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THE EXPLICIT GIBBS-APPELL EQUATION AND GENERALIZED INVERSE FORMS

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Abstract. This paper develops an extended form of the Gibbs-Appell equation and shows that it is equivalent to the generalized inverse equation of motion. Both equations are shown to follow from Gauss's principle. An example to highlight the two equivalent, though different, equations of motion is provided. Conceptual differences between the equations, and differences in their practical application to physical situations are discussed.

We also present in this paper a more general explicit generalized inverse equation of motion than has been hereto obtained. It is shown that many different *forms* of the generalized inverse equation of motion exist, all of which nonetheless are equivalent and *uniquely* determine the accelerations of a constrained mechanical system. The generalized inverse equation of motion retains its structure in any coordinate system.

Introduction. The equations of motion, which are today commonly referred to as the Gibbs-Appell equations, were discovered independently by Gibbs (1879) and Appell (1899). The ability of the equations to describe the evolution of both holonomically and nonholonomically constrained systems without the use of Lagrange multipliers was considered to be a major leap in the development of analytical mechanics. These equations are considered by many to represent the simplest and most comprehensive form of the equations of motion so far discovered (Pars, 1972).

Both Gibbs and Appell used the principle of virtual work to arrive at these equations. Most treatises on analytical dynamics, such as Whittaker (1904) and Pars (1972), derive these equations in the same vein. In this paper we show that the Gibbs-Appell equations can be derived in a more straightforward and explicit manner from Gauss's principle

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(Gauss, 1829). Besides its simplicity, there is an additional advantage of some consequence that follows from such a viewpoint: the equations can be extended to systems in which the nonholonomic constraints depend nonlinearly on the components of the velocities, thereby expanding the compass of their normal applicability, and the equations can be extended, rather simply, to include constraint equations which may not be independent of each other. We shall refer to these equations as the explicit extended Gibbs-Appell equations of motion. The unifying approach provided by Gauss's principle enables us to show that these extended Gibbs-Appell equations and the generalized inverse equations of motion (Udwadia and Kalaba, 1992), though different in form and appearance, are in strict equivalence.

In this paper we also present the various generalized inverse forms of the equation of motion for constrained systems thereby generalizing some previous results. We provide the most general form of the generalized inverse equation of motion for constrained mechanical systems. We show that the Moore-Penrose inverse hereto used in the equation (Udwadia and Kalaba, 1992) is overly restrictive. What is needed is simply *any* $\{1,4\}$-generalized inverse[1] (Udwadia and Kalaba, 1996).

Consider a system of n particles of masses $m_i > 0$, $i = 1, 2, \ldots, n$, in an inertial Cartesian coordinate space. The masses of these n particles will be assumed constant and known. In most of this paper, for expository purposes, we shall restrict ourselves to Cartesian coordinates. We shall see that all the underlying ideas can be illustrated without the unnecessary complications created by generalized coordinates. At the end, we will show how the results can be easily transcribed to apply to generalized coordinates. Let the $3n$ Cartesian coordinates of the n particles be described by the column vector $x = [x_1 x_2 \cdots x_{3n}]^{\mathrm{T}}$.

Rather than consider k given independent nonholonomic constraints, as is usual in deriving the Gibbs-Appell equation (see, for example, Pars (1972), Whittaker (1904), Neimark and Fufaev (1972)) of the form

$$\sum_{j=1}^{3n} c_{ij}(x,t)\dot{x}_j = b_i(x,t), \quad i = 1, 2, \ldots, k, \tag{1}$$

where each equation represents a linear constraint on the components of the velocities $\dot{x}_j$ of the system, we shall allow more general constraints of the form

$$\varphi_i(x, \dot{x}, t) = 0, \quad i = 1, 2, \ldots, k, \tag{2}$$

where each function φ_j is considered to be at least C^3 in its arguments. Furthermore, we do not restrict the k equations to be independent. Differentiating each constraint equation in the set (2) with respect to time once (twice, if the constraint is holonomic and φ_j does not contain $\dot{x}$), we obtain the set of equations

$$A(x, \dot{x}, t)\ddot{x}(t) = b(x, \dot{x}, t) \tag{3}$$

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[1] Given a real matrix X, its Moore-Penrose generalized inverse is the matrix Y that satisfies the conditions: (1) $XYX = X$, (2) $YXY = Y$, (3) XY is symmetric, and (4) YX is symmetric. Any matrix U that satisfies the first and the fourth of these four conditions is referred to as a $\{1,4\}$-generalized inverse of X; similarly any matrix V that satisfies the first, the second, and the fourth of these conditions is called a $\{1,2,4\}$-inverse of X, etc.

where the matrix A is k by $3n$. When we say that a mechanical system is subjected to the constraint (3), we shall mean that the elements of the matrix A and the components of the vector b are known functions of their arguments. We note that the form of the constraint provided by (3) is more general than that usually used in the development of the equations of motion for constrained systems as may be found in Neimark and Fufaev (1972), Whittaker (1904), and Pars (1972).

Let us say that at some time t, we know the position x and the velocity $\dot{x}$ of the constrained system. We now conceive of this n-particle constrained mechanical system in two steps.

