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Sphere rolling on a moving surface: Application of the fundamental equation of constrained motion

Firdaus E. Udwadia [a], Giandomenico Di Massa [b,*]

[a] Civil Engineering, Aerospace and Mechanical Engineering, Mathematics, and Information and Operations Management, University of Southern California, 430K Olin Hall, Los Angeles, CA 90089-14543, United States
[b] Università degli Studi di Napoli FEDERICO II, Dipartimento di Meccanica ed Energetica, Via Claudio, 21, 80125 Napoli, Italy

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ABSTRACT

This paper deals with the general formulation of the problem of a rigid sphere rolling under gravity on an arbitrarily prescribed surface that is moving in an arbitrarily prescribed manner. This is accomplished by using a recently developed modeling paradigm, which is encapsulated in a systematic general three-step procedure. The first step develops the equations of motion of the so-called unconstrained system in which the sphere is decoupled from the surface on which it moves. The novelty in this paper is the inclusion of a zero-mass particle and its associated coordinates in the unconstrained description of the system, whose equations of are trivial to write down since it is assumed that all the coordinates are independent of one another. However, this leads to a singular mass matrix. The second step involves the statement of the constraints that (a) cause the sphere to roll on the surface without slip, (b) cause the zero-mass particle to bind to the surface and to become the point of contact between the sphere and the surface, and (c) ensure that the quaternion describing the rotational motion of the sphere is a unit quaternion. The third step involves the direct application of the Udwadia–Phohomsiri equation that generates the equations of motion for the system. Simulations of the motion of a sphere rolling on a moving parabolic surface are shown illustrating the ease and efficacy with which both the formulation and the numerical results can be obtained.

The systematic modeling procedure used here to study the dynamics of the rolling sphere along with the use of a zero-mass particle opens up new ways for modeling and simulating the dynamical behavior of complex multi-body systems.

1. Introduction

The dynamics of a rigid ball rolling under gravity without slipping on a surface is one of the classical problems of mechanics in which the non-holonomic constraints play an important role, and in which the standard Lagrangian formalism in difficult to apply to readily simulate the dynamical behavior.

One of the first contributions to this problem was published by Lindelöf [1], in which it seemed that the author had completely solved the problem. Some years later, Chaplygin analyzed Lindelöf's paper and found an error. He carried out an investigation of the problem of the motion of a heavy body of revolution on a surface for a number of particular cases [2,3]. In this work Chaplygin derives the integrals of motion of the system. Despite these contributions by Chaplygin, the motion of a ball was practically unstudied until Kilin [4], derived the equations of the motion for a sphere on a plane in

* Corresponding author. Tel.: +39 0817683289.
E-mail addresses: fudwadia@usc.edu (F.E. Udwadia), gdimassa@unina.it (G. Di Massa).

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an inertial frame thereby studying the trajectories of the point of contact between the sphere and the plane. Borisov and Mamaev [5] extend this to a rigid body rolling on a plane and a sphere.

The present article presents a general formulation for obtaining the equations of motion (in an inertial frame) for a rigid sphere rolling on an arbitrarily prescribed moving surface, without slipping. The formulation allows the equations of motion to be easily and efficaciously determined by employing a new way in which the system is modeled. The aim is to develop a formulation that simplifies the modeling effort, while yielding accurate simulations of the dynamics.

The key idea underlying this new approach is the use of a suitable unconstrained system, which then is constrained appropriately to yield the system of interest—in this case, the sphere rolling on a moving surface without slipping. To accomplish this we use the approach of considering a particle of zero mass in our description of the unconstrained system.

We therefore begin with an unconstrained rigid sphere moving in an inertial coordinate frame under the force of gravity, and use an additional zero-mass particle, thereby enrolling more than the minimum number of coordinates needed to specify the configuration of the system [6]. Furthermore, in order to express the rotation of the sphere without encountering singularities in our formulation, we use quaternions, thereby increasing the number of coordinates used to describe the system even further. The equations of motion for this unconstrained system are indeed *trivial* to write down. The zero-mass particle is specifically included in our unconstrained system because it simplifies the derivation of the constraints. These constraints are applied to our unconstrained system so that: (1) the sphere rolls without slipping on the given, moving surface; (2) the zero-mass particle at each instant of time is fixed to the point of contact between the sphere and the surface, so that its motion in time represents the path traced out on the surface by the moving sphere; and, (3) the quaternion four-component column vector represents a physical rotation, and therefore has unit norm. Having described the unconstrained system (in which all the coordinates are considered independent) as above, and the appropriate constraints, the last step in the modeling procedure is the use of the Udwadia–Phohomsiri fundamental equation [7–9] which yields the required equations of motion of the constrained system.

The facility with which these equations are obtained is notable since one has only to provide the equations of motion of the unconstrained system and the constraints. The fundamental equation then automatically generates the correct equations of motion. Several numerical simulations are presented. They show that, despite the simplicity of the chosen geometries, the trajectories of the sphere have interesting non-trivial behavior, which depends in a complex manner on the motion of the surface on which it rolls.

2. The model

Consider an inertial frame of reference Oxyz in which a rigid and homogeneous sphere of radius R and mass m (with center of mass is at C) moves on an arbitrarily prescribed surface (see Fig. 1). The coordinates of C in this inertial frame are (x_C, y_C, z_C), and the surface Γ on which the sphere moves is described by the equation $f(x, y, z, t) = 0$, where the function f is assumed to be a C^2 function of it arguments. We shall also assume that the sphere and the surface are in contact only at one point W with coordinates (ξ, η, ζ), and that the sphere rolls on the surface without slip. Gravity is acting in the negative Z-direction, as shown.

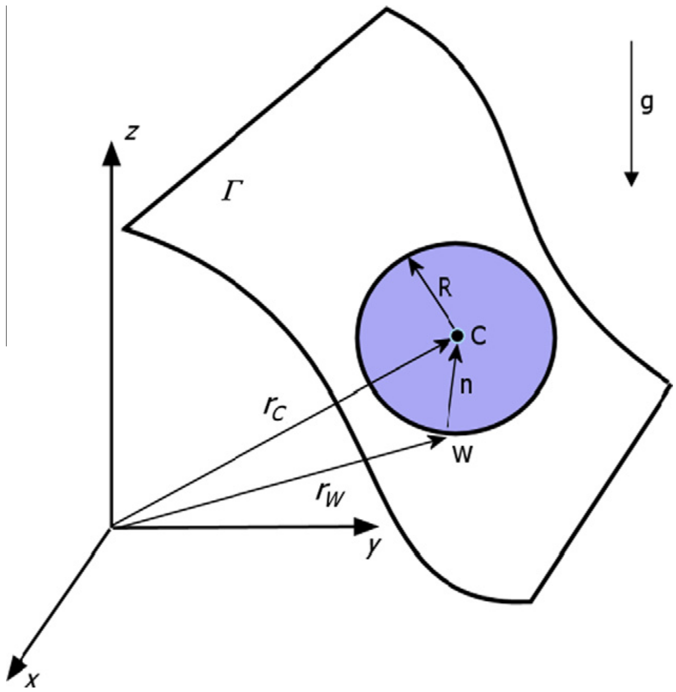

Fig. 1. Rigid sphere of radius R rolling without slipping on the moving surface Γ.

We deduce the equation of the motion of the sphere on an arbitrarily prescribed surface Γ in a simple, three-step fashion [6] .

- *Step 1:* We write the equations of the motion of the unconstrained sphere, free to move under gravity in three-dimensional space and unconstrained by the presence of the surface Γ; we add to this a zero-mass particle and assume that all the coordinates describing this system are independent of one another. We call this the unconstrained system, and obtain its equations of motion.
- *Step 2:* We impose on this unconstrained system appropriate constraints that bind the zero-mass particle to the surface Γ. Furthermore, we suitably constrain its coordinates so that it lies at the point of contact W between the sphere (that rolls without slipping on Γ) and the moving surface.
- *Step 3:* Using the equations of motion of the unconstrained system (obtained in the first step) and the constraints (obtained in the second step) we generate the equations of motion of the constrained system, which then gives the required dynamical description of the sphere rolling on the arbitrarily prescribed moving surface Γ. We do this by using the general Udwadia–Phohomsiri equations of motion that are applicable to systems with singular mass matrices. Application of these equations generate the explicit equations of motion of the constrained system. In what follows we take up each of these three steps.

2.1. Unconstrained system

We begin by decoupling the sphere from its supporting surface. Description of the configuration of the sphere requires six coordinates, three coordinates to describe the location of its center of mass C, and three to describe the orientation of the sphere. In order to avert singularities caused by the use of Euler angles, we use quaternions, thereby adding one additional redundant coordinate for the description of the orientation of the sphere. The orientation of the sphere is then defined by the unit quaternion $u = [u_0, u_1, u_2, u_3]^T = [\cos(\theta/2), e\sin(\theta/2)]$ where $\mathbf{e}(t)$ is a unit vector, which represents the instantaneous axis of rotation, defined in the inertial frame and $\theta(t)$ is the proper rotation about this vector.

To these seven coordinates that describe the configuration of the unconstrained system we add three additional coordinates—the coordinates of a point W which at present may be thought of as a particle of zero mass located at some point (ξ, η, ζ) which is detached from both the sphere and the surface Γ. Later on, we shall attach the point W to the location where the sphere meets the surface, so that the point W will lie at each instant of time on the path traced out on the surface Γ by the sphere as it rolls over Γ. Thus the generic configuration of the system is described by a set of ten variables that are: the three coordinates (x_C, y_C, z_C) of the center of mass of the sphere C (in the inertial frame Oxyz, see Fig. 1); the three coordinates (ξ, η, ζ) of the detached zero-mass point W; and the four components (u_0, u_1, u_2, u_3) of the unit quaternion u that describe the orientation of the sphere. We shall denote this ten-component vector of generalized coordinates by $q = [x_C, y_C, z_C, \xi, \eta, \zeta, u_0, u_1, u_2, u_3]^T$. Moreover, we shall assume, to begin with, that *all* these coordinates are independent of one another, though we well know that the four components of the unit quaternion cannot be independently assigned. We designate the system so obtained as our unconstrained system [8,10].

Since we have assumed that the sphere is not in contact with the moving surface, it is trivial to write the equations of motion of this sphere-particle system.

The kinetic energy of the sphere can be written as

$$T = T_T + T_R \tag{1}$$

where T_T and T_R are the translational energy and the rotational kinetic energy of the sphere respectively. They can be expressed as

$$T_T = \frac{1}{2}m(\dot{x}_C^2 + \dot{y}_C^2 + \dot{z}_C^2) \tag{2}$$

and

$$T_R = 2u^T\dot{E}^T\hat{J}\dot{E}u \tag{3}$$

where $\hat{J}$ is the diagonal matrix $\hat{J} = diag(1, J_1, J_2, J_3)$ and $J_1 = J_2 = J_3 = J$ are the moments of inertia [11]. The matrix E in (3) is an orthogonal matrix defined as:

$$E = \begin{bmatrix} u_0 & u_1 & u_2 & u_3 \\ -u_1 & u_0 & u_3 & -u_2 \\ -u_2 & u_3 & u_0 & u_1 \\ -u_3 & u_2 & -u_1 & u_0 \end{bmatrix} := \begin{bmatrix} u^T \\ - \\ E_1 \end{bmatrix} \tag{4}$$

and the three-components of angular velocity of the sphere in the sphere-fixed frame of reference are given by the column vector $\omega' = 2\dot{E}_1u$. The lower 3 by 4 matrix on the left hand side of the ':=' sign is denoted in (4) by E_1. The potential energy of the sphere is

$$V = mgz \tag{5}$$

Assuming that each of the coordinates describing the configuration of the system is independent of the others, we can then write Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{6}$$

for the sphere where the Lagrangian $L = T - V$.

