Structural Identification and Damage Detection from Noisy Modal Data

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Abstract: In this paper we present a simple, yet powerful, method for the identification of stiffness matrices of structural and mechanical systems from information about *some* of their measured natural frequencies and corresponding mode shapes of vibration. The method is computationally efficient and is shown to perform remarkably well in the presence of measurement errors in the mode shapes of vibration. It is applied to the identification of the stiffness distribution along the height of a simple vibrating structure. An example illustrating the method's ability to detect structural damage that could be highly localized in a building structure is also given. The efficiency and accuracy with which the method yields estimates of the system's stiffness from noisy modal measurement data makes it useful for rapid, on-line damage detection of structures.

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Introduction

Modal testing of structures is an extensive field in civil, aerospace, and mechanical engineering. It is generally used to understand/predict the dynamic behavior of a structure when subjected to low amplitude vibrations. Often modal information is also used to identify/estimate the structural parameters of a system, under the assumption that it has classical normal modes of vibration (Caughey and O'Kelley 1963). Such identification leads to improved mathematical models that can be used in either predicting and/or controlling structural response to dynamic excitations.

Several different approaches to the parameter identification problem have appeared in the literature (Baruch and Itzhak 1978; Udwadia and Ghodsi 1984; Kabe 1985; Wei 1989; Kalaba and Udwadia 1993; Mottershead and Friswell 1993; Kenigsbuch and Halevi 1997; Udwadia and Proskurowski 1998; Koh et al. 2000). One approach is the so-called model updating method. Here a suitable analytical model of a structural system is developed using the equations of motion, and its numerical representation is obtained. Validation of the numerical model through modal testing is then sought. Such tests usually provide some of the frequencies of vibration (usually the lower frequencies) and the corresponding mode shapes. When these frequencies and mode shapes obtained from modal testing are compared with those obtained from the numerical model, they generally do not agree with one another. Discrepancies between the results from experimental testing and theoretical modeling arise due to a variety of reasons: simplifications used in developing the analytical model, uncertainties in the structural description like those in material properties and boundary conditions, and experimental errors during modal testing. The problem of updating a numerical model so that it is as much as possible in conformity with experimental modal test data is referred to as the updating problem, and over the years it has received considerable attention.

In this paper we investigate a direct approach to structural identification through the use of modal test data. No a priori estimates are used. It should be pointed out that such experimental test data is seldom "complete," i.e., all the mode shapes of vibration and the corresponding natural frequencies are seldom available, for there is a practical limit to the range of frequencies that a structural or mechanical system can be tested for. Hence the idea is to obtain suitable models through the use of *incomplete* information, i.e., information on only a limited number of mode shapes and frequencies of vibration. We shall illustrate our method assuming that normal classical modes exist and that the damping factors are small, as is the common occurrence in structural and mechanical systems.

System Model

Consider a structural system modeled by the linear differential equation

$$M\ddot{x} + \hat{C}\dot{x} + \hat{K}x = 0 \tag{1}$$

where $x = n$ by 1 vector, and $M = n$ by n symmetric positive-definite mass matrix, $\hat{K}$ = symmetric stiffness matrix, and $\hat{C}$ = damping matrix. We shall assume that the elements of the mass matrix, M, are sufficiently well known, and that the system is classically damped. We could then rewrite Eq. (1) as

$$\ddot{y} + C\dot{y} + Ky = 0 \tag{2}$$

where $K = M^{-1/2}\hat{K}M^{-1/2}$ and $C = M^{-1/2}\hat{C}M^{-1/2}$.

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Our intention is to investigate the identification of the stiffness matrix from a knowledge of the modal data corresponding to Eq. (2), i.e., from the eigenvectors and eigenvalues of the matrix K. We note that the eigenvectors $\hat{\varphi}_i$ of

$$\hat{K}\hat{\varphi}_i = \lambda_i M \hat{\varphi}_i \tag{3}$$

are related to the eigenvectors φ_i of

$$K\varphi_i = \lambda_i \varphi_i \tag{4}$$

by the relation $\hat{\varphi}_i = M^{-1/2}\varphi_i$, for the eigenvalue λ_i (Caughey and O'Kelley 1963). We shall assume that $\lambda_i \leq \lambda_{i+1}$, $i=1,2,\ldots,(n-1)$.

Often from the analytical model, the "structure" of the matrix K is known. Let us say that the matrix K is a function of the parameter s-vector $k := [k_1\ k_2\ \cdots\ k_s]^T$. Usually, because of the limited connectivity between the subassemblages of a structure, the number of parameters, s, that are required to be identified in the n by n symmetric matrix K is much less than $n(n+1)/2$.