We start with an n-particle *unconstrained* system, subjected to a known impressed force $F(x, \dot{x}, t) = [F_1 F_2 \cdots F_{3n}]^{\mathrm{T}}$. By known force we mean that the functional dependence of each of the $3n$ components of the impressed force F on $x, \dot{x}$, and t is explicitly known. By unconstrained we mean that the number of coordinates describing the system equals the number of degrees of freedom of the system. It is then a simple matter to obtain the acceleration $a(t) = [a_1 a_2 \cdots a_{3n}]^{\mathrm{T}}$ of the unconstrained system at time t by writing down Newton's law for this unconstrained system as

$$Ma(t) = F(x, \dot{x}, t), \tag{4}$$

where the $3n$ by $3n$ matrix M is a diagonal matrix in which the masses m_i of the n particles appear in sets of threes along the diagonal. Since x and $\dot{x}$ are assumed known at time t, and the functional dependence of F on $x, \dot{x}$, and t is assumed known, the right-hand side of Eq. (4) can be explicitly determined, and hence,

$$a(t) = M^{-1} F(x, \dot{x}, t). \tag{5}$$

We next impose the constraint equation (3) on this unconstrained system. Thus the constrained system (at time t) may be thought of as being completely specified by x and $\dot{x}$ at time t, along with the four quantities M, F, A, and b (also evaluated at time t).

We inquire how the acceleration $\ddot{x}(t)$ of the resulting constrained system differs (at time t) from that of the known acceleration $a(t)$ of the unconstrained system. Our aim is to determine the acceleration $\ddot{x}$ at time t *explicitly* in terms of the four quantities M, F, A, and b that describe the constrained mechanical system at that time.

We pursue this line of thinking by invoking Gauss's principle (Gauss, 1829) which states that the acceleration $\ddot{x}(t)$ of the constrained system at each instant of time t is such as to minimize, at time t, the Gaussian

$$G(\ddot{x}) = \tfrac{1}{2}(\ddot{x} - a)^{\mathrm{T}} M (\ddot{x} - a) \tag{6}$$

over all possible $3n$-vectors that satisfy the constraint set (3) at time t. Note that as with the vector F at time t, the elements of the vector b and those of the matrix A are known functions of $x, \dot{x}$, and t; hence they are known at time t. In what follows, for the sake of brevity, we drop the arguments of the various vectors and matrices, unless their presence becomes conceptually helpful. We shall now develop the explicit extended Gibbs-Appell equations starting from this point.

The explicit extended Gibbs-Appell equation. The basic idea that we shall follow is to convert the constrained minimization problem of Gauss into an unconstrained minimization problem by eliminating the dependent components of the acceleration vector $\ddot{x}(t)$.

Since the constrained acceleration vector $\ddot{x}(t)$ at time t satisfies the equation set (3), the components of this acceleration vector are obviously not all linearly independent. Let us assume that the rank of the matrix A at time t is $r \leq k$. We shall assume that the first r columns of the matrix A are independent; if not, we can always re-label the components of the vector $\ddot{x}(t)$ appropriately so that this occurs. We then partition the matrix A and the vector $\ddot{x}(t)$ appropriately, so as to express the constraint equation (3) at time t as

$$A\ddot{x} = [A_{\mathrm{e}} \quad A_{\mathrm{I}}] \begin{bmatrix} \ddot{x}_{\mathrm{e}} \\ \ddot{x}_{\mathrm{I}} \end{bmatrix} = b \tag{7}$$

where the matrices A_{e} and A_{I} are k by r and k by $(3n-r)$ respectively. The subvectors $\ddot{x}_{\mathrm{e}}$ and $\ddot{x}_{\mathrm{I}}$ have r and $(3n-r)$ components respectively. The subscript "e" refers to the subvector that we shall eliminate, as we shall see below, and the subscript "I" refers to the subvector whose components may be taken to be independent.

Equation (7) can be solved for the vector $\ddot{x}_{\mathrm{e}}$ to yield

$$\ddot{x}_{\mathrm{e}} = A_{\mathrm{e}}^{+}(b - A_{\mathrm{I}}\ddot{x}_{\mathrm{I}}), \tag{8}$$

where $A_{\mathrm{e}}^{+} = (A_{\mathrm{e}}^{\mathrm{T}} A_{\mathrm{e}})^{-1} A_{\mathrm{e}}^{\mathrm{T}}$, the superscript "+" denoting the Moore-Penrose (MP) inverse of the matrix A_{e}. Note that the subvector $\ddot{x}_{\mathrm{I}}$ contains the components of $\ddot{x}$ that are independent. We may likewise partition the matrix $M = \mathrm{diag}[M_{\mathrm{ee}}, M_{\mathrm{II}}]$, where M_{ee} and M_{II} are each r by r and $(3n-r)$ by $(3n-r)$ diagonal matrices respectively, and the vector $a = [\, a_{\mathrm{e}}^{\mathrm{T}}, a_{\mathrm{I}}^{\mathrm{T}} \,]^{\mathrm{T}}$. The Gaussian G in expression (6) can now be written as

$$G(\ddot{x}) = \tfrac{1}{2}(\ddot{x}_{\mathrm{e}}^{\mathrm{T}} M_{\mathrm{ee}} \ddot{x}_{\mathrm{e}} + \ddot{x}_{\mathrm{I}}^{\mathrm{T}} M_{\mathrm{II}} \ddot{x}_{\mathrm{I}}) - a_{\mathrm{e}}^{\mathrm{T}} M_{\mathrm{ee}} \ddot{x}_{\mathrm{e}} - a_{\mathrm{I}}^{\mathrm{T}} M_{\mathrm{II}} \ddot{x}_{\mathrm{I}} + \tfrac{1}{2} a^{\mathrm{T}} M a. \tag{9}$$