The resulting unconstrained equations of the motion for the multi-body system comprising the sphere and the zero-mass particle can now be written in matrix form as

$$[M][\ddot{q}] = [Q] \tag{7}$$

The 10 by 10 matrix M has a block diagonal form and is given by

$$[M] = \begin{bmatrix} [M_1]_{3\times 3} & & 0 \\ & [0]_{3\times 3} & \\ 0 & & [M_2]_{4\times 4} \end{bmatrix} \tag{8}$$

where

$$[M_1] = m[I] \tag{9}$$

and [11],

$$[M_2] = \begin{bmatrix} 4(u_0^2 + J(u_1^2 + u_2^2 + u_3^2)) & -4(J-1)u_0u_1 & -4(J-1)u_0u_2 & -4(J-1)u_0u_3 \\ -4(J-1)u_0u_1 & 4(u_1^2 + J(u_0^2 + u_2^2 + u_3^2)) & -4(J-1)u_1u_2 & -4(J-1)u_1u_3 \\ -4(J-1)u_0u_2 & -4(J-1)u_1u_2 & 4(u_2^2 + J(u_0^2 + u_1^2 + u_3^2)) & -4(J-1)u_2u_3 \\ -4(J-1)u_0u_3 & -4(J-1)u_1u_3 & -4(J-1)u_2u_3 & 4(u_3^2 + J(u_0^2 + u_1^2 + u_2^2)) \end{bmatrix} \tag{10}$$

Table 1
Parameters used in simulation for the various Cases Ia–IIId shown in the leftmost column.

<table>
<tr><th></th><th>A_1</th><th>A_2</th><th>A_3</th><th>Ω_1 [rad/s]</th><th>Ω_2 [rad/s]</th><th>Ω_3 [rad/s]</th></tr>
<tr><td>I a</td><td rowspan="4" colspan="3">0.1</td><td rowspan="4">0</td><td>π</td><td rowspan="4">0</td></tr>
<tr><td>I b</td><td>1.5 π</td></tr>
<tr><td>I c</td><td>3 π</td></tr>
<tr><td>I d</td><td>6 π</td></tr>
<tr><td>II a</td><td rowspan="4" colspan="3">0.1</td><td colspan="2">π</td><td rowspan="4">0</td></tr>
<tr><td>II b</td><td colspan="2">1.5 π</td></tr>
<tr><td>II c</td><td colspan="2">3 π</td></tr>
<tr><td>II d</td><td colspan="2">6 π</td></tr>
<tr><td>III a</td><td rowspan="4" colspan="3">0.1</td><td colspan="3">π</td></tr>
<tr><td>III b</td><td colspan="3">1.5 π</td></tr>
<tr><td>III c</td><td colspan="3">3 π</td></tr>
<tr><td>III d</td><td colspan="3">6 π</td></tr>
</table>

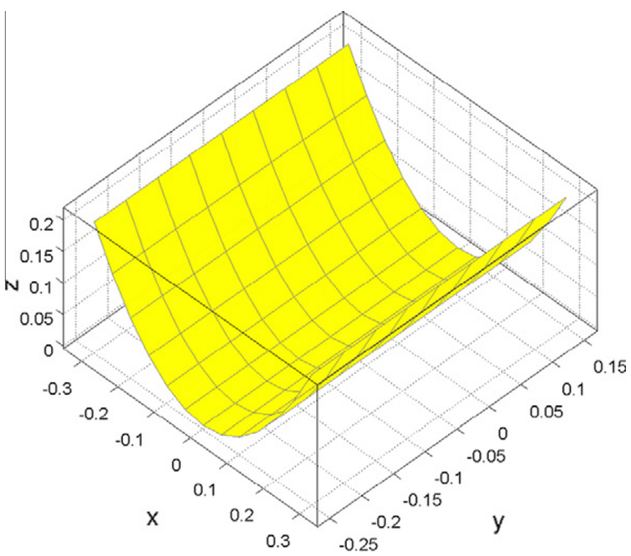

Fig. 2. Shape of parabolic surface Γ that oscillates in the x-, y- and z-direction.

The 3 by 3 zero mass matrix along the diagonal in Eq. (8) corresponds to the motion of the zero-mass particle that we had added to our vector of generalized coordinates. This particle is, up to this point in our development, thought of as being detached from both the sphere and the surface Γ. We note that its inclusion in our description of the unconstrained motion of the dynamical system leads to a mass matrix M that is singular.

The applied generalized force vector Q is:

$$[Q] = \begin{bmatrix} 0 \\ 0 \\ -mg \\ 0 \\ 0 \\ 0 \\ -4u_0\dot{u}_0^2 - 8J(u_1\dot{u}_1 + u_2\dot{u}_2 + u_3\dot{u}_3)\dot{u}_0 + 8(J-\frac{1}{2})(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2)u_0 \\ -4u_1\dot{u}_1^2 - 8J(u_0\dot{u}_0 + u_2\dot{u}_2 + u_3\dot{u}_3)\dot{u}_1 + 8(J-\frac{1}{2})(\dot{u}_0^2 + \dot{u}_2^2 + \dot{u}_3^2)u_1 \\ -4u_2\dot{u}_2^2 - 8J(u_0\dot{u}_0 + u_1\dot{u}_1 + u_3\dot{u}_3)\dot{u}_2 + 8(J-\frac{1}{2})(\dot{u}_0^2 + \dot{u}_1^2 + \dot{u}_3^2)u_2 \\ -4u_3\dot{u}_3^2 - 8J(u_0\dot{u}_0 + u_1\dot{u}_1 + u_2\dot{u}_2)\dot{u}_3 + 8(J-\frac{1}{2})(\dot{u}_0^2 + \dot{u}_1^2 + \dot{u}_2^2)u_3 \end{bmatrix} \tag{11}$$

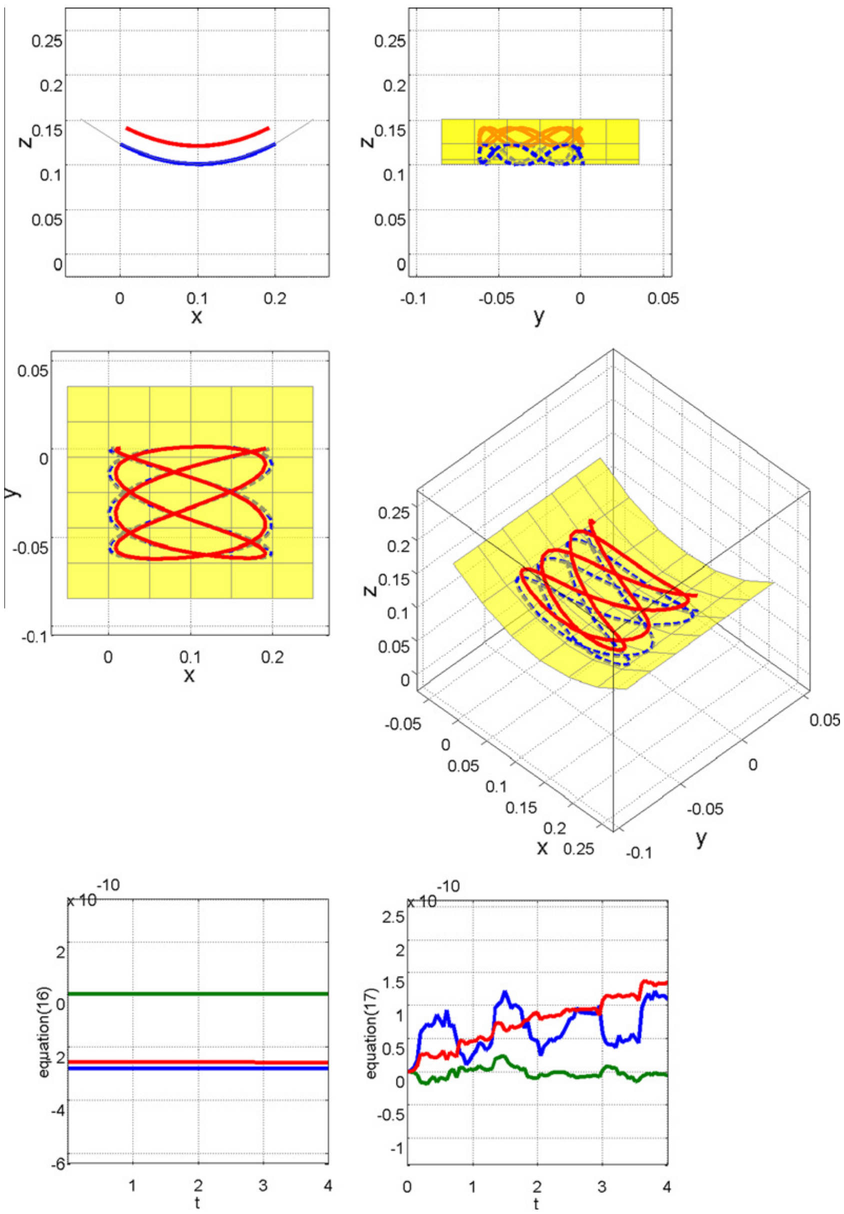

Fig. 3. Dynamical behavior and errors in satisfaction of the constraints for Case Ia.

Eq. (7) yields the unconstrained equations of motion of the sphere. We note that the zero-mass particle has no force applied to it, and so the corresponding three elements of the vector Q in (11) are zero. In the next section we will describe the constraints necessary to be imposed so that this particle lies on the surface and becomes the point of contact between the sphere and the surface, as the sphere rolls. Also, the four components of the quaternion u have been assumed to be independent of one another. Constraints ensuring that the quaternion represents a physical rotation will be given in the next section, and will be enforced in the one that follows it.

2.2. Constraints

In this sub-section we formulate the constraint equations that connect the dependent coordinates (and their derivatives) that are the elements of the vector q so that the constrained system constitutes the sphere rolling on the surface Γ without slipping.

