B_m)v = (B_m - B_mB_m^+B_m)v = (B_m - B_m)v = 0 \tag{22}$$

Also, the minimum norm least-squares solution to the constraint equation $B_m\ddot q_s = b_m$ is given by $\ddot q_s = B_m^+b_m$. When the constraints are consistent, which they must be if the modeling is done correctly, consistency then implies that [1]

$$B_mB_m^+b_m = b_m \tag{23}$$

Premultiplying Eq. (19) by B_m and using Eqs. (22) and (23), one obtains

$$B_m\ddot q_s = B_m(I_n - B_m^+B_m)a_s + B_mB_m^+b_m = b_m$$

thus ensuring that the scaled acceleration vector $\ddot q_s$ satisfies the constraint given by Eq. (13).

From a geometrical viewpoint, Eq. (19) can now be interpreted as follows. Nature appears to enforce the constraint given in Eq. (13) in two steps. First, she projects the unconstrained acceleration vector a_s onto the null space of B_m and then adds the correction vector $B_m^+b_m$ so that the modeling constraint given in Eq. (13) is always satisfied (see Sec. VI for a geometrical viewpoint).

Now, if we were to apply a (generalized) control force Q_c in addition to Q on the system, the equation of motion of the controlled system becomes

$$M\ddot{q} = (Q + Q_c) + Q_m \tag{24}$$

which can be rewritten in terms of the accelerations as

$$\ddot{q}_s = (a_s + \ddot{q}_s^c) + \ddot{q}_s^m \tag{25}$$

Use of the fundamental equation then gives the constraint acceleration $\ddot{q}_s^m$ as

$$\ddot{q}_s^m = B_m^+[b_m - B_m(a_s + \ddot{q}_s^c)] \tag{26}$$

Thus, the equation of motion of the controlled system is

$$\begin{aligned}\ddot{q}_s &= a_s + B_m^+(b_m - B_m a_s) + \ddot{q}_s^c - B_m^+ B_m \ddot{q}_s^c \\ &= (I_n - B_m^+ B_m)a_s + B_m^+ b_m + (I_n - B_m^+ B_m)\ddot{q}_s^c\end{aligned} \tag{27}$$

Equation (27) shows the effect of adding a control force to the system. Comparing Eqs. (20) and (27), the scaled acceleration $\ddot{q}_s$ is modified in the presence of Q_c by the addition of the projection of the scaled control acceleration $\ddot{q}_s^c$ into the null space of B_m. It is important to note that Eq. (27) ensures that the modeling constraints given by Eq. (13) are always satisfied, no matter what the scaled control acceleration $\ddot{q}_s^c$.

IV. Inconsistent Constraints

When the constraints are inconsistent, no acceleration n-vector $\ddot{q}_s$ can be found that can simultaneously satisfy both the constraint sets given by relations (13) and (14). In such a situation, as explained before, it is still required that the modeling constraints be always satisfied; else, the integrity of the proper physical description of the mechanical system will be compromised. One could thus imagine that 1) the controller applies a control force Q_c, and 2) the appropriate constraint force Q_m is created in response to this by the physical mechanical system; one needs to ensure then that, in the presence of Q_c, the mathematical model is in compliance with the proper physical description of the system.

In other words, Q_m may be considered a function of Q_c. The physical mechanical system sees the control force Q_c as an externally applied force and reacts appropriately to it, ensuring that the modeling constraints, which ensure the integrity of the physical description of the system, are always satisfied. Hence, once an explicit expression for Q_m as a function of Q_c and Q is obtained, Q_c can be determined by minimizing the 2-norm of the error e in satisfying the control constraints

$$\|e\| = \|A_c\ddot{q} - b_c\| = \|B_c\ddot{q}_s - b_c\| \tag{28}$$

while simultaneously minimizing the control cost at each instant of time. The equation of the system in the presence of these two forces is given as

$$M\ddot{q} = Q + Q_m + Q_c \tag{29}$$

For this system, for a given impressed force Q and a given control force Q_c, the constraint force Q_m that ensures that 1) the modeling constraints [Eq. (13)] are satisfied and 2) the Gaussian $G(t)$ in Eq. (4) is minimized is given by the fundamental equation of motion.

Result 1: The (generalized) control force Q_c that minimizes 1) the norm of the error in satisfying the control constraints [see Eq. (28)] and 2) simultaneously minimizes the control cost J_c shown in Eq. (8) is

$$Q_c = M^{1/2}\ddot{q}_s^c \tag{30}$$

where the control acceleration $\ddot{q}_s^c$ is given by

$$\ddot{q}_s^c = SB_{cms}^+(b_c - B_c\ddot{u}_s) \tag{31}$$

The various quantities in the preceding equation are, respectively,

$$\begin{aligned}&S = M^{-1/2}W^{-1/2}, \quad B_{cms} = B_c(I_n - B_m^+ B_m)S, \quad \text{and} \\ &\ddot{u}_s = (I_n - B_m^+ B_m)a_s + B_m^+ b_m\end{aligned} \tag{32}$$

Proof: In Sec. III, the equation of motion of the controlled system has been obtained in terms of the accelerations as

$$\ddot{q}_s = a_s + B_m^+(b_m - B_m a_s) + (I_n - B_m^+ B_m)\ddot{q}_s^c \tag{33}$$

If we denote the acceleration of the uncontrolled system [see Eq. (20)] as

$$\ddot{u}_s = (I_n - B_m^+ B_m)a_s + B_m^+ b_m \tag{34}$$

Equation (33) simplifies to

$$\ddot{q}_s = \ddot{u}_s + (I_n - B_m^+ B_m)\ddot{q}_s^c \tag{35}$$

By substituting Eq. (35) into Eq. (28), we obtain

$$\|e\| = \|B_c\ddot{u}_s + B_c(I_n - B_m^+ B_m)\ddot{q}_s^c - b_c\| \tag{36}$$

which can be simplified as

$$\|e\| = \|B_c(I_n - B_m^+ B_m)\ddot{q}_s^c - (b_c - B_c\ddot{u}_s)\| \tag{37}$$

The control cost $J_c(t)$ given in Eq. (8) can be written in terms of the quantity $\ddot{q}_s^c$ as

$$J_c = Q_c^T W Q_c = \ddot{q}_s^{c^T} M^{1/2}W^{1/2}W^{1/2}M^{1/2}\ddot{q}_s^c \tag{38}$$

If we define the quantities

$$S^{-1} \coloneqq W^{1/2}M^{1/2}, \quad \text{and} \quad z_c \coloneqq W^{1/2}M^{1/2}\ddot{q}_s^c = S^{-1}\ddot{q}_s^c \tag{39}$$

then the control cost is simply $J_c = z_c^T z_c$ and $\ddot{q}_s^c = Sz_c$. Minimizing J_c would then mean selecting a vector z_c with a minimum L_2 norm. Note that the matrix S is invertible.

Using these quantities, Eq. (37) reduces to

$$\|e\| = \|B_c(I_n - B_m^+ B_m)Sz_c - (b_c - B_c\ddot{u}_s)\| \tag{40}$$

The problem of finding $\ddot{q}_s^c$ is therefore reduced to finding the vector z_c that minimizes the norm of the error in Eq. (40) and has a minimum norm (since $J_c = \|z_c\|^2$).

Denoting

$$B_{cms} \coloneqq B_c(I_n - B_m^+ B_m)S \tag{41}$$

the solution is simply given by

$$z_c = B_{cms}^+(b_c - B_c\ddot{u}_s) \tag{42}$$

Thus, the explicit expression for the control acceleration $\ddot{q}_s^c$ is

$$\ddot{q}_s^c = Sz_c = SB_{cms}^+(b_c - B_c\ddot{u}_s) \tag{43}$$

Using this relation, the generalized control force is $Q_c = M^{1/2}\ddot{q}_s^c$, as in Eq. (30). □

Result 2: The equation of motion of the dynamical system is given by

$$\ddot{q}_s = a_s + B_m^+(b_m - B_m a_s) + (I_n - B_m^+ B_m)SB_{cms}^+(b_c - B_c\ddot{u}_s) \tag{44}$$

where $\ddot{u}_s \coloneqq (I_n - B_m^+ B_m)a_s + B_m^+ b_m$, $S = M^{-1/2}W^{-1/2}$, and $B_{cms} = B_c(I_n - B_m^+ B_m)S$.

Alternately, the explicit generalized forces Q_c and Q_m are given below in Remark 3.

Proof: Using Eqs. (25) and (26), one obtains

$$\begin{aligned}\ddot{q}_s &= a_s + \ddot{q}_s^c + \ddot{q}_s^m = a_s + \ddot{q}_s^c + B_m^+[b_m - B_m(a_s + \ddot{q}_s^c)] \\ &= a_s + B_m^+(b_m - B_m a_s) + (I_n - B_m^+ B_m)\ddot{q}_s^c\end{aligned} \tag{45}$$

Using Eq. (43) for $\ddot{q}_s^c$ in the last expression on the right gives the result where $\ddot{u}_s \coloneqq (I_n - B_m^+ B_m)a_s + B_m^+ b_m$ by Eq. (34). Alternatively, from $\ddot{q}_s^c$ and $\ddot{q}_s^m$, the (generalized) forces $Q_c = M^{1/2}\ddot{q}_s^c$ and $Q_m = M^{1/2}\ddot{q}_m^c$ can be determined, and the equation of motion for the dynamical system can be described as shown in Eq. (29). □

Remark 1: If the constraints are not consistent, the control given in relation (30) minimizes $\|A_c\ddot{q} - b_c\|$. In other words, we are satisfying the control constraint given in Eq. (7) in the least-square sense while minimizing the control cost $J_c = Q_c^T W Q_c$ at each instant of time.

Remark 2: It must be noted that, for different weighting matrices W, the control force Q_c will be different, but the norm of the error in satisfying the control constraints $\|A_c\ddot{q} - b_c\|$ remains the same and is the minimum possible among all control forces [see Eq. (37)] that satisfy the modeling constraints. The control force that minimizes this norm and minimizes the user-specified control cost J_c given in Eq. (38) is obtained from Eqs. (43) and (30).