For convenience, we denote the first term on the right-hand side with the brackets—the so-called "kinetic energy of accelerations"—by the Gibbs function $S(\ddot{x})$, because it is a function of both the subvectors $\ddot{x}_{\mathrm{I}}$ and $\ddot{x}_{\mathrm{e}}$. Thus, Eq. (9) can be alternately expressed as

$$G(\ddot{x}) = \tfrac{1}{2} S(\ddot{x}) - a_{\mathrm{e}}^{\mathrm{T}} M_{\mathrm{ee}} \ddot{x}_{\mathrm{e}} - a_{\mathrm{I}}^{\mathrm{T}} M_{\mathrm{II}} \ddot{x}_{\mathrm{I}} + \tfrac{1}{2} a^{\mathrm{T}} M a. \tag{10}$$

Using Eq. (8), the subvector $\ddot{x}_{\mathrm{e}}$ may be eliminated from Eq. (9) to yield

$$\begin{aligned} G(\ddot{x}_{\mathrm{I}}) = {} & \tfrac{1}{2}((b - A_{\mathrm{I}}\ddot{x}_{\mathrm{I}})^{\mathrm{T}} A_{\mathrm{e}}^{+\mathrm{T}} M_{\mathrm{ee}} A_{\mathrm{e}}^{+}(b - A_{\mathrm{I}}\ddot{x}_{\mathrm{I}}) + \ddot{x}_{\mathrm{I}}^{\mathrm{T}} M_{\mathrm{II}} \ddot{x}_{\mathrm{I}}) \\ & - a_{\mathrm{e}}^{\mathrm{T}} M_{\mathrm{ee}} A_{\mathrm{e}}^{+}(b - A_{\mathrm{I}}\ddot{x}_{\mathrm{I}}) - a_{\mathrm{I}}^{\mathrm{T}} M_{\mathrm{II}} \ddot{x}_{\mathrm{I}} + \tfrac{1}{2} a^{\mathrm{T}} M a. \end{aligned} \tag{11}$$

We again recognize the first member on the right-hand side with brackets as the "kinetic energy of accelerations", except that now it is expressed in terms of only the vector of independent accelerations, $\ddot{x}_{\mathrm{I}}$. We shall denote this quantity by $\mathcal{S}(\ddot{x}_{\mathrm{I}})$, the "script S" indicating that it is the quantity $S = \frac{1}{2}\ddot{x}^{\mathrm{T}} M \ddot{x}$ expressed in terms of the independent vector $\ddot{x}_{\mathrm{I}}$. Equation (11) now becomes

$$G(\ddot{x}_{\mathrm{I}}) = \mathcal{S}(\ddot{x}_{\mathrm{I}}) - a_{\mathrm{e}}^{\mathrm{T}} M_{\mathrm{ee}} A_{\mathrm{e}}^{+}(b - A_{\mathrm{I}}\ddot{x}_{\mathrm{I}}) - a_{\mathrm{I}}^{\mathrm{T}} M_{\mathrm{II}} \ddot{x}_{\mathrm{I}} + \tfrac{1}{2} a^{\mathrm{T}} M a. \tag{12}$$

We have thus converted the constrained minimization problem stated in Gauss's principle to an unconstrained minimization problem. A necessary condition for the extremum of

(11) with respect to the independent acceleration vector $\ddot{x}_{\rm I}$ is $\frac{\partial G}{\partial \ddot{x}_{\rm I}} = 0$. This yields

$$(M_{\rm II} + R^{\rm T} M_{\rm ee} R)\ddot{x}_{\rm I} - R^{\rm T} M_{\rm ee} A_{\rm e}^{+} b = M_{\rm II} a_{\rm I} - R^{\rm T} M_{\rm ee} a_{\rm e}, \tag{13}$$

where we have denoted $R = A_{\rm e}^{+} A_{\rm I}$. Noting our definition of $\mathcal{S}(\ddot{x}_{\rm I})$, Eq. (13) can also be stated as

$$\frac{\partial S}{\partial \ddot{x}_{\rm I}} = F_{\rm I} - R^{\rm T} F_{\rm e} := P \tag{14}$$

where we have partitioned the known, impressed force vector $F = [\,F_{\rm e}^{\rm T}\; F_{\rm I}^{\rm T}\,]^{\rm T}$ of Eq. (4) into two subvectors $F_{\rm I} = M_{\rm II} a_{\rm I}$ and $F_{\rm e} = M_{\rm ee} a_{\rm e}$. The vectors $F_{\rm e}$ and $F_{\rm I}$ have r and $(3n - r)$ components respectively. Equation (14) is the core that forms the Gibbs-Appell equation of motion for the constrained mechanical system. It results from enforcing the *necessary condition* for the expression in (12) to have an *extremum.* We have shown that Eq. (14) is true when the constraints are of the general form given by Eqs. (2) or (3); furthermore, these constraints need *not* be independent.