(1) Since the quaternion u must be a unit quaternion in order to represent a physical rotation, we require that

$$h_1(q, \dot{q}, t) := u^T u - 1 = u_0^2 + u_1^2 + u_2^2 + u_3^2 - 1. \tag{12}$$

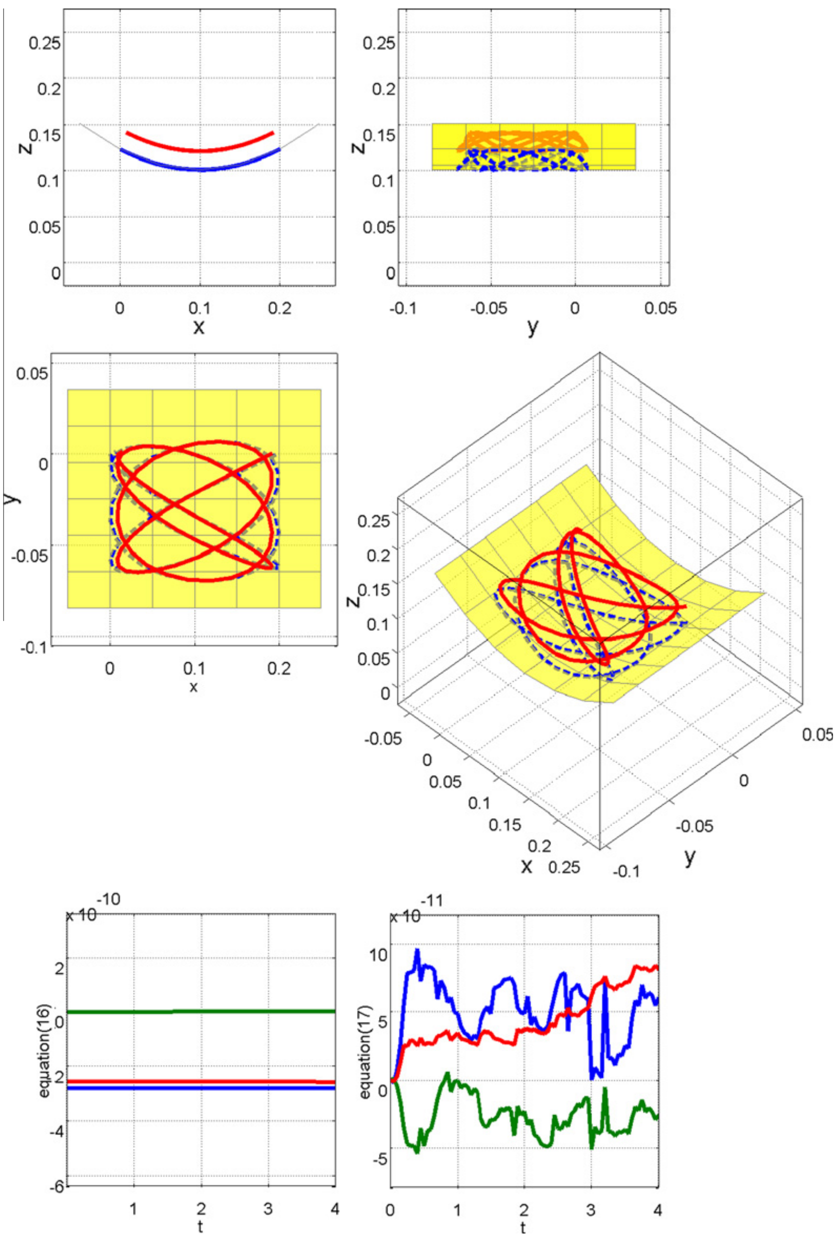

Fig. 4. Dynamical behavior and errors in satisfaction of the constraints for Case Ib.

(2) Next we bind the point W to the moving surface Γ, so that its coordinates (ξ, η, ζ) satisfy the constraint equation

$$h_2(q, \dot{q}, t) := f(x - g_1(t), y - g_2(t), z - g_3(t)) = 0 \tag{13}$$

where the functions $g_i(t), i = 1, 2, 3$ are prescribed functions of time that describe the rigid body motion of the entire surface Γ in the x-, y- and z-directions in an inertial frame of reference.

(3) Let us denote the outward pointing unit normal vector $\hat{n}$ to the surface Γ at the point W at time t (pointing from the point W on Γ towards the center of the sphere C), as shown in Fig. 1, by

$$\hat{n}(W) := \hat{n}(\xi, \eta, \zeta, t) = \nabla f / \|\nabla f\|, \quad \text{where} \quad \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]^T_{(\xi, \eta, \zeta, t)} \tag{14}$$

We next constrain the sphere so that the surface of the sphere and the surface Γ have a common tangent at W at each instant of time. This requires that the distance along the normal to the surface (at W) between the center of the sphere C and the contact point W equals the radius of the sphere, R, and therefore that

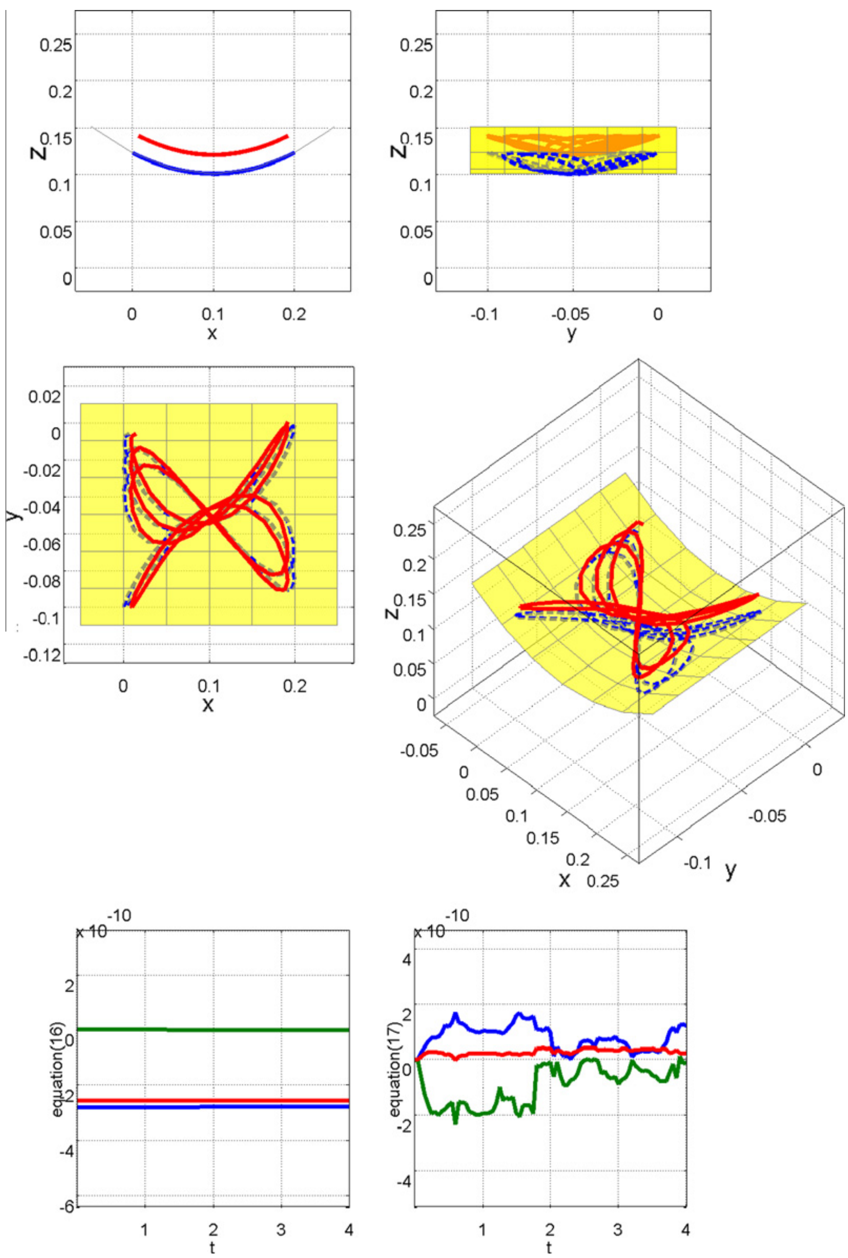

Fig. 5. Dynamical behavior and errors in satisfaction of the constraints for Case Ic.

$$h_{3,4,5}(q, \dot{q}, t) := r_w - r_C + R\hat{n} = 0 \tag{15}$$

where r_w and r_C are the vectors shown in Fig. 1. Eq. (15) can be expressed in components along the inertial reference frame as

$$\begin{bmatrix} \xi - x_C \\ \eta - y_C \\ \zeta - z_C \end{bmatrix} + R \cdot \left[\frac{\nabla f}{\|\nabla f\|}\right]_{\xi,\eta,\zeta,t} = [0] \tag{16}$$

(4) The last constraint enforces the no-slip condition between the sphere and the surface and can be written as

$$h_{6,7,8}(q, \dot{q}, t) := \dot{r}_C - v_{surf} - R\omega \times \hat{n} = 0 \tag{17}$$

where, v_{surf} denotes the velocity vector of the point on the surface Γ coincident with the contact point W that is due to the rigid body motion of the entire surface Γ (see Eq. (13)); its components in the inertial frame are given by $v_{surf} = [\dot{g}_1, \dot{g}_2, \dot{g}_3]^T$. The vector ω in (17) is the angular velocity of the sphere in the inertial frame. The latter can be obtained from the angular velocity ω' in the sphere-fixed frame, defined as $\omega' = 2\dot{E}_1 u$, by using the rotation matrix $[\tilde{R}]'$, which is a function of the 4-component vector u, given by

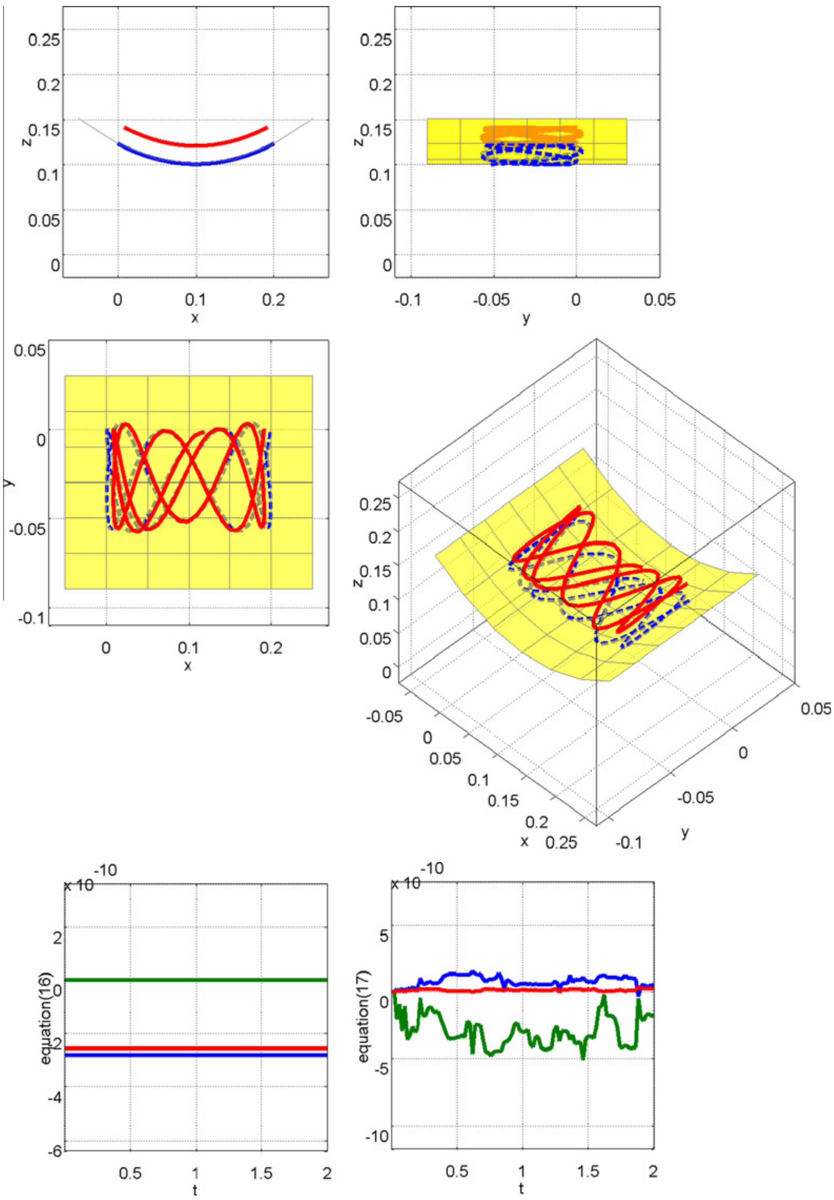

Fig. 6. Dynamical behavior and errors in satisfaction of the constraints for Case Id.