Remark 3: The equation of motion of the controlled system in the final form is

$$M\ddot{q} = Q + Q_m + Q_c \tag{46}$$

The (generalized) control force Q_c is explicitly given using Eqs. (43) and (30) by

$$Q_c = M^{1/2}SB_{cms}^+(b_c - B_c\ddot{u}_s) \tag{47}$$

and the (generalized) constraint force Q_m is explicitly given by using Eqs. (26), (43), and (11) by

$$\begin{aligned}Q_m &= M^{1/2}B_m^+[b_m - B_m(a_s + \ddot{q}_s^c)] \\ &= M^{1/2}[B_m^+(b_m - B_m a_s) - B_m^+ B_m SB_{cms}^+(b_c - B_c\ddot{u}_s)]\end{aligned} \tag{48}$$

It should be observed that the term $(b_m - B_m a_s)$ in the first member on the right-hand side of the preceding equation signifies the extent to which the unconstrained acceleration a_s does not satisfy the modeling constraints. Similarly, in the second member of the preceding equation, the term $(b_c - B_c\ddot{u}_s)$ signifies the extent to which the acceleration $\ddot{u}_s$ of the system in the absence of any control [as given in Eq. (34)] does not satisfy the control constraints. The weighting matrix S is related to weighting matrices used by nature [1] and that prescribed by the control engineer.

Remark 4: As will be explained later in Sec. VI (see Remark 6), the control force given by Eq. (47) can be discontinuous when the rank of matrix B_{cms} changes. If a smoother (generalized) control force is desired, as often required because of actuator (and other control equipment) limitations, an alternative approach to the one taken in Result 1 can be considered. Here, we wish to minimize, at each instant of time, an alternative cost function $\hat{J}_c$ that combines both the quantities minimized in Result 1, the control error $\|e\|$, and the control cost J_c, weighted by a positive parameter μ

$$\hat{J}_c = \|A_c\ddot{q} - b_c\|^2 + \mu J_c \tag{49}$$

The control force that minimizes the preceding control cost can be found (note the capital C in the subscript on Q) as

$$Q_C = M^{1/2}S(B_{cms}^T B_{cms} + \mu I_n)^{-1}B_{cms}^T(b_c - B_c\ddot{u}_s) \tag{50}$$

Observing the form of the cost function in Eq. (49), we can see that a tradeoff is being made between satisfying the control constraint and having a smooth control force. Increasing the value of μ results in a smoother control force but a larger error in satisfying the control constraint. On the other hand, decreasing the value of μ results in smaller control errors but larger variations in control force. The similarity between Eq. (50) and Eq. (47) is seen by noting that, if μ is set to zero in Eq. (50) and the regular inverse is changed to the pseudoinverse, we obtain Eq. (47).

To recap, the expression in Eq. (47) can be used to obtain the control force in the general situation when 1) the control constraints are not consistent with the modeling constraints and 2) the control cost J_c to be minimized is different from the Gaussian, i.e., $W \neq M^{-1}$. This explicit expression for the control force is obtained by minimizing the norm of the error in satisfying the control constraints and minimizing the control cost J_c while ensuring that the modeling constraints are always satisfied. Thus, the control obtained in the current approach has an in-built way of always respecting and satisfying the modeling constraints, which must be satisfied at all instants of time in order to preserve the proper correspondence between the physical system and its mathematical model. This is different from the approach used in [24] where any suitable control force is first found to satisfy the control constraints, and then this control force is made to satisfy the modeling constraints by projecting it onto the space of "permissible" control forces in order to respect the modeling constraints. Next, the situation where both the modeling constraints [Eq. (13)] and the control constraints [Eq. (14)] are consistent is taken up.

V. Consistent Constraints

When both the model constraints and control constraints are consistent, there are two possible situations, because the weighting matrix W could be the same as M^{-1} or it could be different. Consider first the case when they are not the same.

A. Unequal Weighting Matrices: $W \neq M^{-1}$

Such a situation could arise when we might be interested in designing a control system to control a mechanical system that already has some modeling constraints imposed on it, and the control requires the minimization of the norm of the (generalized) control force that is different from the Gaussian.

The main result for this section will be the proof that the force Q_c obtained using Eq. (47) ensures that the control constraints are exactly satisfied by the dynamical system; the modeling constraints are, of course, always satisfied.

Before we state and prove this result, we prove a lemma that will be used later.

Lemma: If, and only if, Eqs. (13) and (14) are consistent, then

$$B_{cms}B_{cms}^+(b_c - B_c B_m^+ b_m) = (b_c - B_c B_m^+ b_m) \tag{51}$$

Proof: The necessary and sufficient condition for both Eqs. (13) and (14) to be consistent is that there exists an n-vector ξ such that

$$B_m\xi = b_m \quad \text{and} \quad B_c\xi = b_c \tag{52}$$

From the first equality in Eq. (52), we know that there exists a vector υ such that [1]

$$\xi = B_m^+ b_m + (I_n - B_m^+ B_m)\upsilon \tag{53}$$

Substituting for b_m from the first equality in Eq. (52) into Eq. (53), we get

$$\xi = B_m^+ B_m \xi + (I_n - B_m^+ B_m)\upsilon \tag{54}$$

Rearranging the terms in Eq. (54), we have

$$(I_n - B_m^+ B_m)\xi = (I_n - B_m^+ B_m)\upsilon \tag{55}$$

Thus, Eq. (53) yields

$$\xi = B_m^+ b_m + (I_n - B_m^+ B_m)\xi \tag{56}$$

Substituting for ξ from Eq. (56) in the second equality of Eq. (52), we get

$$B_c B_m^+ b_m + B_c(I_n - B_m^+ B_m)\xi = b_c \tag{57}$$

Therefore, there exists a vector ξ such that

$$B_c(I_n - B_m^+ B_m)\xi = b_c - B_c B_m^+ b_m \tag{58}$$

Therefore, there exists a vector $\zeta = S^{-1}\xi$ such that

$$B_c(I_n - B_m^+ B_m)S\zeta = b_c - B_c B_m^+ b_m \tag{59}$$

Using the definition of B_{cms} [see Eq. (41)], Eq. (59) reduces to

$$B_{cms}\zeta = b_c - B_c B_m^+ b_m \tag{60}$$

Thus, we conclude that there exists a vector ζ such that Eq. (60) is true; hence,

$$B_{cms}B_{cms}^+(b_c - B_c B_m^+ b_m) = (b_c - B_c B_m^+ b_m) \tag{61}$$

□

Result 3: If Eqs. (13) and (14) are consistent, then the control force Q_c given in Eq. (47) ensures that the control constraints $B_c\ddot{q}_s = b_c$ (or, alternatively, $A_c\ddot{q} = b_c$) are exactly satisfied.

Proof: Using Eq. (33), we have

$$B_c\ddot{q}_s = B_c[a_s + B_m^+(b_m - B_m a_s) + (I_n - B_m^+ B_m)\ddot{q}_s^c] \tag{62}$$

and substituting for $\ddot{q}_s^c$ from Eq. (43), we get

$$\begin{aligned} B_c\ddot{q}_s &= B_c(I_n - B_m^+ B_m)a_s + B_c B_m^+ b_m \\ &\quad + B_c(I_n - B_m^+ B_m)SB_{cms}^+(b_c - B_c\ddot{u}_s) \end{aligned} \tag{63}$$

The third member on the right-hand side of the preceding equation can be simplified as

$$\begin{aligned} &B_c(I_n - B_m^+ B_m)SB_{cms}^+(b_c - B_c\ddot{u}_s) \\ &\quad = B_{cms}B_{cms}^+[b_c - B_c(I_n - B_m^+ B_m)a_s - B_c B_m^+ b_m] \\ &\quad = B_{cms}B_{cms}^+(b_c - B_c B_m^+ b_m) - B_{cms}B_{cms}^+ B_c(I_n - B_m^+ B_m)SS^{-1}a_s \\ &\quad = (b_c - B_c B_m^+ b_m) - B_{cms}B_{cms}^+ B_{cms}S^{-1}a_s \\ &\quad = (b_c - B_c B_m^+ b_m) - B_{cms}S^{-1}a_s \\ &\quad = (b_c - B_c B_m^+ b_m) - B_c(I_n - B_m^+ B_m)a_s \end{aligned} \tag{64}$$

The first equality from the preceding equation follows from the definitions of B_{cms} and $\ddot{u}_s$. In the third equality, we use Eq. (61) along with the definition of B_{cms}. Substituting Eq. (64) into Eq. (63) yields

$$\begin{aligned} B_c\ddot{q}_s &= B_c(I_n - B_m^+ B_m)a_s + B_c B_m^+ b_m + (b_c - B_c B_m^+ b_m) \\ &\quad - B_c(I_n - B_m^+ B_m)a_s \\ &= b_c \end{aligned} \tag{65}$$

□

B. Equal Weighting Matrices: $W = M^{-1}$

Such a situation could arise in real life when studying the effect of an additional set of physical constraints on a mechanical system, and these additional constraints would then show up in the mathematical model as additional modeling constraints. Another example is when we might be interested in designing a control system to control a mechanical system that already has some modeling constraints imposed on it and the weighting matrix W that describes the quadratic control cost is set equal to M^{-1}.

Since $W = M^{-1}$, from the first relation in Eq. (32), we have $S = I_n$. This simplifies various quantities such as

$$\begin{aligned} B_{cms}|_{S=I_n} &\coloneqq B_{cm} = B_c(I_n - B_m^+ B_m), \quad \text{and} \\ \ddot{q}_s^c &= B_{cm}^+(b_c - B_c\ddot{u}_s) \end{aligned} \tag{66}$$

As a result, simpler expressions for the constraint force Q_m and control force Q_c are obtained.

Corollary: The constraint and control forces, Q_m and Q_c, required to enforce the constraints given in Eqs. (3) and (7), and simultaneously minimize the cost functions in Eqs. (4) and (8) when $W = M^{-1}$, are

$$\begin{aligned} Q_m &= M^{1/2}\ddot{q}_s^m = M^{1/2}B_m^+(b_m - B_m a_s - B_m\ddot{q}_s^c), \\ Q_c &= M^{1/2}\ddot{q}_s^c = M^{1/2}[B_c(I_n - B_m^+ B_m)]^+(b_c - B_c\ddot{u}_s) \\ &= M^{1/2}B_{cm}^+(b_c - B_c\ddot{u}_s) \end{aligned} \tag{67}$$

where $\ddot{u}_s$ is given in Eq. (34).

Proof: The first relation is Eq. (26). The second relation is obtained from Eq. (43) by setting $S = I_n$. □

Remark 5: When the control requirements are not satisfied by the system at the initial time ($t = 0$) one can use instead of the constraints $\phi_c(q, \dot{q}, t) = 0$, the modified constraint equations [3]

$$\dot{\phi}_{c_i} + \gamma_i\phi_{c_i} = 0, \qquad \gamma_i > 0 \tag{68}$$

for each control requirement that can be expressed as a nonholonomic constraint, and by the modified constraint equations [3],

$$\ddot{\phi}_{c_i} + \alpha_i\dot{\phi}_{c_i} + \beta_i\phi_{c_i} = 0, \qquad \alpha_i, \beta_i > 0 \tag{69}$$

for each control requirement that can be expressed as a holonomic constraint. In fact, from a numerical point of view, the use of these modified constraints is usually useful, even when the system starts out satisfying the constraint requirements.