To understand more fully the right-hand side of Eq. (14) (which we have defined as P), we use the extended D'Alembert principle (Udwadia and Kalaba, 1995) which says that a virtual displacement vector ν compatible with the constraints (3) is any vector (Udwadia and Kalaba, 1995) that satisfies the relation

$$A\nu = [A_{\rm e} \quad A_{\rm I}] \begin{bmatrix} \nu_{\rm e} \\ \nu_{\rm I} \end{bmatrix} = 0, \tag{15}$$

where we have, as before, partitioned the matrix A; the subvectors $\nu_{\rm e}$ and $\nu_{\rm I}$ have r and $(3n - r)$ components, respectively. This yields the relation

$$\nu_{\rm e} = -A_{\rm e}^{+} A_{\rm I} \nu_{\rm I} = -R\nu_{\rm I}. \tag{16}$$

The virtual work done by the given impressed force then becomes

$$\nu_{\rm I}^{\rm T} F_{\rm I} + \nu_{\rm e}^{\rm T} F_{\rm e} = \nu_{\rm I}^{\rm T}(F_{\rm I} - R^{\rm T} F_{\rm e}) = \nu_{\rm I}^{\rm T} P. \tag{17}$$

We note that the term in brackets in the above equation also shows up on the right-hand side of Eq. (14). Hence this right-hand side is obtained by determining the work done, $\nu_{\rm I}^{\rm T} P$, by the impressed forces under the (independent) virtual displacements $\nu_{\rm I}$ that are compatible with the constraints.

Yet Eq. (14) cannot, in general, stand alone, for though the expression G in Eq. (11) does not contain any components of the vector $\ddot{x}_{\rm e}$ (since this vector was eliminated using Eq. (8)), it does contain, in general, components of the vectors $x_{\rm e}$ and $\dot{x}_{\rm e}$. To complete the system of *differential equations* one would therefore then need to append the equation of constraint (3), so that the complete set of equations would be formed by Eq. (14) and Eq. (3).[2] Using Eqs. (13) and (3), this becomes

$$\begin{bmatrix} A_{\rm e} & A_{\rm I} \\ 0 & (M_{\rm II} + R^{\rm T} M_{\rm ee} R) \end{bmatrix} \begin{bmatrix} \ddot{x}_{\rm e} \\ \ddot{x}_{\rm I} \end{bmatrix} = \begin{bmatrix} b \\ F_{\rm I} - R^{\rm T} F_{\rm e} + R^{\rm T} M_{\rm ee} A_{\rm e}^{+} b \end{bmatrix}, \tag{18}$$

where the matrix $R = A_{\rm e}^{+} A_{\rm I}$. Equation (18) may then be thought of as the explicit extended Gibbs-Appell equation in Cartesian coordinates, applicable to constraints (i)

[2]Actually, it would suffice to include any r independent rows of the equation set (3) that will make the system of equations given in (18) complete.

that are nonholonomic and nonlinear in the velocity components, and (ii) that are not necessarily independent.

The generalized inverse forms of the equation of motion. An alternative approach to minimizing the Gaussian G (at time t) subject to the constraint (3) at time t is to directly solve the constrained minimization problem without first converting it to an unconstrained minimization problem. To do this, it would be more convenient to write the Gaussian in the form

$$G_s(\ddot{x}_s) = \tfrac{1}{2}(\ddot{x}_s - \ddot{a}_s)^{\mathrm{T}}(\ddot{x}_s - \ddot{a}_s) \tag{19}$$

where the subscript s denotes the scaled quantities $\ddot{x}_s = M^{1/2}\ddot{x}$ and $a_s = M^{1/2}a$. The constraint equation (3) at time t can then be expressed as

$$C\ddot{x}_s = b \tag{20}$$

where the matrix $C = AM^{-1/2}$. Thus, Gauss's principle states that the scaled acceleration of the constrained system $\ddot{x}_s$ at time t is obtained by minimizing G_s with respect to all possible (scaled) acceleration vectors $\ddot{x}_s$ that satisfy the constraint (20) at time t.

We determine the solution to this constrained minimization problem by transforming Eqs. (19) and (20) by setting $y = (\ddot{x}_s - a_s)$, so that we now simply require to find that vector y that minimizes

$$G_s = \tfrac{1}{2}y^{\mathrm{T}}y \tag{21}$$

subject to the constraint

$$Cy = b - Ca_s. \tag{22}$$

But this is simply the problem of finding the minimum "length" solution y that satisfies the linear equation (22). The solution to this problem, we know, is uniquely given by (see, for example, Rao, 1973)

$$y = C^{\{1,4\}}(b - Ca_s) \tag{23}$$

where the matrix $C^{\{1,4\}}$ is *any* $\{1,4\}$-generalized inverse of the matrix C. By this we mean any matrix C that satisfies the first and the fourth Moore-Penrose (MP) conditions. The first MP condition is $CC^{\{1,4\}}C = C$, and the fourth is $C^{\{1,4\}}C = [C^{\{1,4\}}C]^{\mathrm{T}}$.