$$[\tilde{R}]' = \begin{bmatrix} u_0^2 + u_1^2 - u_2^2 - u_3^2 & 2(u_1u_2 - u_0u_3) & 2(u_1u_3 + u_0u_2) \\ 2(u_1u_2 + u_0u_3) & u_0^2 - u_1^2 + u_2^2 - u_3^2 & 2(u_2u_3 - u_0u_1) \\ 2(u_1u_3 - u_0u_2) & 2(u_2u_3 + u_0u_1) & u_0^2 - u_1^2 - u_2^2 + u_3^2 \end{bmatrix} \tag{18}$$

so that

$$\omega = [\tilde{R}]' \cdot \omega' = 2 \cdot \begin{bmatrix} u_0\dot{u}_1 + u_1\dot{u}_0 + u_2\dot{u}_3 - u_3\dot{u}_2 \\ u_0\dot{u}_2 - u_2\dot{u}_0 - u_1\dot{u}_3 + u_3\dot{u}_1 \\ u_0\dot{u}_3 - u_3\dot{u}_0 + u_1\dot{u}_2 - u_2\dot{u}_1 \end{bmatrix} := \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \tag{19}$$

Eq. (17) can therefore be written in components along the inertial coordinate system as

$$\begin{bmatrix} \dot{x}_C - \dot{g}_1 \\ \dot{y}_C - \dot{g}_2 \\ \dot{z}_C - \dot{g}_3 \end{bmatrix} - R\tilde{\omega} \cdot \left[\frac{\nabla f}{\|\nabla f\|} \right]_{\xi,\eta,\zeta,t} = [0] \tag{20}$$

where $\tilde{\omega}$ denotes the skew symmetric matrix obtained from the 3-component column vector ω defined in (19). Eqs. (12), (13), (16), and (20) constitute a set of eight (holonomic and non-holonomic) constraint equations that now appropriately

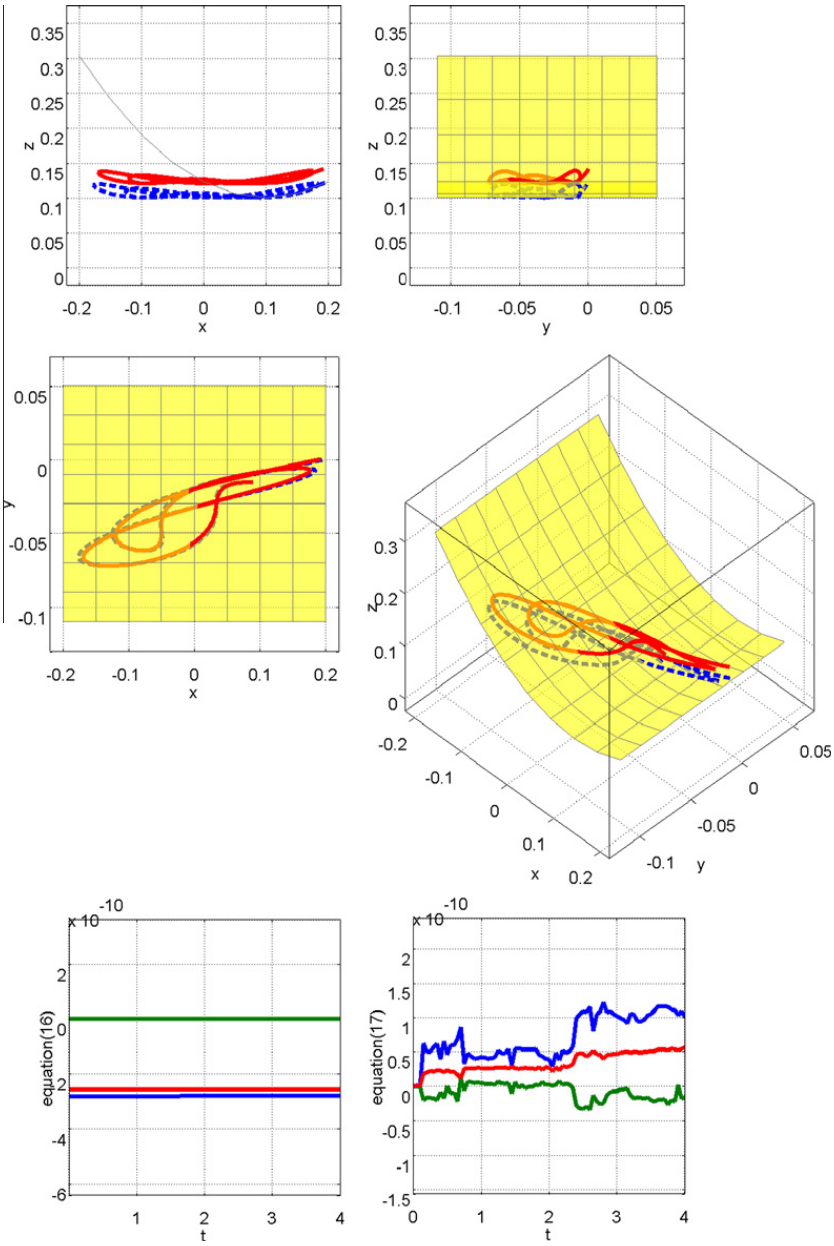

Fig. 7. Dynamical behavior and errors in satisfaction of the constraints for Case IIa.

describe the motion of the sphere as it rolls without slip on the arbitrarily prescribed surface Γ. By differentiation with respect to time, this set of equations can all be expressed in the form [8,9]:

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t) \tag{21}$$

where A is an 8 by 10 matrix. The matrix A is explicitly obtained as:

$$[A] = \begin{bmatrix} \frac{\partial \ddot{h}_1}{\partial \ddot{q}_1} & \cdots & \frac{\partial \ddot{h}_1}{\partial \ddot{q}_{10}} \\ \vdots & & \vdots \\ \frac{\partial \ddot{h}_5}{\partial \ddot{q}_1} & \cdots & \frac{\partial \ddot{h}_5}{\partial \ddot{q}_{10}} \\ \frac{\partial \dot{h}_6}{\partial \ddot{q}_1} & \cdots & \frac{\partial \dot{h}_6}{\partial \ddot{q}_{10}} \\ \vdots & & \\ \frac{\partial \dot{h}_8}{\partial \ddot{q}_1} & \cdots & \frac{\partial \dot{h}_1}{\partial \ddot{q}_{10}} \end{bmatrix} \tag{22}$$

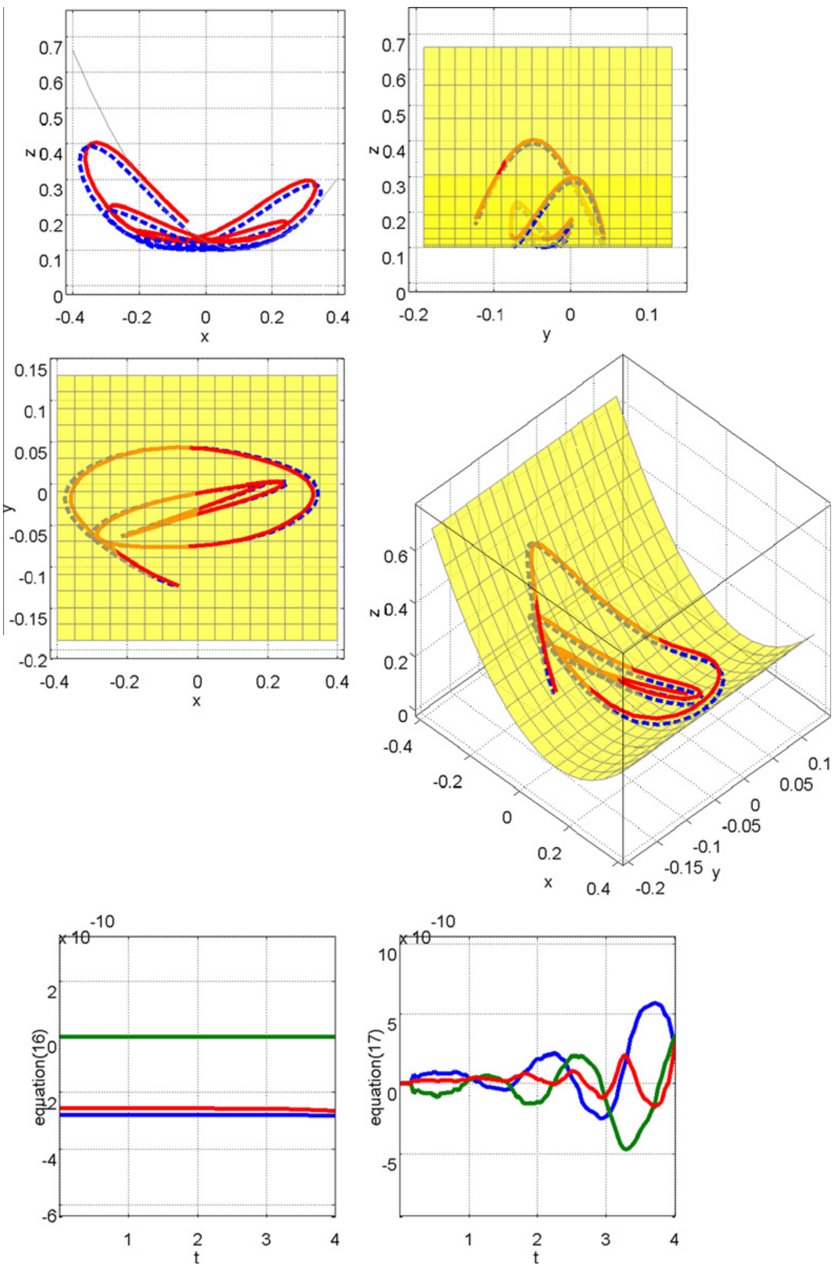

Fig. 8. Dynamical behavior and errors in satisfaction of the constraints for Case IIb.

and the column vector b is given by

$$\{b\} = \begin{Bmatrix} \sum_{i=1}^{10} \frac{\partial(\ddot{h}_1)}{\partial \ddot{q}_i} \ddot{q}_i - \ddot{h}_1 \\ \vdots \\ \sum_{i=1}^{10} \frac{\partial(\ddot{h}_5)}{\partial \ddot{q}_i} \ddot{q}_i - \ddot{h}_5 \\ \sum_{i=1}^{10} \frac{\partial(\dot{h}_6)}{\partial \ddot{q}_i} \ddot{q}_i - \dot{h}_6 \\ \vdots \\ \sum_{i=1}^{10} \frac{\partial(\dot{h}_8)}{\partial \ddot{q}_i} \ddot{q}_i - \dot{h}_8 \end{Bmatrix} \tag{23}$$

The explicit expressions for the elements of the matrix A and the column vector b can be obtained for any given surface Γ as shown in the Appendix.