VI. Geometric Explanation of the Control Approach

In this section, a geometric interpretation of the method is provided. For ease of exposition, B_m and B_c are taken to be row vectors and modeling constraints are taken to be consistent with the control constraints. Figure 1 shows the general representation of the unconstrained system in R^2. O is the origin in the scaled acceleration space, and the vector $\boldsymbol{OA}$ represents the scaled unconstrained acceleration a_s. The modeling constraint is represented by the plane P_m described by the equation $B_m\ddot{q}_s = b_m$. Any vector that starts at the origin and whose head lies on this plane satisfies the modeling constraint.

Any vector that lies wholly in the plane P_m lies in the null space of the matrix B_m. Similarly, the control constraint is represented by the plane P_c for which the equation is $B_c\ddot{q}_s = b_c$. These two constraint

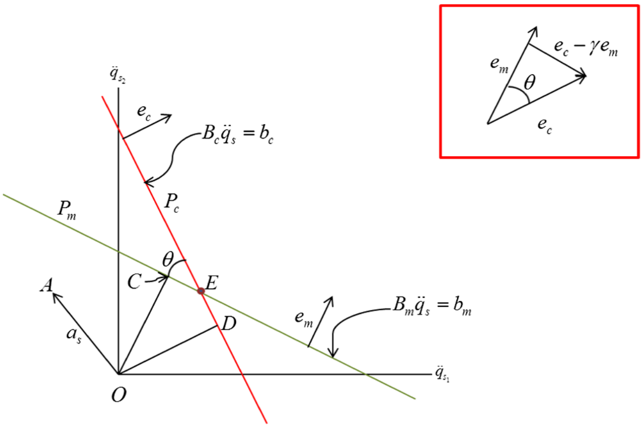


Fig. 1 Representation of the unconstrained system and the constraints in scaled acceleration space.

planes P_m and P_c (in R^2) intersect in the point E at which the modeling and the control constraint are both exactly satisfied.

The lines OC and OD are perpendicular to the planes P_m and P_c, and the unit vectors along these lines are given by

$$e_m = \frac{B_m^T}{\|B_m\|}, \quad \text{and} \quad e_c = \frac{B_c^T}{\|B_c\|} \tag{70}$$

respectively. Both e_m and e_c are column vectors.

The inset in Fig. 1 (top right corner) shows a closer view of these unit vectors. The angle between them is θ and, denoting its cosine by γ, we have

$$\gamma := \cos\theta = e_m^T e_c \tag{71}$$

The vector γe_m is thus the projection of vector e_c along e_m, and $e_c - \gamma e_m$ is a vector perpendicular to e_m (and thus is in a direction along the plane P_m) with magnitude $(1-\gamma^2)^{1/2}$. Hence, one can write $e_c - \gamma e_m = (1-\gamma^2)^{1/2} e_{c,m}$, where $e_{c,m}$ is a unit vector orthogonal to e_m. Being orthogonal, we note that $e_m \cdot e_{c,m} = 0$.

We begin by considering the situation when only modeling constraints are present, as shown in Fig. 2. The acceleration of the uncontrolled system is then given by $\ddot{u}_s = a_s + \ddot{q}_s^m$. The vector $\boldsymbol{OA}$ represents a_s, and the vector $\boldsymbol{AB} = B_m^+(b_m - B_m a_s) = \ddot{q}_s^m$. $\boldsymbol{AB}$ is in the direction of B_m^T, and therefore in the direction of e_m, which is perpendicular to the plane P_m. The vector $\boldsymbol{OB}$ represents the acceleration of the uncontrolled system $\ddot{u}_s$ [see Eqs. (16–20)]. It is the vector sum of 1) the projection $\boldsymbol{CB}$ of the acceleration vector a_s along the plane P_m, and 2) the vector $\boldsymbol{OC}$ that equals $B_m^+ b_m$, which is perpendicular to the plane P_m and is the shortest

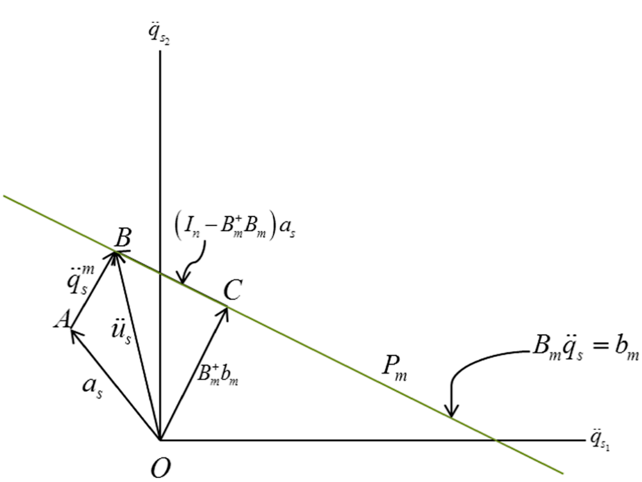


Fig. 2 Representation of the uncontrolled system with only modeling constraints.

distance from O to the plane. In brief, Fig. 2 shows that $\ddot{u}_s = \boldsymbol{OB} = \boldsymbol{OC} + \boldsymbol{CB} = \boldsymbol{OA} + \boldsymbol{AB}$.

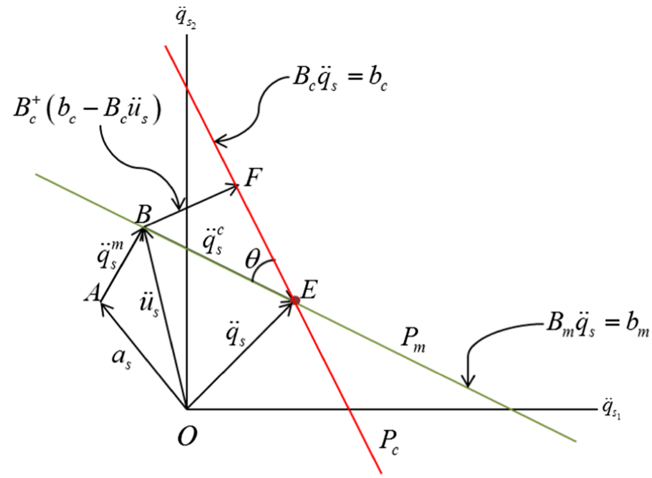


Fig. 3 Representation of the controlled system with both modeling and control constraints.

Figure 3 shows the situation when both modeling and control constraints are present. For simplicity, the case when $W = M^{-1}$ is considered. From Eq. (67), the scaled control acceleration $\ddot{q}_s^c$ is given by

$$\ddot{q}_s^c = [B_c(I_n - B_m^+ B_m)]^+ (b_c - B_c \ddot{u}_s) \tag{72}$$

Since

$$B_m^+ B_m = \frac{B_m^T}{B_m B_m^T} B_m = \frac{\|B_m\| e_m}{\|B_m\|^2} \|B_m\| e_m^T = e_m e_m^T \tag{73}$$

and

$$\begin{aligned} B_c(I_n - B_m^+ B_m) &= B_c - B_c B_m^+ B_m = \|B_c\| e_c^T - \|B_c\| e_c^T e_m e_m^T \\ &= \|B_c\|(e_c^T - \gamma e_m^T) = \|B_c\|(1-\gamma^2)^{1/2} e_{c,m}^T \end{aligned} \tag{74}$$

The vector $[B_c(I_n - B_m^+ B_m)]^+$ is therefore given by

$$[B_c(I_n - B_m^+ B_m)]^+ = \frac{1}{\|B_c\|(1-\gamma^2)^{1/2}} e_{c,m} \tag{75}$$

Thus, $\ddot{q}_s^c$ is a vector in the direction of $e_{c,m}$ for which the length is [using Eqs. (72) and (75)]

$$\|\ddot{q}_s^c\| = \frac{\left(b_c - B_c \ddot{u}_s\right)}{\|B_c\|(1-\gamma^2)^{1/2}} \tag{76}$$

In Fig. 3, the vector $\boldsymbol{BF}$ that is orthogonal to the plane P_c is given by

$$\boldsymbol{BF} = B_c^+(b_c - B_c \ddot{u}_s) = (B_c^T / \|B_c\|^2)(b_c - B_c \ddot{u}_s) \tag{77}$$

Hence, its length is

$$l = \frac{(b_c - B_c \ddot{u}_s)}{\|B_c\|} \tag{78}$$

Therefore, the distance of the intersection point E of the two planes P_m and P_c from B in the direction $e_{c,m}$ is obtained from the right triangle BEF as

$$BE = \frac{l}{\sin\theta} = \frac{l}{(1-\gamma^2)^{1/2}} = \frac{(b_c - B_c \ddot{u}_s)}{\|B_c\|(1-\gamma^2)^{1/2}} \tag{79}$$

which is exactly the length of the vector $\ddot{q}_s^c$, as was found in Eq. (76). Thus, $\ddot{q}_s^c = \boldsymbol{BE}$.

From Eq. (12), we have

$$\ddot{q}_s = a_s + \ddot{q}_s^m + \ddot{q}_s^c \tag{80}$$

and using Eq. (67), we find that

$$\ddot{q}_s^m = B_m^+(b_m - B_m a_s) - B_m^+ B_m \ddot{q}_s^c = B_m^+(b_m - B_m a_s) = \boldsymbol{AB} \tag{81}$$

where the last equality follows from $e_m \cdot e_{c,m} = 0$. Since vector e_m is along the vector B_m^T while $\ddot{q}_s^c$ is along the vector $e_{c,m}$, the two vectors B_m^T and $\ddot{q}_s^c$ are orthogonal, and their inner product $B_m \ddot{q}_s^c$ must equal zero. Therefore, $\ddot{q}_s^m$ is simply $\boldsymbol{AB}$ and, from Eq. (80), we obtain

$$\ddot{q}_s = a_s + \ddot{q}_s^m + \ddot{q}_s^c = \boldsymbol{OA} + \boldsymbol{AB} + \boldsymbol{BE} = \boldsymbol{OE} \tag{82}$$

Thus, the scaled acceleration of the system is represented by $\boldsymbol{OE}$, where point E is exactly at the point of intersection of the two planes P_m and P_c. Hence, Eq. (82) shows that the acceleration $\ddot{q}_s$ of the controlled system satisfies both the modeling and control constraints exactly.

The approach taken in [24] does not offer such a guarantee when the control constraints are consistent with the modeling constraints. Using the approach given in [24], the vector that starts at point B in Fig. 3 and moves along the plane P_m may not, in general, reach the point of intersection E of the two planes; consequently, though the modeling constraint is satisfied, the control requirement (constraint) may not be.