Noting the definition of the vectors y, a_s and the matrix C, we then have the following explicit equation of motion of the constrained system at time t:

$$\ddot{x} = a + M^{-1/2}(AM^{-1/2})^{\{1,4\}}(b - Aa). \tag{24}$$

Premultiplying (24) by the matrix M and using Eq. (4), we obtain

$$M\ddot{x} = F + M^{1/2}(AM^{-1/2})^{\{1,4\}}(b - Aa), \tag{25}$$

where the superscript $\{1,4\}$ again denotes *any* $\{1,4\}$-inverse of the matrix $AM^{-1/2}$. Equations (24) and (25) appear to be the most general and direct form of the equations of motion for constrained systems.

They may be further transformed by noting that for any matrix X, *one* choice of $X^{\{1,4\}}$ is always $X^{\mathrm{T}}(XX^{\mathrm{T}})^{\{1\}}$ (Rao, 1973). Equation (25) then yields

$$M\ddot{x} = F + A^{\mathrm{T}}(AM^{-1}A^{\mathrm{T}})^{\{1\}}(b - Aa), \tag{26}$$

where the superscript $\{1\}$ denotes *any* $\{1\}$-inverse of the matrix $AM^{-1}A^{\mathrm{T}}$. The second members on the right-hand sides of Eqs. (25) and (26) explicitly provide the force of constraint engendered by the imposition of the constraint (3).

We note that neither the $\{1,4\}$-inverse nor the $\{1\}$-inverse of a given matrix X is unique. In fact, for a given matrix X, the *infinite* number of different matrices qualify in satisfying either the first MP condition, or both the first and fourth MP conditions for X (Rao and Mitra, 1972). We are therefore at liberty to use *any one* of these matrices in Eqs. (24)–(26). With *each choice* of a $\{1,4\}$-inverse in Eqs. (24) and (25), or a $\{1\}$-inverse in Eq. (26), we will arrive at a *different form* of the explicit generalized inverse equation of motion of the constrained system. Yet all these different forms are equivalent. For though the $\{1,4\}$-inverse and the $\{1\}$-inverse of the matrices in Eqs. (24)–(26) are not unique, the second members in the right-hand sides of these equations are always uniquely determined (Rao, 1973). Hence the acceleration at time t of the constrained system is indeed uniquely determined.

Equivalence of the extended Gibbs-Appell equation and the generalized inverse forms. It should be observed that both the extended Gibbs-Appell equation and the generalized inverse equation have been deduced from Gauss's principle. This principle requires that the acceleration of the constrained system at each instant of time t be such as to minimize the Gaussian (at that time) while satisfying the constraints (at that time). However, the approaches used to arrive at the two sets of equations are quite different.

To obtain the generalized inverse forms of the equation of motion, we resort to a direct constrained minimization of the Gaussian, which leads to a unique vector $\ddot{x}(t)$ that *minimizes* the Gaussian. The extended Gibbs-Appell equation, however, arises from enforcing the necessary condition for the *extremization* of the unconstrained Gaussian. This unconstrained Gaussian is obtained after the dependent acceleration vector $\ddot{x}_{\mathrm{e}}$ is eliminated from G in favor of the vector $\ddot{x}_{\mathrm{I}}$. To show the equivalence of the two sets of equations, we must then show that the value of $\ddot{x}_{\mathrm{I}}$ given by Eq. (13) does indeed minimize the right-hand side of the expression in (12), and further, that it is unique. We next take up these issues.

(i) *Uniqueness*: Assume that there are two acceleration vectors $\ddot{x}_{\mathrm{I}}$ and $\tilde{\ddot{x}}_{\mathrm{I}}$ both of which satisfy Eq. (13) at time t. Then their difference satisfies the equation (note that $x(t)$ and $\dot{x}(t)$ are assumed to be known)

$$(M_{\mathrm{II}} + R^{\mathrm{T}}M_{\mathrm{ee}}R)(\ddot{x}_{\mathrm{I}} - \tilde{\ddot{x}}_{\mathrm{I}}) = 0. \tag{27}$$

But the matrix M_{II} is positive definite and the symmetric matrix $R^{\mathrm{T}}M_{\mathrm{ee}}R$ is positive semidefinite. Hence their sum is a positive definite matrix. Equation (27) then implies that $\ddot{x}_{\mathrm{I}} = \tilde{\ddot{x}}_{\mathrm{I}}$.

(ii) *Minimum*: The Hessian matrix of G given by the expression in (11) (with respect to the components of the vector $\ddot{x}_I$) is again the matrix $(M_{II} + R^T M_{ee} R)$ which we have shown is positive definite, and hence the extremum is a minimum.

(iii) *Equivalence*: Thus the explicit extended Gibbs-Appell equation and the explicit generalized inverse forms of the equation of motion both solve the same constrained minimization problem which is embodied in Gauss's principle, and hence are equivalent.