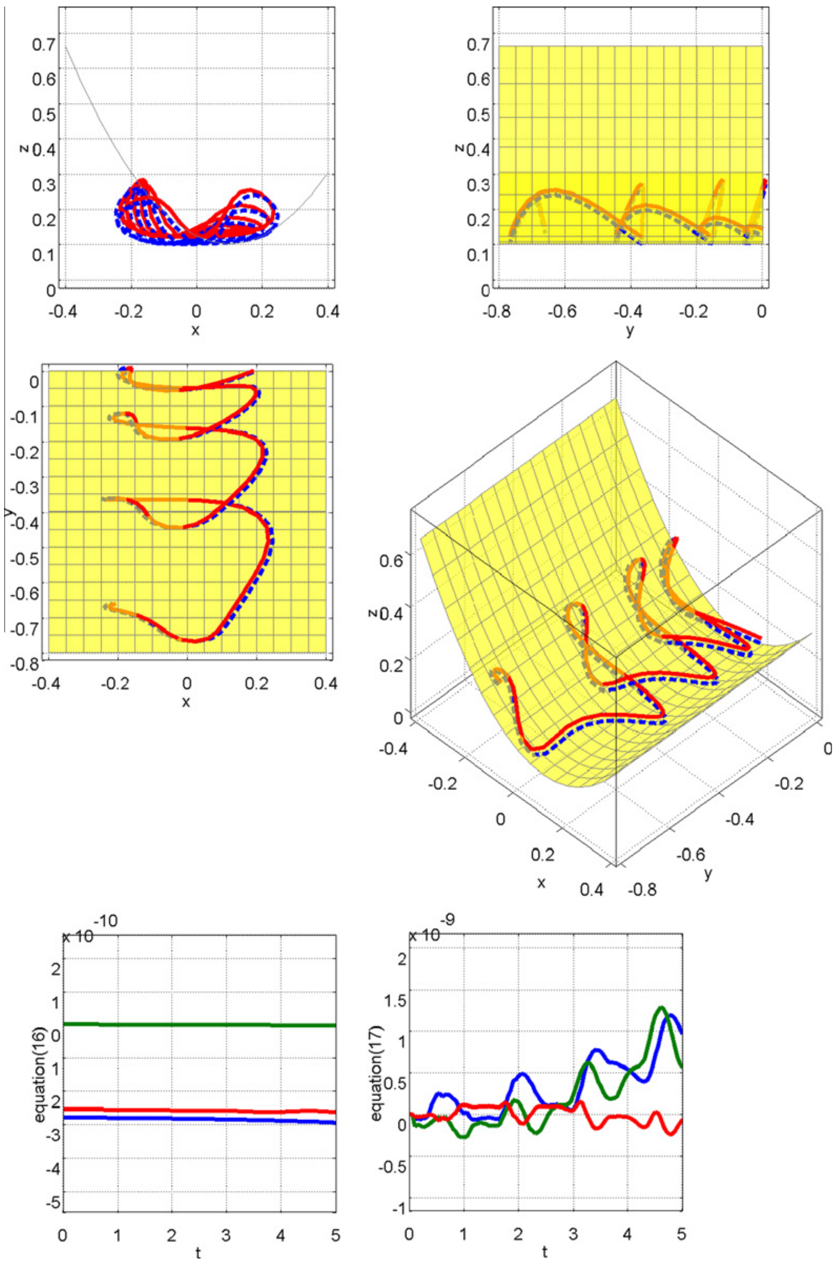

Fig. 9. Dynamical behavior and errors in satisfaction of the constraints for Case IIc.

2.3. The constrained equation of motion

Eq. (7) now constitutes the unconstrained equation of motion of the system, and Eq. (21) specifies the constraints. Since the matrix $[M]$ is singular, we need to use the Udwadia–Phohomsiri equation [7] to get the equation of motion for the constrained system. Hence, the acceleration of the system is given by [7–12]

$$[\ddot{q}] = \begin{bmatrix} [I - A^{+}A]M \\ A \end{bmatrix}^{+} \cdot \begin{bmatrix} Q \\ b \end{bmatrix} \tag{24}$$

where P^{+} denotes the Moore–Penrose inverse of the rectangular matrix P. We note that [7] for the above equation to be valid we require the matrix $[M|A^{T}]$ to have full rank (i.e., rank = 10). This rank condition serves as a check on whether we have modeled our system correctly, because it is also the condition required for the acceleration $\ddot{q}$ to be unique – a condition that must be fulfilled for all physical systems in classical mechanics.

Closed form expressions for the elements of M and Q are explicitly given in Eqs. (8)–(11); those for A and b, for a prescribed surface, may be obtained as in Eqs. (34)–(40) given in the Appendix. Having thus obtained in closed form the

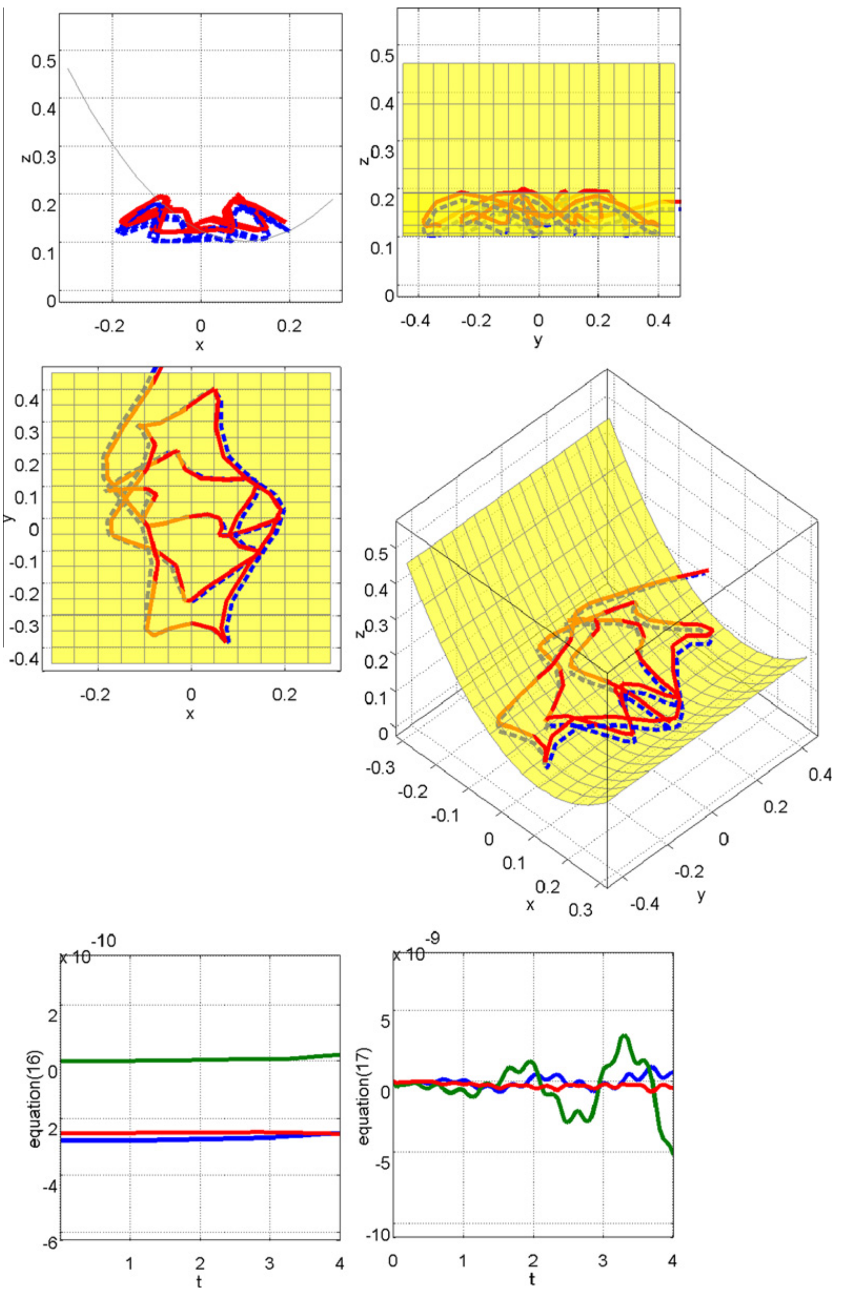

Fig. 10. Dynamical behavior and errors in satisfaction of the constraints for Case IId.

matrices M and A, and the column vectors Q and b, the required equations of motion describing the constrained system are generated simply by substitution in the right-hand side of Eq. (24). These ten coupled equations, which describe the motion of the constrained system, are highly nonlinear, non-autonomous, complex, and long, and upon use of the explicit expressions for M, A, Q, and b, they are directly obtained in the Maple environment. They run into several pages, and have not been given here. Their complexity precludes any direct analytical analysis and/or understanding, and it appears that insight into the dynamics of this system can be more fruitfully obtained through simulations, which we present next.

3. Numerical example

In this section we show a numerical example of the procedure developed earlier. We consider a time dependent surface Γ described by the equation (Fig. 2):

$$f(x,y,z,t) = \frac{\bar{x}^2}{a_0^2} - \frac{\bar{z}}{b_0^2} = 0, \quad \bar{y} = 0 \tag{25}$$

where

$$\bar{x} = x - A_1 \cos(\Omega_1 t), \quad \bar{y} = y - A_2 \cos(\Omega_2 t)\bar{z} = z - A_3 \cos(\Omega_3 t) \tag{26}$$

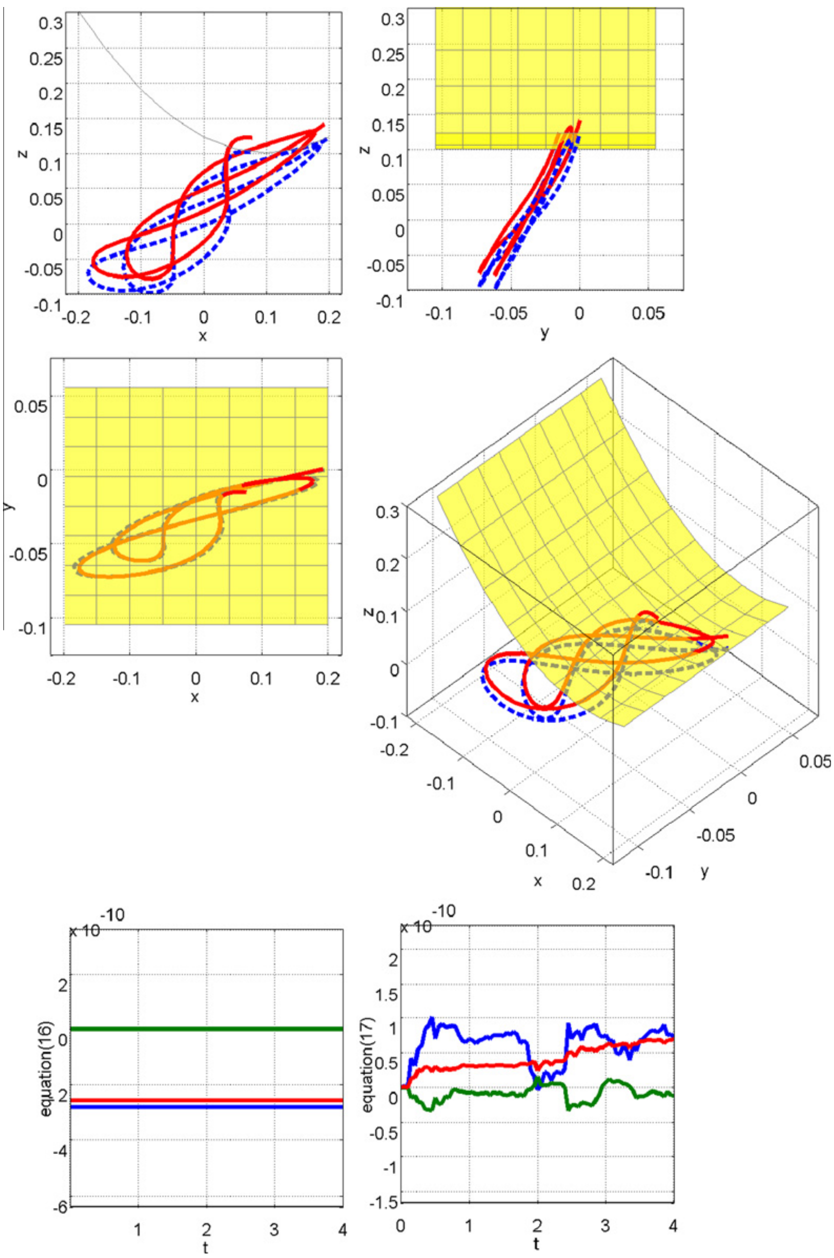

Fig. 11. Dynamical behavior and errors in satisfaction of the constraints for Case IIIa.