Remark 6: In Fig. 3, if the angle θ between the two planes P_m and P_c is very small, and the intersection point E can be quite far to the right (or the left); one could then roughly say that the constraints are approaching a condition of inconsistency. Such a scenario could occur in practice when the control is underactuated, and the system could even move from a regime in which the constraints are consistent to a regime in which they are inconsistent, or vice versa. In such cases, the magnitude of the control force represented by vector BE in Fig. 3 can be quite large (in fact, going to infinity in the limit, as the angle θ in Fig. 3 tends to zero) and can vary rapidly. This causes a problem because, in practice, infinitely large forces cannot be applied on systems. Also, if the angle θ switches (in the vicinity of zero) from $+\varepsilon$ to $-\varepsilon$, rapid variations in the control force will again result. Even if one were to adopt a strategy in which the forces are cut off at a certain magnitude and "saturation forces" are applied to the system in lieu of the actual control forces computed using Eq. (47), the system might switch regimes at a fast rate, causing the actuators to deteriorate in their performance. In such circumstances, it is desirable to obtain a smoother control force using Eq. (50). The use of this approach to obtain a smoother control is illustrated in an example in the following section (see Sec. VII.C).

VII. Numerical Examples

A. Example 1

Consider a "dumbbell" system consisting of two point masses m_1 and m_2 connected by a massless rigid bar of length l. The coordinates of mass m_1 in an inertial frame of reference are x_1, y_1, z_1, and those of mass m_2 are x_2, y_2, z_2. The z coordinate points vertically upward. The body is acted upon by the downward force of gravity, so the equation of motion of the unconstrained system (the point masses without the bar) is

$$M\ddot{q} = Q \tag{83}$$

where $q \coloneqq [x_1, y_1, z_1, x_2, y_2, z_2]^T$; M is the symmetric positive definite mass matrix $M \coloneqq \mathrm{diag}(m_1, m_1, m_1, m_2, m_2, m_2)$: and Q is the impressed force vector due to gravity, $Q = [0, 0, -m_1 g, 0, 0, -m_2 g]^T$. The equation of motion of the unconstrained system can also be expressed using scaled accelerations as

$$\ddot{q}_s = a_s \tag{84}$$

where the scaled acceleration $\ddot{q}_s$ is given as

$$\ddot{q}_s \coloneqq M^{1/2}[\ddot{x}_1, \ddot{y}_1, \ddot{z}_1, \ddot{x}_2, \ddot{y}_2, \ddot{z}_2]^T \tag{85}$$

the scaled unconstrained acceleration a_s is

$$a_s = [0, 0, -\sqrt{m_1}g, 0, 0, -\sqrt{m_2}g]^T \tag{86}$$

The rigid bar is modeled using the modeling constraint

$$\phi_m \coloneqq (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - l^2 = 0 \tag{87}$$

which needs to be satisfied at all instants of time. The modeling constraint can be put in the desired form by twice differentiating Eq. (87) with respect to time to obtain

$$A_m \ddot{q} = b_m \tag{88}$$

where the row vector A_m is

$$A_m = [(x_1 - x_2), (y_1 - y_2), (z_1 - z_2), -(x_1 - x_2), -(y_1 - y_2), -(z_1 - z_2)] \tag{89}$$

and the scalar b_m is

$$b_m = -(\dot{x}_1 - \dot{x}_2)^2 - (\dot{y}_1 - \dot{y}_2)^2 - (\dot{z}_1 - \dot{z}_2)^2 \tag{90}$$

Alternatively, Eq. (88) can be expressed in terms of scaled accelerations as $B_m \ddot{q}_s = b_m$, where

$$B_m \coloneqq \left[\frac{(x_1 - x_2)}{\sqrt{m_1}}, \frac{(y_1 - y_2)}{\sqrt{m_1}}, \frac{(z_1 - z_2)}{\sqrt{m_1}}, -\frac{(x_1 - x_2)}{\sqrt{m_2}}, -\frac{(y_1 - y_2)}{\sqrt{m_2}}, -\frac{(z_1 - z_2)}{\sqrt{m_2}}\right] \tag{91}$$

In the presence of only the modeling constraint, the equation of motion of the system is modified as

$$M\ddot{q} = Q + Q_m \tag{92}$$

or, equivalently (see Sec. III),

$$\ddot{q}_s = a_s + \ddot{q}_s^m = a_s + B_m^+(b_m - B_m a_s) \coloneqq \ddot{u}_s \tag{93}$$

In the preceding equation, $\ddot{q}_s^m$ is the scaled constraint acceleration and $\ddot{u}_s$ is the scaled uncontrolled system acceleration.

We wish to control the system so that it satisfies the trajectory requirement (control objective)

$$\phi_c \coloneqq x_1 + \frac{y_1}{2} - \frac{z_1^2}{4} = 0 \tag{94}$$

The equation of motion of the dynamical system in the presence of both the modeling constraint and the control requirement (constraint) is then given by

$$M\ddot{q} = Q + Q_m + Q_c \tag{95}$$

Note that the Q_m in Eq. (95) is now not the same as that in Eq. (92).

Since our initial conditions may not lie on this trajectory, the control objective is modified to [3]

$$\ddot{\phi}_c + \beta\dot{\phi}_c + \alpha\phi_c = 0 \tag{96}$$

where $\alpha, \beta > 0$ are constants. It should be noted that, even in cases where the system starts on the manifold $\phi_c = 0$, use of the modified constraint in Eq. (96) improves the computational stability.

On simplifying Eq. (96), the control constraint is obtained in the form

$$A_c \ddot{q} = b_c \tag{97}$$

where

$$A_c = \left[1, \frac{1}{2}, -\frac{z_1}{2}, 0, 0, 0\right], \quad \text{and} \quad b_c = \frac{\dot{z}_1^2}{2} - \beta\dot{\phi}_c - \alpha\phi_c \tag{98}$$

The equation of motion of the controlled system in terms of the scaled accelerations is

$$\ddot{q}_s = a_s + \ddot{q}_s^m + \ddot{q}_s^c \tag{99}$$

We choose the weighting matrix $W = M^{-1}$, so that the control cost minimized at each instant of time is $J_c = Q_c^T M^{-1} Q_c$. Thus, $S = M^{-1/2} W^{-1/2} = I_n$, and the scaled control acceleration $\ddot{q}_s^c$ is then obtained using Eqs. (67) and (66) as

$$\ddot{q}_s^c = [B_c(I - B_m^+ B_m)]^+ (b_c - B_c \ddot{u}_s) = B_{cm}^+ (b_c - B_c \ddot{u}_s) \tag{100}$$

Also, the scaled constraint acceleration $\ddot{q}_s^m$ is given by

$$\ddot{q}_s^m = B_m^+ [b_m - B_m(a_s + \ddot{q}_s^c)] \tag{101}$$

Thus, the control force and the constraint force are explicitly obtained, respectively, as

$$Q_c = M^{1/2} \ddot{q}_s^c = M^{1/2} B_{cm}^+ (b_c - B_c \ddot{u}_s) \tag{102}$$

and

$$\begin{aligned} Q_m &= M^{1/2} \ddot{q}_s^m = M^{1/2} B_m^+ [b_m - B_m(a_s + \ddot{q}_s^c)] \\ &= M^{1/2} B_m^+ [b_m - B_m\{a_s + B_{cm}^+ (b_c - B_c \ddot{u}_s)\}] \end{aligned} \tag{103}$$

where $\ddot{u}_s$ is given in Eq. (93). For the numerical simulation, the parameter values chosen are

$$m_1 = 1, \quad m_2 = 2, \quad \beta = 1, \quad \alpha = 10, \quad l = 2, \quad g = 9.81 \tag{104}$$

and the initial conditions are chosen as

$$q(0) = [0, 0, 0, 2, 0, 0]^T, \qquad \dot{q}(0) = [0, 0, 0, 0, 0, 0]^T \tag{105}$$

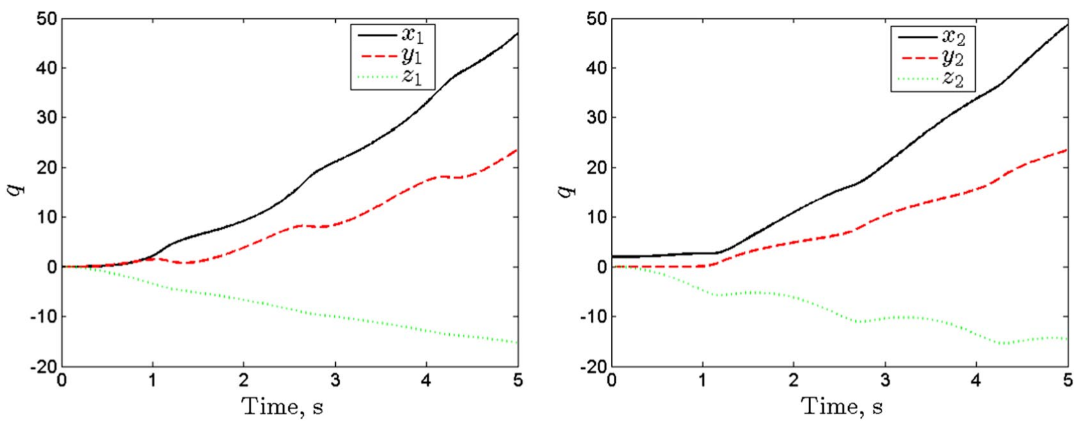
The equation of motion of the controlled system given by Eq. (95) has been numerically integrated for 5 s using ode45 on the MATLAB platform with a relative error tolerance of 10^{-8} and an absolute error tolerance of 10^{-12}. The simulation results are presented in Figs. 4–6.

Figure 4 shows the time history of the response of the system, and Fig. 5 shows the control force computed using Eq. (102). These control forces minimize the Gaussian at each instant of time. In this case, the constraints are consistent, and it is ensured that the controlled system exactly satisfies both the control and the modeling constraints.