We assume that each element of the matrix K is a linear combination of the parameters k_i, so that each of the n equations in the equation set $K\varphi_i = \lambda_i\varphi_i$ is linear in these parameters. Hence each row of Eq. (4) is linear in both the parameters k_i and in the n components of the eigenvector φ_i. One can then rewrite Eq. (4) as

$$\Phi_i k = \lambda_i \varphi_i, \tag{5}$$

where the elements of the n by s matrix Φ_i are linear functions of the components of the n-vector φ_i. In what follows, we shall denote the jth component of the n-vector φ_i by φ_i^j.

Modal test data provides the *measured* frequencies and the *measured* mode shapes of vibration, yielding λ_i^m, $i=1,2,\ldots,r$, and the corresponding eigenvectors φ_i^m, $i=1,2,\ldots,r$, where some $r \leq n$. We denote experimental data by the superscript m. Were Eq. (5) to be true for the measured modal data, we would then obtain

$$\Phi_i^m \tilde{k} = \lambda_i^m \varphi_i^m, \quad i=1,2,\ldots,r, \tag{6}$$

giving us an estimate $\tilde{k}$ of the parameter s-vector k that contains the s parameters k_i, $i=1,2,\ldots,s$, which the stiffness matrix K is a function of Eq. (6) can be expressed in a more compact form by stacking the measured data from each of the r modes, as

$$B\tilde{k}_r := \begin{bmatrix} \Phi_1^m \\ \Phi_2^m \\ . \\ . \\ \Phi_r^m \end{bmatrix} \tilde{k}_r = \begin{bmatrix} \lambda_1^m\varphi_1^m \\ \lambda_2^m\varphi_2^m \\ . \\ . \\ \lambda_r^m\varphi_r^m \end{bmatrix} = b_r \tag{7}$$

where the matrix $B = rn$ by s, and the vector $b = rn$-vector.

The minimum-norm-least-squares solution to this system of Eq. (7) is simply given by

$$\tilde{k}_r = B^+ b_r \tag{8}$$

where B^+ stands for the Penrose generalized inverse of the matrix B, see Udwadia and Kalaba (1996) for details on the properties of generalized inverses of matrices, and their computation. The subscripts "r" indicate that measured modal data from r modes is "stacked" together in Eq. (7).

To illustrate the above equations, we consider a building structure modeled by a simple n-degree-of-freedom system (see Fig. 1) subjected to horizontal base motion. The mass matrix M is taken to be the identity matrix, and the structure is assumed to be lightly damped. Though effects like soil–structure interaction may be important in understanding the structural dynamics of such building structures, to illustrate our ideas we shall assume that the structure is resting on a rigid base, so that we can focus purely on our ability to estimate the constant stiffness matrix of the structure from modal data. Also, several mechanical systems are often modeled by such a "chain" of springs and masses, and hence we use this as a prototypical system.

The equation of motion is described by Eq. (1), and the matrix $\hat{K} = K$ is tridiagonal and has the form

$$K = \begin{bmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & . & . & . & \\ & & . & . & . \\ & & & -k_n & k_n \end{bmatrix} \tag{9}$$

One would like to estimate the parameter n-vector $k = [k_1\ k_2\ \cdots\ k_n]^T$ from modal test data.

The matrix Φ_i in Eq. (5) now becomes the upper triangular banded matrix

$$\Phi_i = \begin{bmatrix} \varphi_i^1 & -\varphi_i^{2,1} & & & & \\ & \varphi_i^{2,1} & -\varphi_i^{3,2} & & O & \\ & & . & . & & \\ & O & & . & . & \\ & & & & \varphi_i^{n-1,n-2} & -\varphi_i^{n,n-1} \\ & & & & & \varphi_i^{n,n-1} \end{bmatrix} \tag{10}$$

where $\varphi_i^{p,q} = (\varphi_i^p - \varphi_i^q)$. In this special case one can explicitly obtain the inverse of the matrix Φ_i and solve Eq. (5) to yield

$$k = \lambda_i \Phi_i^{-1} \varphi_i$$

where the upper-triangular matrix Φ_i^{-1} is given by

$$\Phi_i^{-1} = \begin{bmatrix} 1/\varphi_i^1 & 1/\varphi_i^1 & 1/\varphi_i^1 & . & . & & 1/\varphi_i^1 \\ & 1/\varphi_i^{2,1} & 1/\varphi_i^{2,1} & . & . & & 1/\varphi_i^{2,1} \\ & & . & & . & . & . \\ & & & & . & . & . \\ & & & & 1/\varphi_i^{n-1,n-2} & & 1/\varphi_i^{n-1,n-2} \\ & & & & & & 1/\varphi_i^{n,n-1} \end{bmatrix} \tag{11}$$

Were the eigenvalue λ_i obtained from measurements of the frequency of vibration of the ith normal mode, and the eigenvector φ_i obtained from experimental measurements of the ith mode shape of vibration of the building structure, we would obtain an *estimate* of the parameter n-vector k as

$$\tilde{k} = \lambda_i^m [\Phi_i^m]^{-1} \varphi_i^m \tag{12}$$

where we have replaced all the φ_i^j's in Eq. (11) by their measured values, $(\varphi_i^j)^m$. While this estimate may be adequate when the mode shapes of vibration are assumed to be noise-free (a situation that never arises in practice), as shown below, it quickly deteriorates in the presence of measurement noise.