Thus the surface Γ can rigidly translate along the x-, y- and z-direction respectively with a harmonic velocity in each direction given by

$$\mathbf{v}_{surf} = -[A_1\Omega_1 \sin(\Omega_1 t), A_2\Omega_2 \sin(\Omega_2 t), A_3\Omega_3 \sin(\Omega_3 t)]^T \tag{27}$$

In the following we report the results of the numerical integration of the equation of motion (24) for a steel sphere ($m = 0.2634$; $J = 4.21E-5$; $R = 0.02$) rolling on the surface Γ given by Eq. (25) with $a_0 = 2$ and $b_0 = 3$. The simulations are carried out for different amplitudes, A_i, and frequencies, Ω_i, of motion of the surface in each direction, and their values are shown in Table 1.

For all the simulations, the sphere is taken to be initially at rest. Its initial position is taken to be $\xi_0 = 0.20$ and $\eta_0 = 0$ for all cases in the simulation sets I and II (see Table 1), and $\xi_0 = 0.30$ and $\eta_0 = 0$ for all cases in the simulation set III. The other coordinates at $t = 0$ are evaluated so that they satisfy the constraint equations. Matlab's *ode45* integrator is used with a relative error tolerance of *1.0E−9* and an absolute error tolerance of *1.0E−13*. The condition that the rank of $[M|A^T]$ equals 10 is checked at each time during the simulation. The simulation is shown for a duration of 4 s. As seen from the figures that follow, this duration is sufficient to exhibit the variety and complexity of the dynamical behavior for each of the cases shown in Table 1 without making the complex trajectories that are generated overly difficult to decipher.

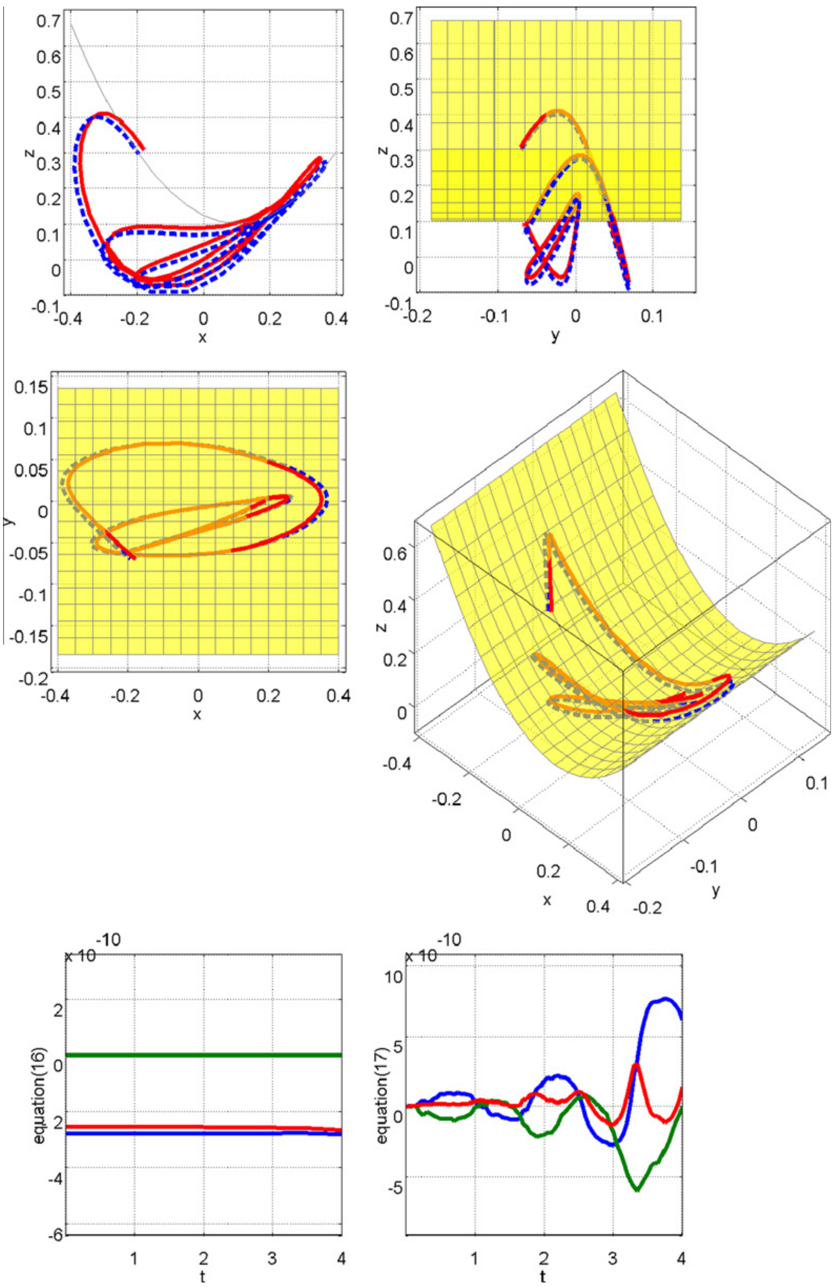

Fig. 12. Dynamical behavior and errors in satisfaction of the constraints for Case IIIb.

Figs. 3–14 show the dynamical behavior of the system corresponding to each of the cases in the leftmost column in Table 1. The trajectory of the center of mass of the ball C is shown in each of these figures by a solid line, and the corresponding trajectory of the point of contact W of the sphere with the moving surface is shown by a dashed line. Errors in the satisfaction of the constraints (16) and (17) are also shown for each case, and we find that these errors are of the same order of magnitude as the relative error tolerances set for our numerical integration. In each of the figures the errors related to Eq. (16) are color coded as follows: the blue line gives the error is satisfaction of the constraint h_3, the green line gives the error in the satisfaction in the constraint h_4, and the red line gives the error in the satisfaction of the constraint h_5. Similarly, the errors in the satisfaction of the constraints related to Eq. (17) are color coded so that the blue line corresponds to the error in h_6, the green line to h_7, and the red line to h_8. Explicit expressions for these errors are given in Eqs. (32) and (33) in the Appendix.

Fig. 3 shows the results of the simulation, when the surface Γ moves only along the y-direction with a frequency $\Omega_1 = \pi$ rads/s. The motion of the sphere appears to be periodic. Figs. 4–6 show the change in the motion when the frequency of motion of the surface is increased to 1.5π, 3π, and 6π rads/s, respectively. We note that the motion of the sphere appears to remain periodic though its nature can be widely different (compare especially Figs. 5 and 6). Fig. 7 shows a switch to what appears to be aperiodic behavior of the sphere when the surface Γ is excited in both the x- and the y-directions simultaneously by a harmonic excitation at a frequency of π rads/s (see Table 1 also). This behavior appears to persist as the frequency of the excitation of the surface Γ in both the x- and y-directions is increased as shown in Figs. 8–10. Fig. 9 shows

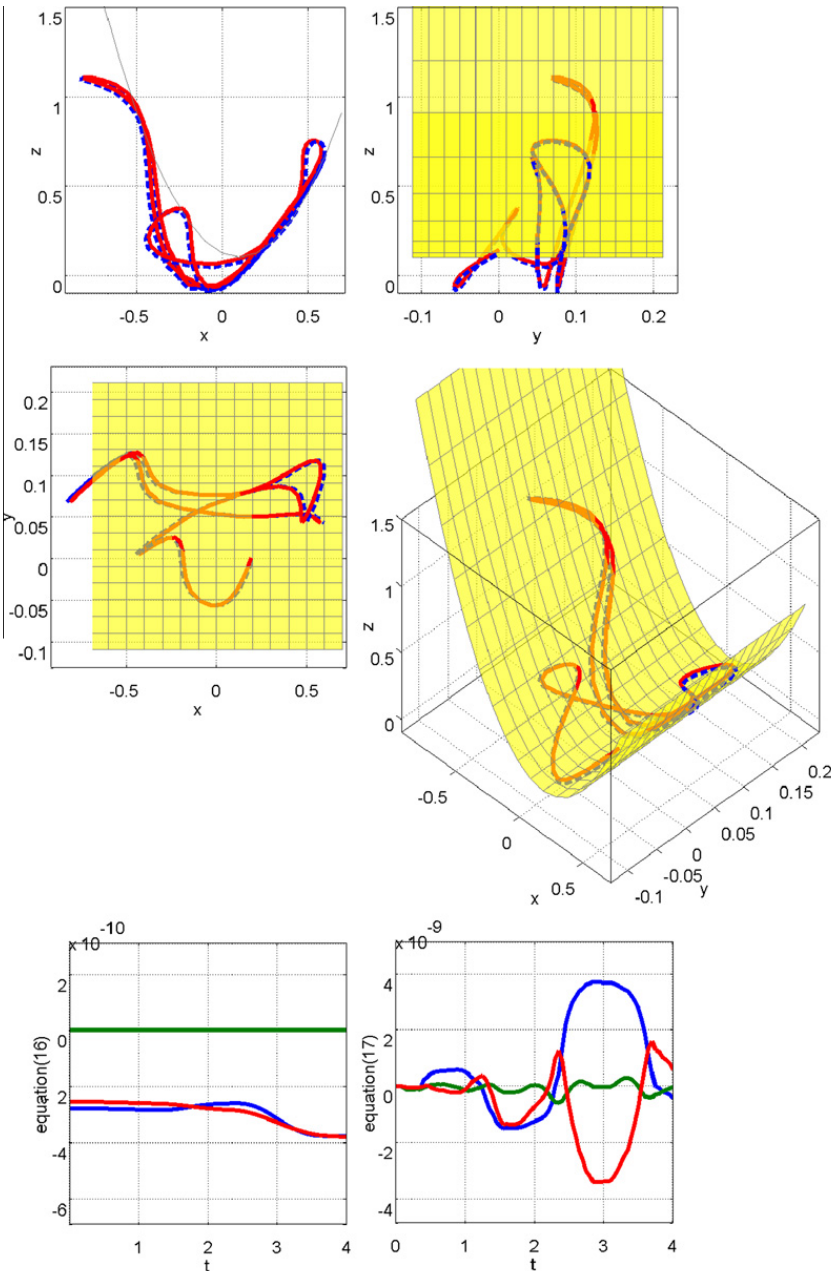

Fig. 13. Dynamical behavior and errors in satisfaction of the constraints for Case IIIc.

an interesting behavior in which the sphere appears to 'walk' in the x-direction when the surface moves harmonically at a frequency $\Omega_1 = \Omega_2 = 3\pi$ rads/s, resulting in, what appears to be, unbounded motion in the x-direction.

When the surface is harmonically moved in all three directions, the motion of the sphere becomes considerably more complex (see Figs. 11–14). We observe that for $\Omega_1 = \Omega_2 = \Omega_3 = \pi$ rads/s, comparing Figs. 7 and 11, the addition of harmonic motion in the z-direction of the surface Γ does not seem to change the projected motion of the sphere on the x–y plane. The same is true for when $\Omega_1 = \Omega_2 = \Omega_3 = 1.5\pi$ as seen by comparing Figs. 8 and 12. However, for higher frequencies as seen from Figs. 9 and 13, when $\Omega_1 = \Omega_2 = \Omega_3 = 3\pi$ rads/s, the projected motions in the x–y plane are no longer similar. The additional harmonic motion in the z-direction of the surface Γ causes the entire three-dimensional motion of the sphere to change. A similar result can be adduced by comparing Figs. 10 and 14 for which the frequency of motion of the surface Γ is 6π rads/s in each of the three directions. Thus the motion of the sphere is strongly dependent on the frequencies of motion of the surface Γ, and whether this surface moves along more than one direction. The motion of the sphere is seen to be complex and it appears to range across a broad regime of behaviors, from closed periodic orbits, to aperiodic bounded orbits, to aperiodic unbounded ones.