Figure 6a shows the error in satisfying the modeling constraint e_m defined as

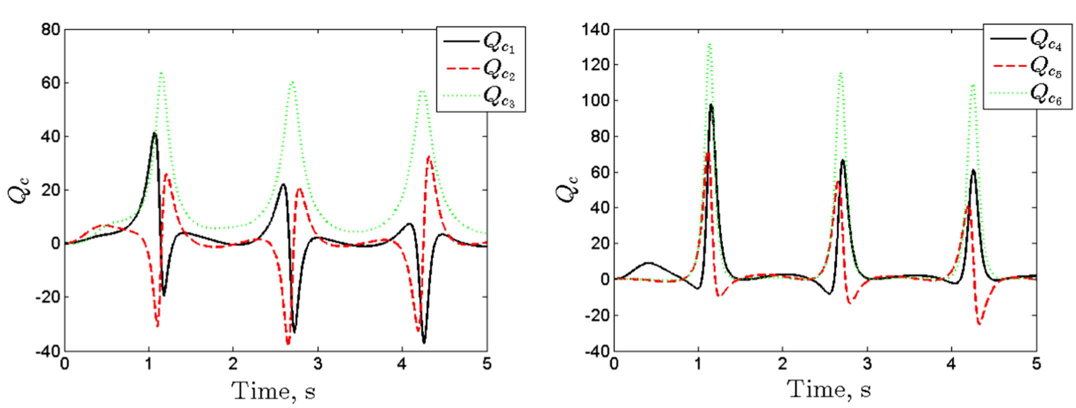
$$e_m(t) := A_m \ddot{q} - b_m = B_m \ddot{q}_s - b_m \tag{106}$$

Similarly, Fig. 6b shows the variation of error in satisfying the control constraint e_c



a) b)

Fig. 4 Time history of response: a) first three degrees of freedom, and b) last three degrees of freedom.



a) b)

Fig. 5 Time history of the control force: a) first three components, and b) last three components.

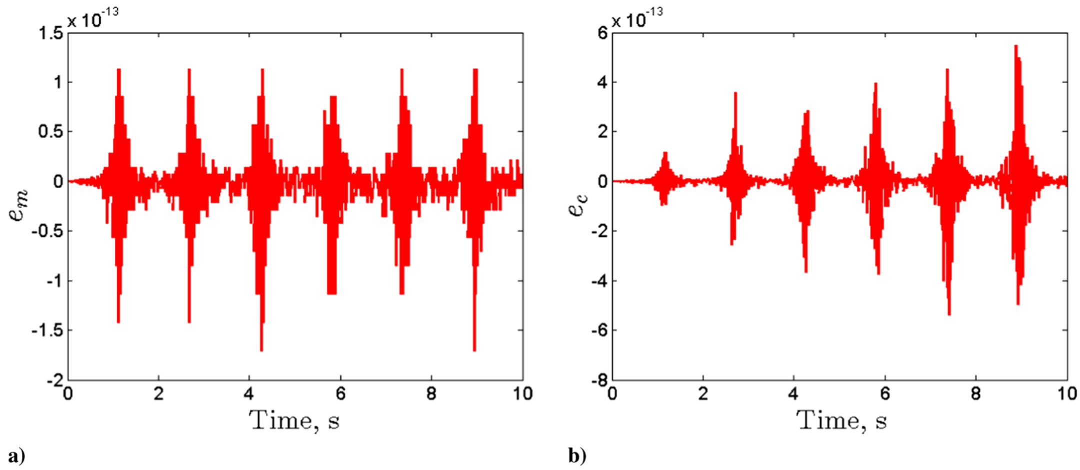


Fig. 6 Error in satisfying the constraints: a) modeling constraint, and b) control constraint.

$$e_c(t) := A_c\ddot{q} - b_c = B_c\ddot{q}_s - b_c \tag{107}$$

with time. As seen from the figure, these errors are of $\mathcal{O}(10^{-13})$ and their order of magnitude is comparable to the error tolerances used in the numerical integration.

B. Example 2

We next consider an underactuated control problem. Consider a cart–pendulum system consisting of a cart of mass m_c that can roll on a frictionless surface along the X axis and a point mass m_p connected to the cart at its center of mass O with a massless link of length l, as shown in Fig. 7. The system has two degrees of freedom: x is the displacement of the cart along the X axis, and θ is the angle of the link from Y axis measured in the counterclockwise direction.

1. *Model of Cart-Pendulum System*

The equation of motion of the system is given by

$$M\ddot{q} = Q \tag{108}$$

where

$$q(t) = [x(t), \theta(t)]^T, \quad M = \begin{bmatrix} m_c + m_p & m_p l\cos(\theta) \\ m_p l\cos(\theta) & m_p l^2 \end{bmatrix}, \text{ and}$$

$$Q = \begin{bmatrix} m_p l\dot{\theta}^2\sin(\theta) \\ -m_p g l\sin(\theta) - c(\dot{\theta}, \theta) \end{bmatrix} \tag{109}$$

The nonlinear damping in the "θ equation" is denoted by $c(\dot{\theta}, \theta)$. For illustrative purposes, in what follows, we shall consider two types of damping functions:

$$c(\dot{\theta}, \theta) = \varepsilon\dot{\theta}|\dot{\theta}|^2, \qquad \varepsilon \geq 0 \tag{110}$$

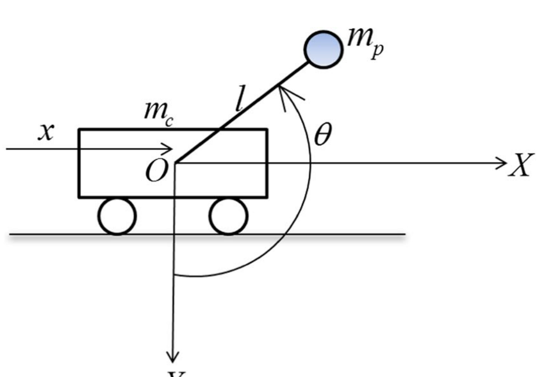


Fig. 7 Cart–pendulum system.

and

$$c(\dot{\theta}, \theta) = \delta\dot{\theta}/|\dot{\theta}| \tag{111}$$

where $\delta \geq 0$ is the coefficient of friction.

Equation (110) models nonlinear damping and Eq. (112) models Coulomb friction.

We wish to control this system so that the pendulum bob has a desired trajectory $\theta = \theta_d(t)$ while applying a control force only to the cart, and without applying any control torque on the link. Thus, this is an example of an underactuated mechanical system.

This control objective can be achieved using two constraints: 1) an "underactuation constraint" that ensures that the control torque on the link is zero, and 2) a control constraint that ensures that the angular position of the bob has the desired trajectory $\theta(t) = \theta_d(t)$.

The underactuation constraint cannot be violated at any time and can therefore be deemed as a modeling constraint in our current framework. Then, the modeling constraint is given by

$$A_m\ddot{q} = b_m, \quad A_m = [m_p l\cos(\theta) \quad m_p l^2], \quad \text{and}$$
$$b_m = -m_p g l\sin(\theta) - c(\dot{\theta}, \theta) \tag{112}$$

The preceding constraint is nothing but the second equation of motion [see Eq. (109)]. The control constraint used is

$$\ddot{\theta} + \alpha_1(\dot{\theta} - \dot{\theta}_d) + \beta_1(\theta - \theta_d) = \ddot{\theta}_d(t), \quad \alpha_1, \beta_1 > 0 \tag{113}$$

which can be put it the standard form

$$A_c\ddot{q} = b_c \tag{114}$$

by defining

$$A_c := [0 \quad 1] \quad \text{and} \quad b_c = -\alpha_1(\dot{\theta} - \dot{\theta}_d) - \beta_1(\theta - \theta_d) + \ddot{\theta}_d \tag{115}$$

In the preceding relations, we use the stabilization parameters α_1 and β_1 because the system does not start out satisfying the desired control requirements [3].

The controlled system is then described by the equation

$$M\ddot{q} = Q + Q_m + Q_c \tag{116}$$

in which Q_m and Q_c are explicitly obtained in closed form in Sec. IV. Hence, the constraint force Q_m and the control force Q_c are obtained using Eqs. (47) and (48) for a given weighting matrix W.

2. *Numerical Results*

The equation of motion of the controlled system given in Eq. (116) is numerically integrated using ode15s on the MATLAB platform with a relative error tolerance of 10^{-12} and an absolute error tolerance of 10^{-13} throughout the various simulations in this subsection.

We first consider the case when the objective is to have the pendulum bob start from rest from a position above the cart and come to rest in the "inverted pendulum" position so that $\theta_d = \pi$ (see Fig. 7).

Due to the presence of a singularity at $\theta = \pi/2$ in the control force obtained, one can control the system to come to rest at any angle $\pi/2 \le \theta_d \le 3\pi/2$ if the system starts from rest so that $\pi/2 \le \theta(0) \le 3\pi/2$.

The various parameters (in consistent SI units) are chosen as

$$m_c = 2, \quad m_p = 1, \quad l = 1, \quad g = 9.81, \quad \alpha_1 = 0.2, \quad \text{and} \quad \beta_1 = 1 \tag{117}$$

and the initial conditions are chosen as

$$q(0) = [0, \quad 1.35\pi]^T \quad \text{and} \quad \dot{q}(0) = [0, \quad 0]^T \tag{118}$$

1) For underactuated control with nonlinear viscous damping and $\varepsilon = 0.01$, the weighting matrix is chosen as $W = \mathrm{diag}([1, 1])$.

The results of the simulation are shown in Figs. 8–10.

Figure 8a shows the time history of the angle θ measured in degrees. As seen in the figure, its value stabilizes at the desired value of 180 deg. The time history of the cart's displacement $x(t)$ is shown in Fig. 8b. As expected, this displacement is large; the velocity of the cart obviously becomes a constant once the bob reaches its final position θ_d.

The magnitude of the error ($\theta - \theta_d$, not shown) at the end of 300 s is found to be of $\mathcal{O}(10^{-12})$. Figure 9 shows the control force required to be applied to the cart as well as the torque to be applied to the link. This torque is seen to be of $\mathcal{O}(10^{-13})$, which is commensurate with error tolerances used in the numerical integration, indicating that the control torque is effectively zero, as required by the underactuated nature of the control.

One could, if desired, place greater emphasis on minimizing the control torque by simply changing the weighting matrix by increasing its second diagonal element so that $W = \mathrm{diag}([1, 100])$. Since $Q_c^T W Q_c$ is minimized, this new W matrix places greater emphasis on minimizing the torque. When the simulation is run again, keeping all the parameters unchanged except for the matrix W, we obtain Fig. 10. As seen in the lower panel, the torque on the link has been further reduced by an order of magnitude, essentially demonstrating that the underactuation constraint given by Eq. (112)

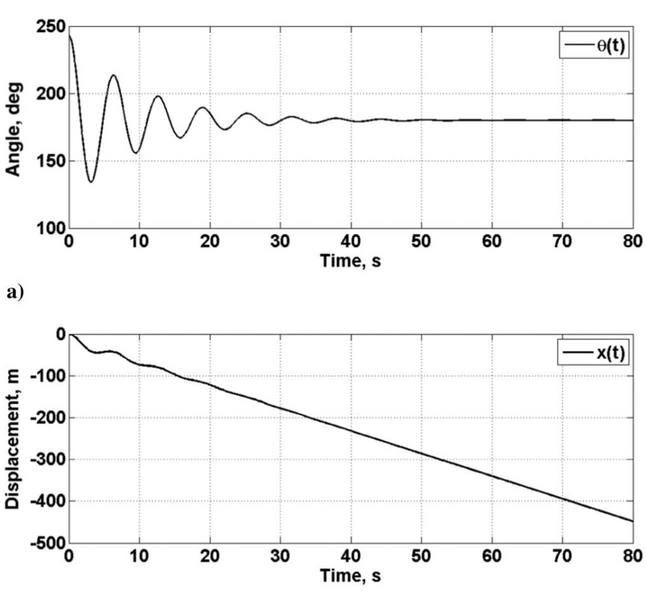


Fig. 8 Time history of response: a) angle θ in degrees, and b) displacement $x(t)$ in meters.