An example. To illustrate the two sets of equations, we consider a variant of a problem first considered by Appell (1911). A particle of unit mass moves in three-dimensional Cartesian space and is subjected to the force of gravity acting in the downward Z-direction (Z-axis is taken pointing upwards). We want to determine the equation of motion of the particle when it is constrained by the relation

$$\dot{x}^2 + \dot{y}^2 - \dot{z}^2 = 2\alpha h(x, y, z, t), \tag{28}$$

where $h(x, y, t)$ is a given, known smooth function of its arguments and α is a given constant. We assume that the initial conditions, say at time $t = t_0$, are given and are such that they are consistent with the constraint (28). (Appell's paper is entirely devoted to this problem. In it, he takes α to be zero.)

The matrix $M = I_3$, and the unconstrained acceleration a of the particle is given by

$$a = F = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}. \tag{29}$$

Differentiating with respect to time, the constraint equation may be expressed as

$$\begin{bmatrix} \dot{x} & \dot{y} & -\dot{z} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \alpha(h_x \dot{x} + h_y \dot{y} + h_z \dot{z} + h_t), \tag{30}$$

so that the matrix $A = [\, \dot{x}\ \dot{y}\ -\dot{z} \,]$, and $b = \alpha(h_x \dot{x} + h_y \dot{y} + h_z \dot{z} + h_t)$. Hence, the Moore-Penrose inverse $A^+ = \frac{1}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)} [\, \dot{x}\ \dot{y} - \dot{z} \,]^T$, and the generalized inverse equation of motion of the constrained particle can then be simply and directly written down as (see Eq. (24))

$$\ddot{x} = a + A^+(b - Aa), \tag{31}$$

which becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} + \frac{\alpha(h_x \dot{x} + h_y \dot{y} + h_z \dot{z} + h_t) - \dot{z} g}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)} \begin{bmatrix} \dot{x} \\ \dot{y} \\ -\dot{z} \end{bmatrix}. \tag{32}$$

Note that Eq. (24) allows any $\{1,4\}$-inverse to be used; in particular, we have used the MP inverse of A, which is the $\{1,2,3,4\}$-inverse of A and hence qualifies as a $\{1,4\}$-inverse as well. Also, the constraint force engendered by the presence of the constraint (28) is *directly* given by the second member on the right-hand side in Eq. (32).

We now compare this procedure with the explicit Gibbs-Appell equation. Here we first need to determine the rank of the matrix A, which in this case is unity (we do not consider the case when $\dot{x} = \dot{y} = \dot{z} = 0$). We next choose $\ddot{x}_e = \ddot{x}$, and $\ddot{x}_I = [\, \ddot{y}\ \ddot{z} \,]^T$. Using the constraint equation (30) we see that $A_e = \dot{x}$ ($\dot{x} \neq 0$), and $A_I = [\, \dot{y}\ -\dot{z} \,]$. Also, from

Eq. (29), the vector $F_e = 0$ and $F_I = [0\ -g]^T$. Noting that $A_e^+ = 1/\dot{x}$, Eq. (18) then becomes

$$\begin{bmatrix} \dot{x} & \dot{y} & -\dot{z} \\ 0 & 1+\frac{\dot{y}^2}{\dot{x}^2} & -\frac{\dot{y}\dot{z}}{\dot{x}^2} \\ 0 & -\frac{\dot{y}\dot{z}}{\dot{x}^2} & 1+\frac{\dot{z}^2}{\dot{x}^2} \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \alpha(h_x\dot{x}+h_y\dot{y}+h_z\dot{z}+h_t) \\ \alpha(h_x\dot{x}+h_y\dot{y}+h_z\dot{z}+h_t)\dot{y}/\dot{x}^2 \\ -g-\alpha(h_x\dot{x}+h_y\dot{y}+h_z\dot{z}+h_t)\dot{z}/\dot{x}^2 \end{bmatrix}. \tag{33}$$

We note that, in practice, when the number of constraints is large, the determination of the rank of A and a proper breakdown of the vector $\ddot{x}$ into its subvectors $\ddot{x}_e$ and $\ddot{x}_I$ may not be a trivial matter and will usually require a re-labeling of the coordinates in the problem. Also, the determination of the constraint force created by the presence of the constraint (28) is indeed not easily apparent from Eq. (33). It should be emphasized, as seen in this example, that though Eqs. (32) and (33) are *equivalent* they are *not the same.*

The equations in terms of generalized coordinates. The general procedure to obtain the extended Gibbs-Appell equation and the generalized inverse forms of the equations of motion may be applied when use is made of generalized coordinates rather than Cartesian coordinates. Let us say that we have g generalized coordinates to describe the unconstrained motion of the system and that we again have k constraints of the form given by Eq. (2). Rather than burden the reader by going through the entire procedures in detail, we focus on where and how the differences occur, and then proceed directly to provide the final results. We begin with the generalized inverse forms of the equation of motion.