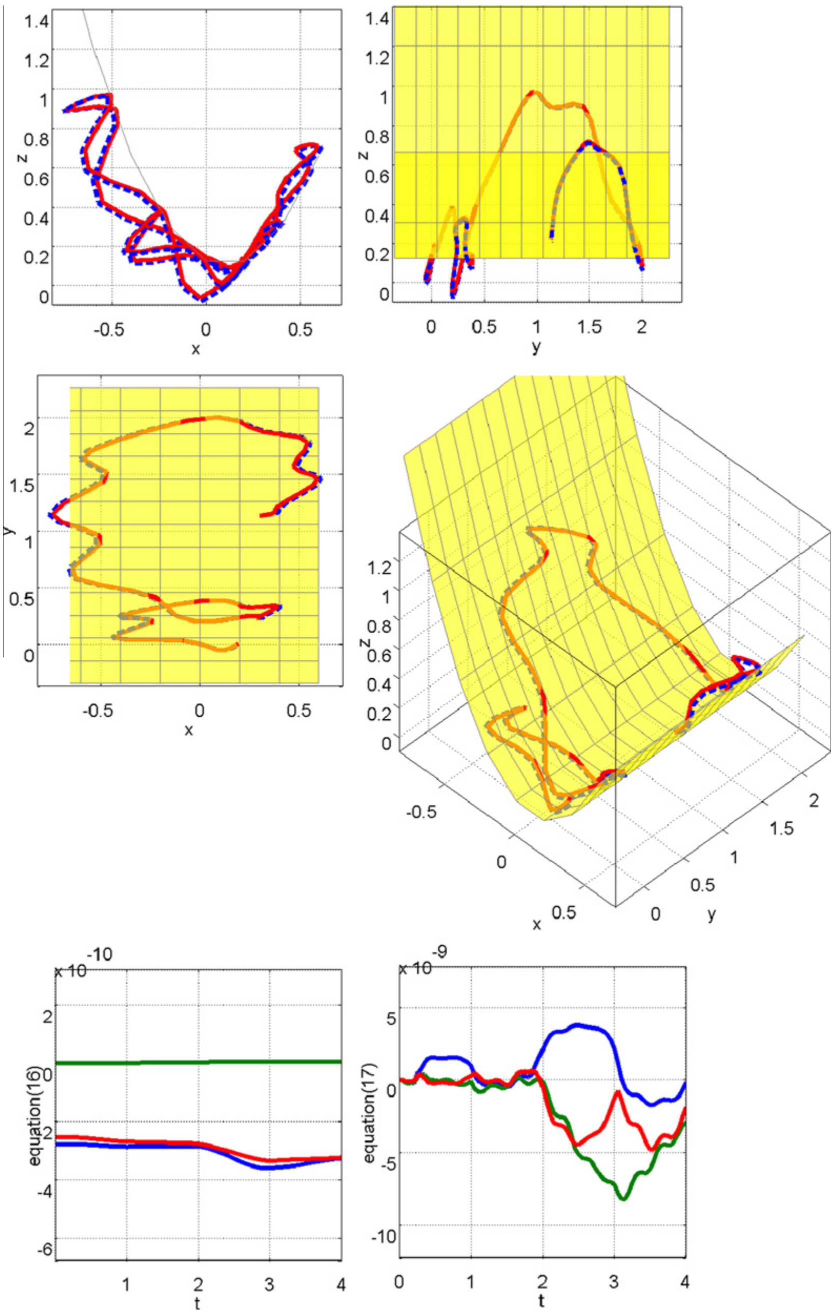

Fig. 14. Dynamical behavior and errors in satisfaction of the constraints for Case IIId.

Fig. 15 represents the numerical simulation of the motion of the same sphere starting from rest (with $\xi_0 = 0.30$ and $\eta_0 = 0$) on the same surface as before (described by (25)), except that the motion of the surface in each of the x- and y-directions is now made up of two frequency components so that

$$\begin{aligned} \bar{x} &= x - [A_{11}\cos(\Omega_{11}t) + A_{12}\cos(\Omega_{12}t)] \\ \bar{y} &= y - [A_{21}\cos(\Omega_{21}t) + A_{22}\cos(\Omega_{22}t)] \\ \bar{z} &= z \end{aligned} \tag{28}$$

where,

$$A_{11} = 0.05, \quad \Omega_{11} = \pi, \quad A_{12} = 0.1, \quad \Omega_{12} = 1.5\pi, \quad A_{21} = 0.1, \quad \Omega_{21} = \pi, \quad A_{22} = 0.05, \quad \Omega_{22} = 6\pi \tag{29}$$

The color coding of the errors in the satisfaction of the constraints given in Eqs. (16) and (17) is the same as before. This simulation clearly reveals that the motion of the sphere can become much more complex when more frequency components are present in the motion of the surface Γ on which the sphere moves.

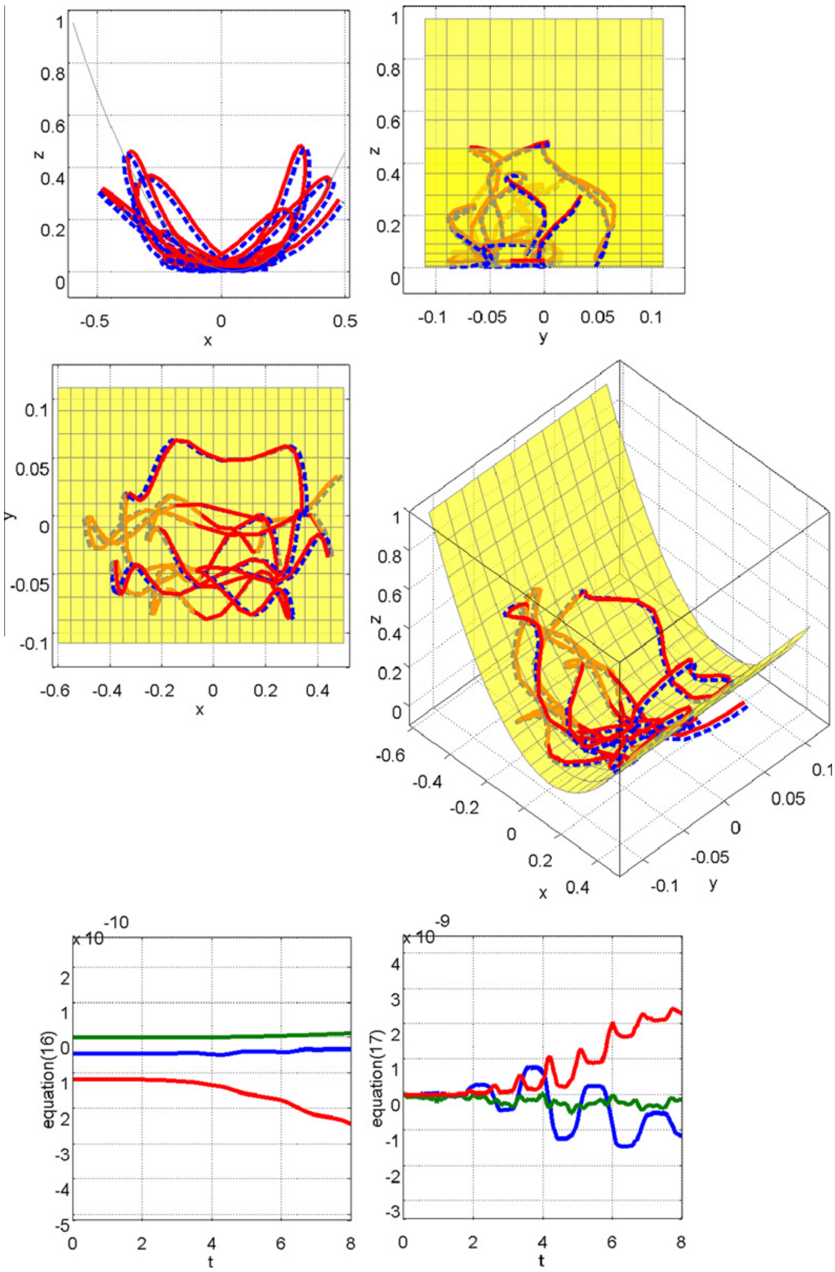

Fig. 15. Dynamical behavior and errors in satisfaction of the constraints when the motion of the surface is described by Eqs. (28) and (29).

4. Conclusions

This paper deals with the development and formulation of the equations of motion for a sphere rolling under gravity on an arbitrarily specified surface that undergoes arbitrarily prescribed motion. The contributions of this paper are the following.

1. A systematic three step modeling approach is used to facilitate the modeling of the system, which is viewed as a constrained mechanical system. The approach is simple to use and implement and generates the equations of motion of the system in a straightforward manner. More than the minimum number of coordinates are used to describe the configuration of the dynamical system, thereby providing considerable convenience in the modeling process. No use is made of the notion of Lagrange multipliers throughout the development.
2. While the use of quaternions automatically causes an over-parametrization when describing the orientation of a rigid body (the sphere), the use of a zero-mass particle in the development of the equations of motion is novel and appears not to have been attempted before. The use of such a particle has the advantage of allowing one to write the appropriate constraints almost trivially. Thus both the equations of motion pertinent to the unconstrained system and the equations describing the constraints can be written with considerable ease and facility.
3. More generally, in multi-body dynamics, constraints play a very significant and near-central role. Often these constraints can be most easily written in terms of the coordinates of one or more points that are not part of the generalized coordinates that describe the configuration of a multi-body system. For example, this is what happens here when dealing with our rigid sphere; the coordinates of the point of contact between the sphere and the surface are of central importance in stating the constraints, and yet the coordinates of this point of contact do not directly appear in the set of generalized coordinates that specify the configuration of the sphere. In such cases, as exemplified here, considerable simplification in developing the model for a multi-body system can result by *incorporating such points directly into the description of the configuration* of the multi-body system (and its dynamics) by placing zero-mass particles at these points.
4. As shown here, the suitable inclusion of a such a zero-mass particle, which is placed at a particular point, in the description of a multi-body system can permit, in a simple manner, the direct determination of the motion of that point as the system evolves in time. Since the motion of the zero-mass particle is included directly in the dynamical equations of motion of the system, the solution of these equations yields the motion of the point. The path traced out on the surface by the point of contact between the sphere and the surface, as the sphere rolls over it, is thus obtained directly.
5. The inclusion of zero-mass particles in the modeling of complex mechanical systems while simplifying the description of corresponding the unconstrained systems (and the constraints) causes them to have singular mass matrices. Recent results in analytical dynamics [7] can handle such situations when appropriately modeled, thereby opening up new and easier ways for accurately modeling and simulating the dynamical behavior of complex mechanical systems.
6. As seen from the simulations, there is considerable complexity in the motions of a rigid sphere rolling under gravity on a simple parabolic surface that moves sinusoidally. The motions of this apparently simple system appear to span a wide regime of behaviors, from closed periodic orbits, to aperiodic bounded orbits, to aperiodic unbounded ones.

Appendix

We provide in this appendix the explicit equations for the constraints for the surface described by Eq. (25), elements of the 8 by 10 matrix A and the 10 components column vector $\{b\}$.

1. The explicit constraint equations corresponding to Eqs. (12) and (13) are respectively

$$h_1(q,\dot{q},t) := u_0^2 + u_1^2 + u_2^2 + u_3^2 - 1 \tag{30}$$

$$h_2(q,\dot{q},t) := \frac{[\xi - A_1\cos(\Omega_1 t)]^2}{a^2} - \frac{\zeta - A_2\cos(\Omega_2 t)}{b^2} = 0 \tag{31}$$

To reduce notational complexity, in what follows, we denote in the above expression the scalars a: = a_0, and b: = b_0 (see Eq. (25)).