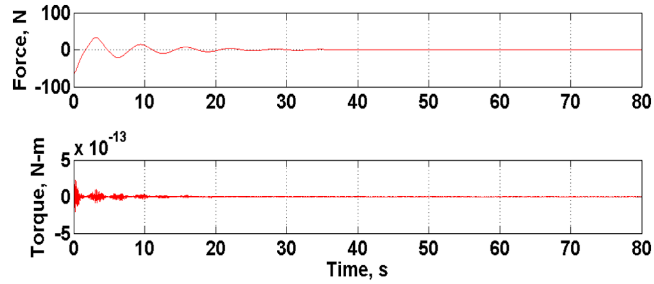


Fig. 9 Time history of control force on the cart and control torque on the link, $W = \mathrm{diag}([1,1])$.

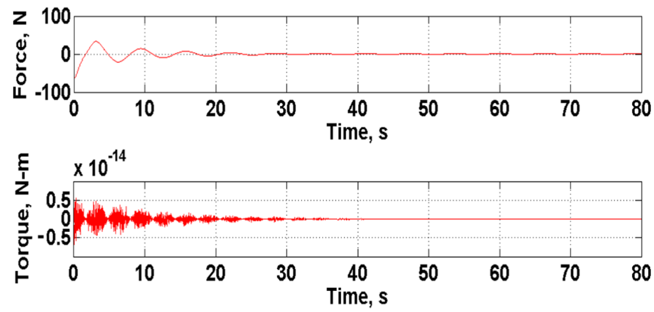


Fig. 10 Time history of control force on the cart and control torque on the link, $W = \mathrm{diag}([1,100])$.

is satisfied to an extent smaller than the error tolerances used in the numerical integration, and the system is indeed underactuated. For the sake of brevity, we do not show the time histories of θ and x, which are identical to those shown in Fig. 8.

2) For underactuated control with Coulomb friction and $\delta = 0.4$, the weighting matrix is chosen as $W = \mathrm{diag}([1, 100])$.

To avoid numerical problems in integration with variable time-step integrators when using the discontinuous signum function, we implement Coulomb damping by approximating

$$c(\dot{\theta}, \theta) = \delta\dot{\theta}/|\dot{\theta}| \approx \delta \tanh(\delta_1 \dot{\theta}), \quad \text{with } \delta = 0.4, \quad \delta_1 = 80{,}000 \tag{119}$$

Using the same parameter values as before, the results for the first 80 s of response are shown in Figs. 11 and 12. For brevity, we do not show the graph of $x(t)$, which looks similar to Fig. 8b. At $t = 300$ s, the error $\theta - \theta_d$ is found to be of $\mathcal{O}(10^{-13})$.

Figure 13 shows the force applied to the cart between 10 and 80 s on an expanded scale where the effect of Coulomb damping is clearly visible.

We now consider the case when the objective is to have the pendulum bob start from rest from a position above the cart and control only the cart so that the bob oscillates sinusoidally about a mean position θ_m with (angular) amplitude θ_{amp} and frequency ω. The bob oscillates in an inverted position above the cart.

The control requirement is then given by

$$\theta_d(t) = \theta_m + \theta_{\mathrm{amp}} \cos(\omega t) \tag{120}$$

Relation (120) is thus used in Eqs. (113–115).

For numerical computations, the same parameter values as in Eq. (117) and the same initial conditions as in Eq. (118) are used.

3) For underactuated control with nonlinear viscous damping and $\varepsilon = 0.01$, the weighting matrix is chosen as $W = \mathrm{diag}([1, 100])$.

In Eq. (120), the parameters values $\theta_m = \pi$, $\theta_{\mathrm{amp}} = \pi/4$, and $\omega = 2\pi/10$ are used to describe the required trajectory of the bob. Thus, the bob is required to oscillate about the vertical inverted pendulum position with an amplitude of $\pi/4$ rad and a period of 10 sec.

Figure 14 shows the time histories $\theta(t)$ and $\dot{x}(t)$ over a time interval of 200 s. As seen, the underactuated system tracks the trajectory

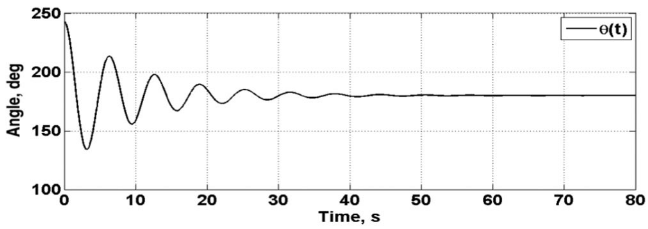


Fig. 11 Time history of angle θ, measured in degrees.

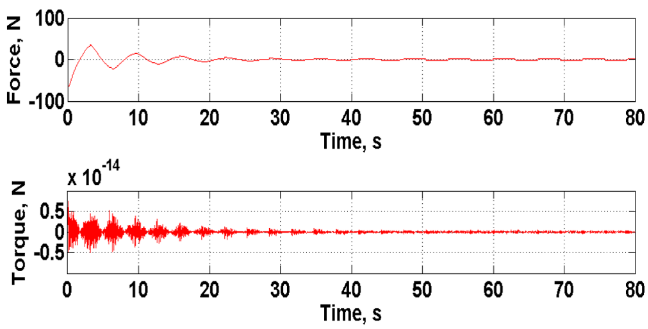


Fig. 12 Time history of control force on the cart and control torque on the link, $W = \mathrm{diag}([1,100])$.

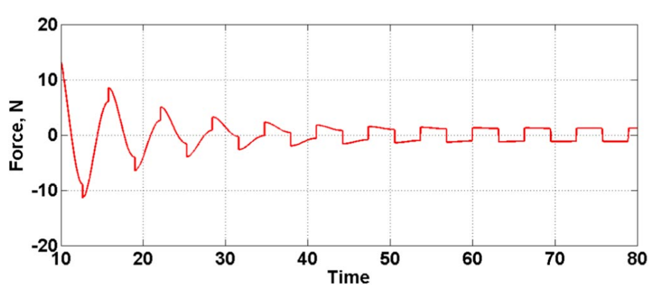


Fig. 13 Control force applied to the cart over the interval [10, 80] seconds in the presence of Coulomb damping.

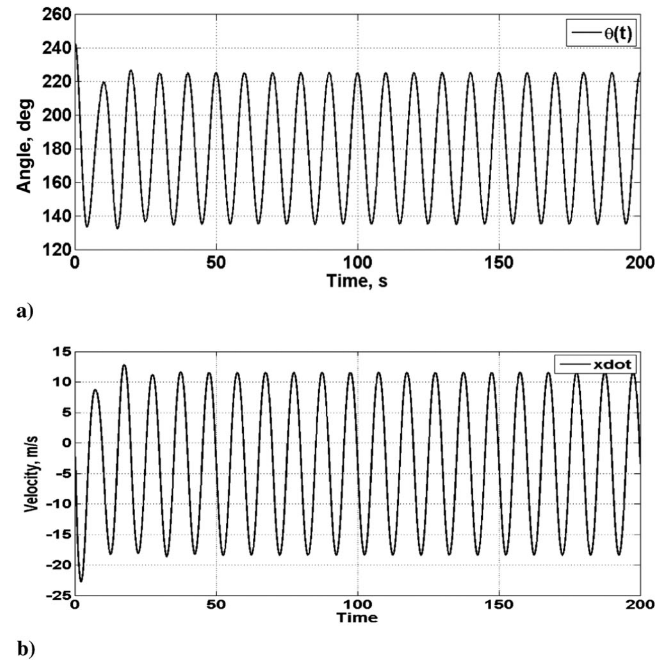


Fig. 14 Time history of response: a) angle θ in degrees, and b) velocity $\dot{x}(t)$ in meters per second.

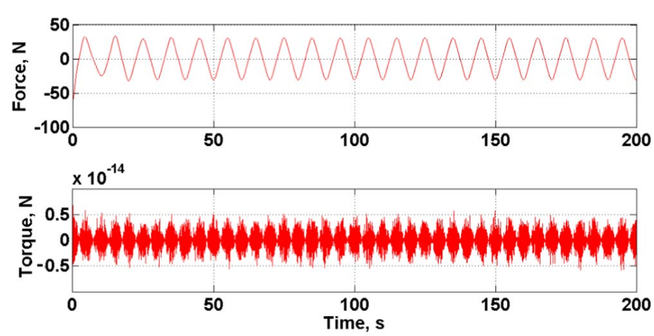


Fig. 15 Time history of control force on the cart and control torque on the link.

requirement given in Eq. (120) well. The tracking error $\theta - \theta_d$ at the end of 300 s is found to be of $\mathcal{O}(10^{-11})$. The top panel of Fig. 15 shows the control force needed to be applied to the cart to adequately control the bob; the torque (see bottom panel of Fig. 15) required is smaller than the tolerance used in integrating the equations of motion, and it shows that the system is underactuated.

4) For underactuated control with Coulomb friction, the approximation given in Eq. (119) is used with $\delta = 0.4$ and $\delta_1 = 80{,}000$. The weighing matrix is chosen as $W = \mathrm{diag}([1, 100])$.

In Eq. (120), the parameters values $\theta_m = 19\pi/18$, $\theta_{\mathrm{amp}} = \pi/4$, and $\omega = 2\pi/5$ are used to describe the required trajectory of the bob. Thus, the pendulum is required to oscillate in an inverted position about a line at an angle $\theta = 190$ deg (see Fig. 7), which is 10 deg to the left of the vertical, with an amplitude of $\pi/4$ rad and a period of 5 s. As stated before, the underactuated control here has a limitation, and it will work as long as the motion of the bob is at all times above the horizontal. The controlled response of the system is shown in Fig. 16a. As expected, the oscillations of the inverted pendulum bob are seen to occur about a line at an angle of 190 deg, and the bob oscillates about this line sinusoidally through an angle of 45 deg (that is, the oscillations are between the angles of 145 and 235 deg) with a period of 5 s. Figure 16b shows the error, $\theta(t) - \theta_d(t)$, in tracking the desired trajectory in the presence of Coulomb friction over the time interval [0, 100] s. At $t = 300$ s, the tracking error is found to be of $\mathcal{O}(10^{-9})$.