Were x a generalized coordinate g-vector, Eq. (4) would now be obtained using, in general, Lagrange's equations of motion. Also, the g by g matrix M would no longer be a diagonal matrix whose elements are constants but would, in general, be a positive definite (symmetric) matrix whose elements would be functions of x and t (see, for example, Pars, 1972). Since Gauss's principle is valid in generalized coordinates (Udwadia and Kalaba, 1994), the rest of the steps in our previous derivation will follow mutatis mutandis and the same equations of motion (24)–(26) will again be obtained, except that now the unconstrained acceleration g-vector a at time t is defined using the relation $a(t) = M^{-1}(x,t)F(x,\dot{x},t)$. Thus the explicit generalized inverse forms of the equation of motion retain their structure in both Cartesian and generalized coordinates.

Let us now turn to the extended Gibbs-Appell equation. The same salient differences in the matrix M and Eq. (4) indicated in the last paragraph would ensue when using the generalized coordinate g-vector $\ddot{x}$. Again, we need to determine the rank r of the matrix A, which is now a k by g matrix, in Eq. (3). Further, we need to partition the vector $\ddot{x} = [\ddot{x}_e^T\ \ddot{x}_I^T]^T$ and correspondingly the matrix A as in Eq. (7). The positive definite matrix $M(x,t)$ would then need to be partitioned as

$$M(x,t) = \begin{bmatrix} M_{ee} & M_{eI} \\ M_{Ie} & M_{II} \end{bmatrix} \tag{34}$$

where $M_{ee}(x,t)$ is an r by r matrix. The Gaussian G of expression (6) may now once again be expressed in terms of only the independent acceleration vector $\ddot{x}_I$ by using Eq.

(8). The condition $\frac{\partial G}{\partial \ddot{x}_{\rm I}} = 0$ for the extremum of $G(\ddot{x}_{\rm I})$ then yields

$$(M_{\rm II} + R^{\rm T} M_{\rm ee} R - M_{\rm Ie} R - R^{\rm T} M_{\rm Ie}^{\rm T})\ddot{x}_{\rm I} - (R^{\rm T} M_{\rm ee} - M_{\rm Ie})A_{\rm e}^{+} b = F_{\rm I} - R^{\rm T} F_{\rm e}. \tag{35}$$

The left-hand side may be written as before as $\frac{\partial \mathcal{S}}{\partial \ddot{x}_{\rm I}}$, so that Eq. (35) becomes

$$\frac{\partial \mathcal{S}}{\partial \ddot{x}_{\rm I}} = F_{\rm I} - R^{\rm T} F_{\rm e} := P. \tag{36}$$

Here, $\mathcal{S}(\ddot{x}_{\rm I})$ is again the quantity $\frac{1}{2}\ddot{x}^{\rm T} M \ddot{x}$ expressed in terms of the independent acceleration vector $\ddot{x}_{\rm I}$ through the use of Eq. (8). Equation (36) along with (3) then yields the Gibbs-Appell equation.

The explicit Gibbs-Appell equation in generalized coordinates is obtained by appending Eq. (3) to Eq. (35).[3] We thus obtain

$$\begin{bmatrix} A_{\rm e} & A_{\rm I} \\ 0 & (M_{\rm II} + R^{\rm T} M_{\rm ee} R - M_{\rm Ie} R - R^{\rm T} M_{\rm Ie}^{\rm T}) \end{bmatrix} \begin{bmatrix} \ddot{x}_{\rm e} \\ \ddot{x}_{\rm I} \end{bmatrix} = \begin{bmatrix} b \\ F_{\rm I} - R^{\rm T} F_{\rm e} + (R^{\rm T} M_{\rm ee} - M_{\rm Ie}) A_{\rm e}^{+} b \end{bmatrix}, \tag{37}$$

where again $R = A_{\rm e}^{+} A_{\rm I}$. Comparing Eqs. (18) and (37) we thus find that, in general, the explicit Gibbs-Appell equation in generalized coordinates is *different* in form from that obtained when using Cartesian coordinates. We observe, though, that when the matrix $M_{\rm Ie} = 0$, Eq. (37) reduces to Eq. (18).

Conclusions and discussion. We summarize our results as below.

(1) Part of the reason, we believe, that the Gibbs-Appell equation is not commonly used to describe the constrained motion of mechanical systems is perhaps because it is not stated in an explicit form as is, say, Lagrange's equation of the first kind. We have in this paper obtained the explicit extended Gibbs-Appell equation of motion applicable to mechanical systems (i) where the constraints are nonlinearly dependent on the generalized velocities, and/or (ii) where the constraints may not necessarily be independent. Despite an extensive literature search, the authors have not come across the explicit extended Gibbs-Appell equation presented in the simple form derived in this paper. By explicit, we mean the equation of motion is obtained in terms of the four quantities M, F, A, and b that describe the constrained system. We hope that this equation will find wider applicability, now that it is presented in an explicit form.

(2) We have obtained general forms of the explicit generalized inverse equation of motion and shown that there are *many equivalent forms* of this equation (actually an infinite number!). Thus previous results, which relied solely on the use of the MP generalized inverse, have been extended. We have shown here that the equation of motion hitherto obtained (Udwadia and Kalaba, 1992) is correct even when using a far less restrictive generalized inverse, namely the $\{1,4\}$-inverse instead of the MP-inverse (or the $\{1,2,3,4\}$-inverse). One may suspect some computational advantages to accrue from this relaxation of the type of generalized inverse used in the more refined description of constrained motion obtained in this paper.