The equations corresponding to Eq. (16) are

$$\begin{aligned} h_3(q,\dot{q},t) &:= \frac{2R[\xi - A_1\cos(\Omega_1 t)]}{\sqrt{\frac{4b^4[\xi - A_1\cos(\Omega_1 t)]^2 + a^4}{b^4}}} - \xi + x_C = 0, \\ h_4(q,\dot{q},t) &:= \eta - y_C = 0 \quad \text{and} \\ h_5(q,\dot{q},t) &:= \frac{R}{\sqrt{\frac{4b^4[\xi - A_1\cos(\Omega_1 t)]^2 + a^4}{a^4}}} - \xi + x_C = 0 \end{aligned} \tag{32}$$

and those corresponding to Eq. (17) are

$$
\begin{aligned}
h_6(q,\dot q,t) &:= \frac{2R(u_0\dot u_2-u_2\dot u_0-u_1\dot u_3+u_3\dot u_1)}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{a^4}}}-\dot x_C-A_1\Omega_1\sin(\Omega_1 t)=0\\
h_7(q,\dot q,t) &:= \frac{-4R(u_0\dot u_3-u_3\dot u_0+u_1\dot u_2-u_2\dot u_1)[\xi-A_1\cos(\Omega_1 t)]}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{b^4}}}+\\
&\quad -\frac{2R(u_0\dot u_1-u_1\dot u_0+u_2\dot u_3-u_3\dot u_2)}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{a^4}}}-\dot y_C-A_2\Omega_2\sin(\Omega_2 t)=0 \quad \text{and}\\
h_8(q,\dot q,t) &:= \frac{4R(u_0\dot u_2-u_2\dot u_0-u_1\dot u_3+u_3\dot u_1)[\xi-A_1\cos(\Omega_1 t)]}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{b^4}}}-\dot z_C-A_3\Omega_3\sin(\Omega_3 t)=0
\end{aligned}
\tag{33}
$$

Using these expressions we next present the elements of the matrix A and the column vector b.

2. The matrix A is explicitly given by

$$
[A]=\begin{bmatrix}
\frac{\partial\ddot h_1}{\partial\ddot q_1} & \cdots & \frac{\partial\ddot h_1}{\partial\ddot q_{10}}\\
\vdots & & \vdots\\
\frac{\partial\ddot h_5}{\partial\ddot q_1} & \cdots & \frac{\partial\ddot h_5}{\partial\ddot q_{10}}\\
\frac{\partial\dot h_6}{\partial\ddot q_1} & \cdots & \frac{\partial\dot h_6}{\partial\ddot q_{10}}\\
\vdots & & \vdots\\
\frac{\partial\dot h_8}{\partial\ddot q_1} & \cdots & \frac{\partial\dot h_8}{\partial\ddot q_{10}}
\end{bmatrix}
:=\left[\frac{[A_1]_{5\times 10}}{[A_2]_{3\times 8}[A_3]_{3\times 2}}\right]
\tag{34}
$$

where

$$
[A_1]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 2u_0 & 2u_1 & 2u_2 & 2u_3\\
0 & 0 & 0 & \frac{2[\xi-A_1\cos(\Omega_1 t)]}{a^2} & 0 & -\frac{1}{b^2} & 0 & 0 & 0 & 0\\
1 & 0 & 0 & \frac{2R}{\left[\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{a^4b^4}\right]^{\frac{3}{2}}a^2b^4}-1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & \frac{4R[\xi-A_1\cos(\Omega_1 t)]}{\left[\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{a^4b^4}\right]^{\frac{3}{2}}a^4b^2} & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}
\tag{35}
$$

$$
[A_2]=\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & -\frac{2u_2R}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{a^4}}} & \frac{2u_3R}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{a^4}}}\\
0 & -1 & 0 & 0 & 0 & 0 & \frac{2R[2u_3b^2[\xi-A_1\cos(\Omega_1 t)]+u_1a^2]}{\sqrt{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}} & \frac{2R[2u_2b^2[\xi-A_1\cos(\Omega_1 t)]+u_0a^2]}{\sqrt{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}}\\
0 & 0 & -1 & 0 & 0 & 0 & -\frac{4u_2R[\xi-A_1\cos(\Omega_1 t)]}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{b^4}}} & \frac{4u_3R[\xi-A_1\cos(\Omega_1 t)]}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{b^4}}}
\end{bmatrix}
\tag{36}
$$

and

$$
[A_3]=\begin{bmatrix}
\frac{2u_0R}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{a^4}}} & -\frac{2u_1R}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{a^4}}}\\
-\frac{2R[2u_1b^2[\xi-A_1\cos(\Omega_1 t)]-\frac{1}{2}u_3a^2]}{\sqrt{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}} & -\frac{2R[2u_0b^2[\xi-A_1\cos(\Omega_1 t)]+u_2a^2]}{\sqrt{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}}\\
\frac{4u_0R[\xi-A_1\cos(\Omega_1 t)]}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{b^4}}} & -\frac{4u_1R[\xi-A_1\cos(\Omega_1 t)]}{\sqrt{\frac{4b^4[\xi-A_1\cos(\Omega_1 t)]^2+a^4}{b^4}}}
\end{bmatrix}
\tag{37}
$$

The column vector b is explicitly given by

$$\{b\} = \begin{Bmatrix} \sum_{i=1}^{10} \frac{\partial(\ddot{h}_1)}{\partial \ddot{q}_i} \ddot{q}_i - \ddot{h}_1 \\ \vdots \\ \sum_{i=1}^{10} \frac{\partial(\ddot{h}_5)}{\partial \ddot{q}_i} \ddot{q}_i - \ddot{h}_5 \\ \sum_{i=1}^{10} \frac{\partial(\dot{h}_6)}{\partial \ddot{q}_i} \ddot{q}_i - \dot{h}_6 \\ \vdots \\ \sum_{i=1}^{10} \frac{\partial(\dot{h}_8)}{\partial \ddot{q}_i} \ddot{q}_i - \dot{h}_8 \end{Bmatrix} := \begin{Bmatrix} b_1 \\ b_1 \end{Bmatrix} \tag{38}$$

where,

$$\{b_1\} = \left\{ \begin{array}{c} -2\dot{u}_0^2 - 2\dot{u}_1^2 - 2\dot{u}_2^2 - 2\dot{u}_3^2 \\ \hdashline \frac{2A_1^2\Omega^2[c\cos(\Omega_1 t)-1]-2A_1\Omega_1[2\dot{\xi}\sin(\Omega_1 t)+\Omega\xi\cos(\Omega_1 t)]-2\dot{\xi}^2}{a^2} + \frac{A_3\Omega_3^2\cos(\Omega_3 t)}{b^2} \\ \hdashline \dfrac{16a^4b^6R \begin{bmatrix} A_1^3\Omega_1^2[\cos(\Omega_1 t)^3 - \frac{3}{2}\cos(\Omega_1 t)] + \\ -A_1^2\Omega_1[3\dot{\xi}\cos(\Omega_1 t)\sin(\Omega_1 t) + \frac{1}{2}\Omega_1\xi\cos(\Omega_1 t)^2 - \frac{3}{2}\xi\Omega_1] + \\ +A_1\left[-\frac{3}{2}\left[\dot{\xi}^2 + \left(\frac{1}{3}\xi^2 + \frac{1}{12}\frac{a^2}{b^2}\right)\Omega_1^2\right]\cos(\Omega_1 t) + 3\xi\dot{\xi}\Omega_1\sin(\Omega_1 t)\right] + \\ +\frac{3}{2}\xi\dot{\xi}^2 \end{bmatrix}}{\left[4b^2[\xi - A_1\cos(\Omega_1 t)]^2 + a^4\right]^{\frac{3}{2}}} \\ \hdashline 0 \\ \hdashline \dfrac{4b^4A^2R \begin{bmatrix} 4A_1^4\Omega_1^2b^4[-\cos(\Omega_1 t)^4 + 2\cos(\Omega_1 t)^2] + \\ +4A_1^3\Omega_1b_1^4\cos(\Omega_1 t)[4\Omega_1\xi + \Omega_1\xi + \Omega\xi\cos(\Omega_1 t)^2 + 4\dot{\xi}\cos(\Omega_1 t)\sin(\Omega_1 t)\sin(\Omega_1 t)] + \\ +A_1^2\left[a^4\Omega_1^2[2\cos(\Omega_1 t)^2 - 1] + b^4\begin{bmatrix} 8\dot{\xi}^2\cos(\Omega_1 t)^2 + 4\xi^2\Omega_1^2\cos(\Omega_1 t)^2 + 8\Omega_1^2\xi^2 + \\ -32\xi\dot{\xi}\cos(\Omega_1 t)\sin(\Omega_1 t) \end{bmatrix}\right] + \\ +a_1\left[a^4\Omega_1[\Omega_1\xi\cos(\Omega_1 t) + 2\dot{\xi}\sin(\Omega_1 t)] + b^4\begin{bmatrix} 16\xi\dot{\xi}\cos(\Omega_1 t) = 4\Omega_1^2\xi^3\cos(\Omega_1 t) \\ -16\xi^2\dot{\xi}\Omega_1\sin(\Omega_1 t) \end{bmatrix}\right] \end{bmatrix}}{[4b^2[\xi - A_1\cos(\Omega_1 t)]^2 + a^4]^{\frac{3}{2}}} \end{array} \right\} \tag{39}$$

and

$$\{b_2\} = \left\{ \begin{array}{c} \dfrac{\begin{bmatrix} A_1b^2a^2\Omega_1^2\cos(\Omega_1 t)[4b^2[\xi - A_1\cos(\Omega_1 t)]^2 + a^4]^{\frac{3}{2}} + \\ +8Rb^4a^2[\xi - A_1\cos(\Omega_1 t)][\dot{\xi} + A_1\Omega_1\sin(\Omega_1 t)](u_0\dot{u}_2 - u_2\dot{u}_0 - u_1\dot{u}_3 - u_3\dot{u}_1) \end{bmatrix}}{[4b^2[\xi - A_1\cos(\Omega_1 t)]^2 + a^4]^{\frac{3}{2}}} \\ \hdashline \dfrac{a^2b^2\begin{bmatrix} 4b^4\Omega_2A_2\cos(\Omega_2 t)]\xi - A_1\cos(\Omega_1 t)]^2 + \\ +8Rb(u_1\dot{u}_0 - u_0\dot{u}_1 - u_3\dot{u}_2 - u_2\dot{u}_3)[\xi - A_1\cos(\Omega_1 t)][\dot{\xi} + A_1\Omega_1\sin(\Omega_1 t)] + \\ 4Ra^2(u_0\dot{u}_3 - u_3\dot{u}_0 - u_1\dot{u}_2 - u_2\dot{u}_1)[\dot{\xi} + A_1\Omega_1\sin(\Omega_1 t)] + \\ +A_2\Omega_2a^4\cos(\Omega_2 t)\sqrt{4b^2[\xi - A_1\cos(\Omega_1 t)]^2 + a^4} \end{bmatrix}}{[4b^2[\xi - A_1\cos(\Omega_1 t)]^2 + a^4]^{\frac{3}{2}}} \\ \hdashline \dfrac{\begin{bmatrix} A_3\Omega_3^2\cos(\Omega_3 t)[4b^2[\xi - A_1\cos(\Omega_1 t)]^2 + a^4]^{\frac{3}{2}} + \\ +4Rb^2a^4[\dot{\xi} + A_1\Omega_1\sin(\Omega_1 t)](u_1\dot{u}_3 - u_3\dot{u}_1 - u_0\dot{u}_2 - u_2\dot{u}_0) \end{bmatrix}}{4b^2[\xi - A_1\cos(\Omega_1 t)]^2 + a^4]^{\frac{3}{2}}} \end{array} \right\} \tag{40}$$

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