The control force on the cart is shown in Fig. 17 (top) over the interval of [0, 30] s (instead of over the 100 s interval) so one can see the effect of Coulomb damping on the system as observed at the troughs and valleys where the force appears to jump. As seen in the bottom figure, the torque on the link is effectively zero; it is less than the error tolerances used in the numerical integration.

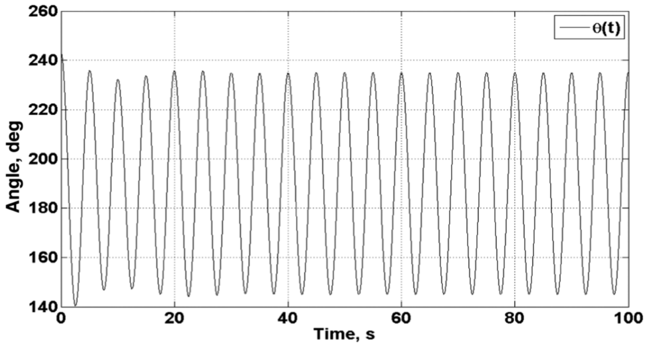
C. Example 3

For the third example, let us consider an underactuated surface vessel of mass m and rotational moment of inertia $\hat{J}$, depicted in Fig. 18, which is to be controlled to follow a circular trajectory. The system is underactuated because forces can only be applied to the vessel along the vessel's ε_1 axis.

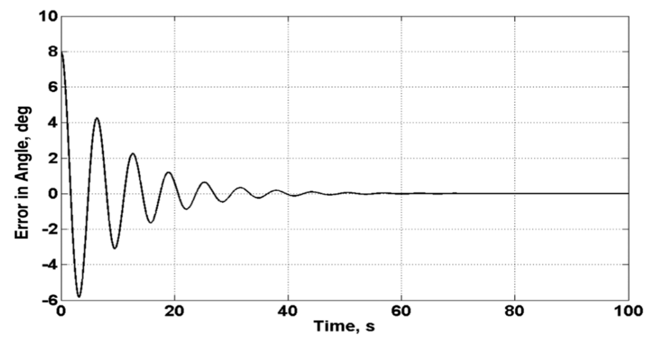
The position of the center of mass in the global coordinate frame (XY) is (x, y). A body-fixed coordinate frame $\varepsilon_1, \varepsilon_2$ is attached at the center of mass. The orientation of the vehicle is determined by the angle θ between the X axis and the ε_1 axis, as shown in Fig. 18. We use two quaternions $u_0 = \cos(\theta/2)$ and $u_1 = \sin(\theta/2)$ to describe the orientation to avoid any singularities that may arise [24]. Thus, the configuration of the system is described by the vector $q = [R^T, u^T]^T$, where $R = [x, y]^T$ represents the position of center of mass of the vehicle and $u = [u_0, u_1]^T$ is the unit quaternion representing the orientation of the vessel.

The rotation matrix used to transform vectors from the body-fixed frame to the global frame is obtained in terms of quaternions as

$$T = [T_1, T_2] = \begin{bmatrix} u_0^2 - u_1^2 & -2u_0u_1 \\ 2u_0u_1 & u_0^2 - u_1^2 \end{bmatrix} \tag{121}$$



a)



b)

Fig. 16 Time history of a) angle θ in degrees, and b) error in tracking the trajectory $\theta(t) - \theta_d(t)$.

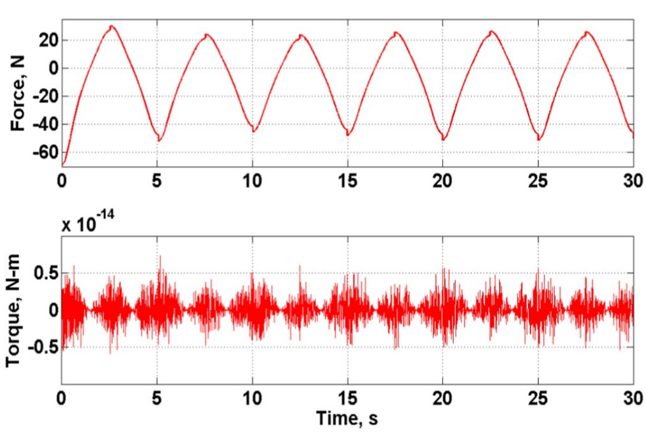


Fig. 17 Time history of control force on the cart and control torque on the link.

In the preceding, T_1 and T_2 are first and second columns of the orthonormal rotation matrix. The equations of motion of the unconstrained system (see [29]) are obtained using Lagrange's method as

$$M\ddot{q} = Q \tag{122}$$

where M is the symmetric positive definite mass matrix, and Q is the generalized force vector given by

$$M = \begin{bmatrix} mI_2 & 0 \\ 0 & 4E^T JE \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 \\ 8\dot{E}^T J\dot{E}u \end{bmatrix} \tag{123}$$

In the preceding, m is the mass of the vessel, I_2 is the 2-by-2 identity matrix,

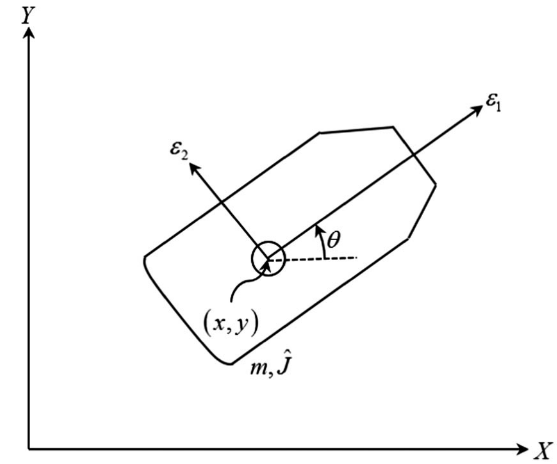


Fig. 18 Surface vessel of mass m and moment of inertia $\hat{J}$.

$$E = \begin{bmatrix} u_0 & u_1 \\ -u_1 & u_0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} J_0 & 0 \\ 0 & \hat{J} \end{bmatrix} \tag{124}$$

J_0 is an arbitrary positive scalar. Typically, these vehicles have a thruster that can apply a force only in the ε_1 direction. They do not have the capability to apply a force along the ε_2 direction. This underactuation requirement is then written in the form of a modeling constraint as

$$-2u_0 u_1 \ddot{x} + (u_0^2 - u_1^2)\ddot{y} = T_2^T \ddot{R} = 0 \tag{125}$$

In addition, the system also needs to satisfy the unit quaternion constraint

$$u^T u = 1 \tag{126}$$

Equations (125) and (126) are put together to form the modeling constraint equation of the form given by Eq. (3), where

$$A_m = \begin{bmatrix} 0 & u^T \\ T_2^T & 0 \end{bmatrix} \quad \text{and} \quad b_m = \begin{bmatrix} -\dot{u}^T \dot{u} \\ 0 \end{bmatrix} \tag{127}$$

It should be noted that the underactuation requirement is seen as a modeling constraint rather than a control constraint because this is a requirement that cannot be violated at any time. The control objective is to drive the system to follow a circular trajectory described by

$$\phi_1 = x - \cos t = 0, \quad \phi_2 = y - \sin t = 0 \tag{128}$$

In addition, we also want to enforce the condition that the ε_2 direction of the vehicle is oriented normal to the desired circular trajectory so that the vehicle can track the trajectory. This condition is written as

$$\phi_3 := T_2^T \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} = 2u_0 u_1 \cos(t) - (u_0^2 - u_1^2)\sin(t) = 0 \tag{129}$$

Since our initial conditions may not satisfy these constraints, the control objective is modified to

$$\ddot{\phi}_i + \alpha_i \dot{\phi}_i + \beta_i \phi_i = 0, \qquad i = 1 \cdots 3 \tag{130}$$

where $\alpha_i, \beta_i > 0$ are constants. Control constraints of the form $A_c \ddot{q} = b_c$ are derived from Eq. (130), where

$$A_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2u_0 \sin(t) + 2u_1 \cos(t) & 2u_0 \cos(t) + 2u_1 \sin(t) \end{bmatrix} \quad \text{and} \quad b_c = \begin{bmatrix} -\cos t - \alpha_1 \dot{\phi}_1 - \beta_1 \phi_1 \\ -\sin t - \alpha_2 \dot{\phi}_2 - \beta_2 \phi_2 \\ b_3 - \alpha_3 \dot{\phi}_3 - \beta_3 \phi_3 \end{bmatrix} \tag{131}$$

In Eq. (131),

$$b_3 = 2(\dot{u}_0^2 - \dot{u}_1^2)\sin t - 4\dot{u}_0\dot{u}_1\cos t + 4\dot{u}_0 u_1 \sin t + 4u_0\dot{u}_1\sin t + 4\dot{u}_0 u_0\cos t - 4\dot{u}_1 u_1 \cos t - (u_0^2 - u_1^2)\sin t + 2u_0 u_1 \cos t \tag{132}$$

These constraints, when enforced, ensure that the vehicle executes motion along a unit circle defined by $\phi_c \coloneqq [\phi_1, \phi_2]^T = 0$. In this example, the constraints change from being inconsistent to consistent frequently. So, we use the modified formulation mentioned in Remark 4 of Sec. IV. The equation of motion of the controlled system is given by

$$M\ddot{q} = Q + Q_m + Q_C \tag{133}$$

We choose to minimize the Gaussian; hence, matrix S in this case is simply the identity matrix. The control force Q_C is then given by Eq. (50) as

$$Q_C = M^{1/2}(B_{cm}^T B_{cm} + \mu I_n)^{-1} B_{cm}^T (b_c - B_c \ddot{u}_s) \tag{134}$$

where

$$B_{cm} = B_c(I_n - B_m^+ B_m) \quad \text{and} \quad \ddot{u}_s = (I_n - B_m^+ B_m)a_s + B_m^+ b_m \quad \text{with } a_s = M^{-1/2}Q \tag{135}$$

and the force Q_m is given by

$$Q_m = M^{1/2} B_m^+ (b_m - B_m a_s - B_m \ddot{q}_s^c); \qquad \ddot{q}_s^c = M^{-1/2} Q^C \tag{136}$$

For numerical simulation, let us choose the parameters $m = 5$, $J_0 = 1$, $\hat{J} = 10$, and $\mu = 10^{-4}$, and the initial conditions as $q(0) = [3, -2, \cos(\pi/2), \sin(\pi/2)]^T$ and $\dot{q}(0) = [0, 0, 0, 0]^T$. With these parameters, Eq. (133) is numerically integrated using the ode15s package in MATLAB with a relative error tolerance of 10^{-8} and an absolute error tolerance of 10^{-12}. The results of the simulation are shown in Figs. 19–22.