[3]Here again, it would suffice to include any r independent rows of the equation set (3) that will make the system of equations given in (37) complete.

(3) We have shown that both the extended Gibbs-Appell equation of motion and the generalized inverse forms can be derived from Gauss's principle. Both equations yield the solution to the same constrained minimization problem stated by Gauss. They are thus shown to be equivalent.

(4) *Though equivalent, it is shown that the two sets of equations are not the same.* Nor is the approach to obtaining them in actual practice identical for a given physical mechanical system. Even when using the explicit Gibbs-Appell equation provided in this paper, the rank of the matrix A is required to be first determined. It should be noted that this may not be a trivial task when the number of constraint equations exceeds even, say, ten. From a computational standpoint also, the determination of the rank of a matrix is prone to numerical problems. Furthermore, before the equation can be used the acceleration components need to be categorized into the subvectors $\ddot{x}_e$ and $\ddot{x}_I$, and the matrix A needs to be partitioned appropriately. In actual practice, this would usually entail a re-labeling of the coordinates, an inconvenience at the very least. The approach relies on converting a constrained minimization problem to an unconstrained minimization problem through the elimination of the dependent acceleration components. This underlying idea conceptually leads to an "unequal" treatment of the coordinates through the selection of certain "preferred" acceleration components ($\ddot{x}_I$) in terms of which the minimization is then done, and the elimination of other acceleration components ($\ddot{x}_e$). And yet, after this minimization is carried out (as, for example, in (13)), in general one is required to append the equation of constraint (3), which reintroduces both the components $\ddot{x}_e$ and $\ddot{x}_I$, to complete the system of equations (as, for example, in (18)).

It should be noted that the explicit Gibbs-Appell equation represents only the *necessary condition* for the Gaussian G to achieve an *extremum*.

(5) On the other hand, the explicit generalized inverse forms of the equation of motion stem from the *direct* solution of the constrained minimization problem of Gauss. From a conceptual viewpoint, no elimination is contemplated. These forms are therefore obtained directly without the need to identify any independent or dependent acceleration components. No "preferred" set of acceleration component is therefore used, nor required. No re-labeling of coordinates is involved, and the rank of the matrix A is not required to be found. Furthermore, one directly obtains the constraint force vector brought into play by the presence of the constraint equation (3). This is *explicitly* provided by the second members on the right-hand sides of Eqs. (25) and (26). As seen in the example, the explicit determination of the constraint force vector is far less apparent from the extended Gibbs-Appell equation. This appears to be the price that must be paid for "preferring" certain acceleration components, and eliminating others—a key feature of the Gibbs-Appell approach.

(6) The generalized inverse forms of the equation of motion retain their structure in any coordinate system. The structure of the explicit Gibbs-Appell equation changes when using generalized coordinates instead of Cartesian coordinates.

REFERENCES

[1] P. Appell, Traité de Mécanique Rationelle, Third edition, Paris, 1911

[2] P. Appell, *Exemple de Mouvement d'un Point Assujeti a une Liason Exprimée par une Relation Non Lineare Entre les Composantes de la Vitesse*, Comptes Rendus, 1911, pp. 48–50

[3] C. F. Gauss, *Über ein neues allgemeines Grundgesetz der Mechanik*, Journal für die Reine und Angewandte Mathematik **4**, 232–235 (1829)

[4] W. J. Gibbs, *On the fundamental formulae of dynamics*, Amer. Jour. Math. **2**, 49–64 (1879)

[5] R. E. Kalaba and F. E. Udwadia, *Lagrangian mechanics, Gauss's principle, quadratic programming, and generalized inverses: New equations for nonholonomically constrained discrete mechanical systems*, Quart. Appl. Math. **52**, 229–241 (1994)

[6] Y. Neimark and N. Fufaev, *Dynamics of Nonholonomic Systems*, Amer. Math. Soc. Translations, vol. 33, 1972

[7] L. Pars, *A Treatise on Analytical Dynamics*, Ox Bow Press, Connecticut, Second Printing, 1972

[8] R. Penrose, *A generalized inverse of matrices*, Proc. Camb. Philos. Soc. **51**, 406–313 (1955)

[9] C. R. Rao, *Linear Statistical Inference and Applications*, Wiley, 1973

[10] C. Rao and S. Mitra, *Generalized Inverse of Matrices and Its Applications*, John Wiley, New York, 1972

[11] F. E. Udwadia and R. E. Kalaba, *A New Perspective on Constrained Motion*, Proc. Roy. Soc. London **439**, 407–410 (1992)

[12] F. E. Udwadia and R. E. Kalaba, *Analytical Dynamics: A New Approach*, Cambridge University Press, London, 1996

[13] F. E. Udwadia, R. E. Kalaba, and E. Hee-Chang, *Equations of motion for constrained mechanical systems and the extended D'Alembert's principle*, Quart. Appl. Math. **55**, 321–331 (1997)

[14] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge, Fourth Edition, 1937