Figure 19a shows the projection of the phase portrait on the xy plane. It can be seen that the vessel goes around the unit circle as required by the control objectives. Figure 19b shows the time history of the control forces obtained using Eq. (134). The control forces are smooth functions of time. Had Eq. (67) been used instead of Eq. (50) to determine the control force [see Eq. (134)], because the constraints switch from being consistent to being inconsistent, a bang–bang type of control would have resulted.

Figure 20a shows the time history of the control forces along the ε_1 and ε_2 directions, respectively, denoted by $\hat{Q}_{C_1}$ and $\hat{Q}_{C_2}$. They are easily obtained using the rotation matrix as

$$\hat{Q}_{C_1} = T_1^T \begin{bmatrix} Q_{C_1} \\ Q_{C_2} \end{bmatrix} \quad \text{and} \quad \hat{Q}_{C_2} = T_2^T \begin{bmatrix} Q_{C_1} \\ Q_{C_2} \end{bmatrix} \tag{137}$$

As can be seen from the figure, the control force along the ε_2 direction is zero at all times, thus satisfying the underactuation constraint. The force $\hat{Q}_{C_1}$ is the force applied by the thruster on the vessel.

The control torque applied on the surface vessel at time t can be computed as (see [29])

$$\tau(t) = \frac{1}{2}[-u_1(t), u_0(t)] \begin{bmatrix} Q_{C_3}(t) \\ Q_{C_4}(t) \end{bmatrix} \tag{138}$$

Figure 20b shows the control torque applied on the system as a function of time in order to keep the orientation of the system such that the ε_2 direction is perpendicular to the desired trajectory.

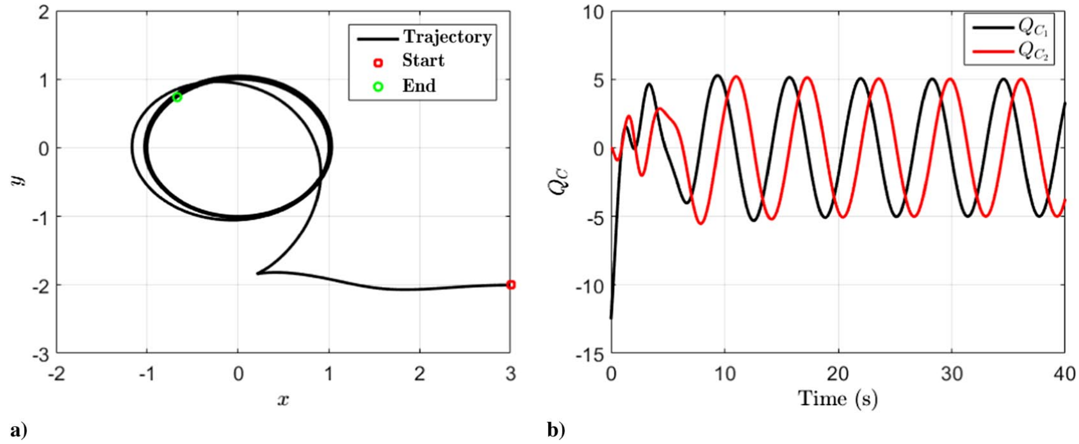


Fig. 19 Plots showing a) projection of phase portrait on xy plane, and b) time history of control force Q_C.

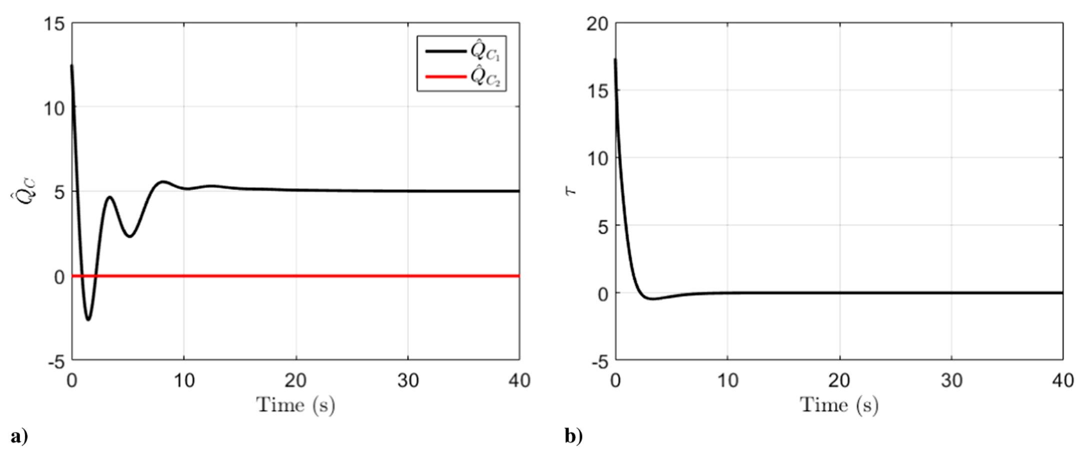


Fig. 20 Time history of a) control force in ε_1 and ε_2 directions, and b) control torque τ.

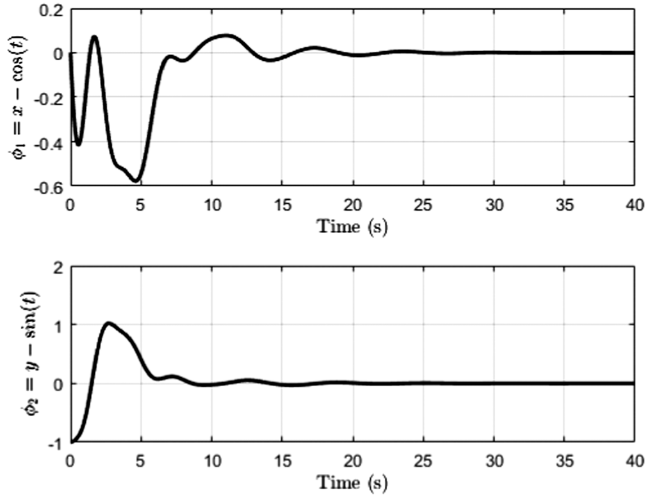


Fig. 21 Time history of error in control constraint: $\phi_i = 0, i = 1,2$.

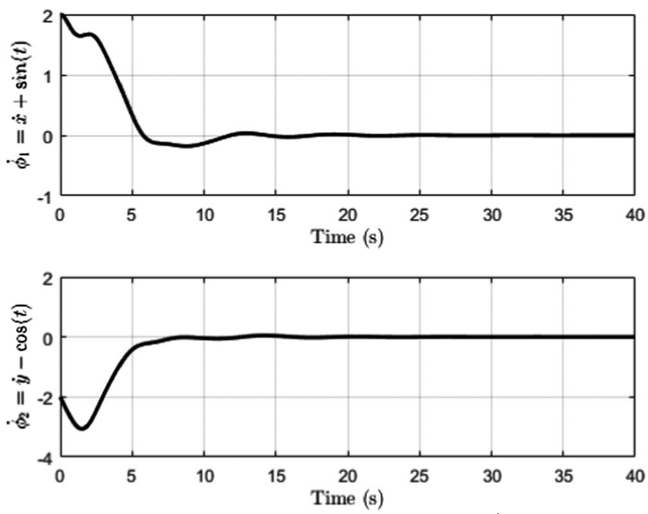


Fig. 22 Time history of error in control constraint: $\dot{\phi}_i = 0, i = 1,2$.

Figure 21 shows the time history of error in satisfying the control requirement $\phi_i = 0, i = 1, 2$, which goes to zero over time. Figure 22 shows the time history of the error in meeting the requirement $\dot{\phi}_i = 0$, $i = 1, 2$. The simulations show that the method is effective and produces smooth control forces even when the constraints switch frequently from being consistent to being inconsistent.

VIII. Conclusions

A unified approach has been proposed to obtain the generalized control force as well as the modeling constraint force for a mechanical system when control objectives (requirements) are prescribed and a user-desired control cost is desired to be minimized. The control objectives are cast as constraints that are imposed on the system. The approach ensures that the modeling constraints, which pertain to the proper description of the physical mechanical system, are always satisfied.

The proposed approach obtains the generalized control force in closed form such that the following is true:

1) The modeling requirements (constraints) are always exactly satisfied, irrespective of the control objectives (constraints) desired.

2) When the control requirements and the modeling requirements are consistent with each other, both the control and the modeling requirements are exactly satisfied.

3) If the control requirements are not consistent with the modeling requirements, then the control requirements are satisfied to the extent possible in the sense that the L_2 norm of the error in their satisfaction is minimized.

4) The generalized control force obtained always minimizes a user-desired control cost at each instant of time.

The approach does not involve simplifying assumptions such as linearizations and/or approximations of the dynamical system or the constraints. No a priori structure is imposed on the controller.

A geometric explanation of the control methodology is provided to enhance its understanding. When full-state control is used and the control requirements are feasible, then the modeling and the control constraints are consistent. For underactuated systems, these two varieties of constraints can at times become inconsistent. In fact, when the system has inconsistent constraints, the dynamics could cause the constraints to alternate between consistency and inconsistency. For such situations of underactuated control, a modified approach that reduces large variations in the control force is provided. For the sake of brevity, this modification to the general approach that is developed herein is only briefly described, though its use is illustrated by way of an example. A forthcoming communication will explore this aspect in greater detail.

Three numerical examples are considered. The first example deals with a simple two-mass dumbbell, and it has pedagogical significance. The second deals with an underactuated cart–pendulum system; the motion of the pendulum is controlled by applying forces only to the cart. Underactuation is modeled as a constraint on the system, and this constraint is interpreted in the framework that is developed in this paper as an inviolable modeling constraint. Nonlinear viscous damping and Coulomb friction are considered in the motion of the pendulum bob, and their effect on the control force that is applied to the cart is demonstrated. The underactuated system is controlled so that the bob that starts from rest from a position above the cart 1) comes to rest (stably) vertically above it, and 2) oscillates sinusoidally about a (mean) direction with a given (angular) amplitude of oscillation and given frequency. The third example also deals with an underactuated mechanical system that has the tendency to become inconsistent at frequent intervals of time. The modified control approach that minimizes a weighted combination of control cost and control error is used to produce smooth control forces. This example shows that, in practical scenarios, it is possible to trade off accuracy in enforcing the control constraint for gains in performance as measured by smoothness of control. Although there is certainly much that remains to be done in the area of underactuated control, the last two examples demonstrate the great simplicity, ease, and high accuracy with which closed-form generalized control forces can be obtained for highly nonlinear dynamical systems using the methodology developed in the paper.

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