Were frequency and mode shape data obtained for modes $i=1,2,\ldots,r$, we would form the matrix B as shown in Eq. (7) and

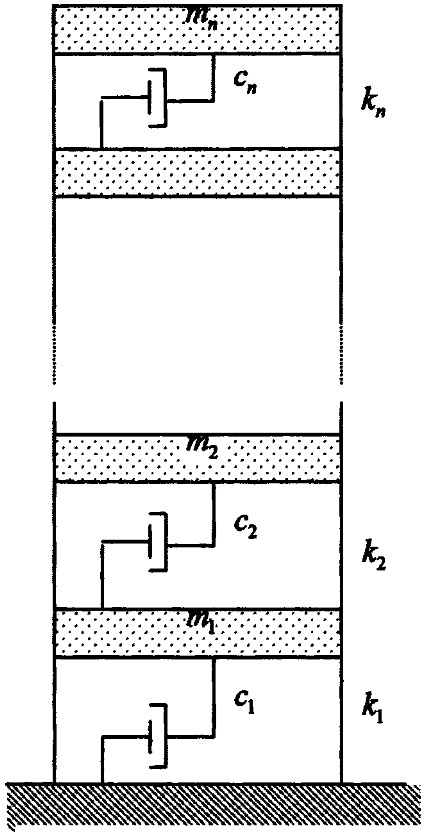

Fig. 1. Model of a simple building structure

obtain the estimate of the parameters as indicated in Eq. (8). We illustrate this using the following numerical example.

Numerical Example

Consider a building structure modeled by a nine-degree-of-freedom system ($n=9$) whose true (exact) story stiffnesses are taken to be (we assume the numerical data to be in consistent units)

$$k_1 = 2{,}000;\quad k_2 = 2{,}050;\quad k_3 = 2{,}090;\quad k_4 = 2{,}050;\quad k_5 = 1{,}950;$$

$$k_6 = 1{,}990;\quad k_7 = 1{,}920;\quad k_8 = 1{,}900;\quad \text{and } k_9 = 1{,}900$$

The stiffness parameters are *purposely* chosen to have a small range of numerical values so that the power of the identification scheme in discriminating between these close stiffness values is assessed.

The fundamental frequencies of vibration corresponding to the 9 by 9 stiffness matrix shown in Eq. (9) are then: 1.1810, 3.4637, 5.6707, 7.7577, 9.5685, 11.1865, 12.4205, 13.3962, 14.0528 cycles/s. We begin by assuming that the experimentally determined frequencies and the mode shapes of vibration from modal testing are accurate (noiseless). Thus $\lambda_i=\lambda_i^m$, $\varphi_i=\varphi_i^m$, $i=1,2,\ldots,r$, where r=total number of resonant frequencies and mode shapes obtained from the test data.

Fig. 2(a) shows the percentage error in estimates of the parameter vector k whose components are the story stiffnesses. The error in the estimate, $\tilde{k}_i$, of the story stiffness k_i is defined as $(\tilde{k}_i-k_i)/k_i$. It is assumed that only the lowest four modes of vibration are measured, along with the lowest four modal frequencies. The figure shows that the stiffness distribution can be very accurately determined using Eq. (12) above, for i equal to 1–4. However, the addition of measurement noise alters the situation considerably.

During modal testing it is customary to assume that the frequencies of vibration are accurately determined, and that it is in the determination of the amplitudes of the mode shapes that the experimental errors arise. This assumption is by and large valid because the frequency of shakers, even at resonance, can be quite accurately controlled. Accordingly, the data is simulated by adding noise to each modal amplitude, and the "measured" component "j" of the ith-mode shape is taken to be

$$(\varphi_i^j)^m = \varphi_i^j(1 + \alpha_{\text{noise}}\xi) \tag{13}$$

where ξ is a uniformly distributed random number between −1 and +1. Fig. 1(b) shows results with $\alpha_{\text{noise}}=5\%$ indicating the dramatic increase in the error when Eq. (12) is used in estimating $\tilde{k}$ when using noisy mode shape data.

We next illustrate the improvement that is created by using Eq. (8) where the estimate is obtained by using the generalized inverse of the matrix B^+ for values of r ranging from 1 to 4. Fig. 2(c) shows the progressive improvement in the estimates of the stiffness with the addition of information about each successive mode of vibration. The simultaneous use of data from all four modes ($r=4$) shows that there is a substantial reduction in the percentage error in the estimation of the stiffness. The maximum percentage error is now less than 10%.

Iterative Improvements of the Stiffness Estimates in the Presence of Measurement Noise

Having obtained the minimum length-least-squares estimate of the parameters by using the generalized inverse of the matrix B, we now attempt to improve this estimate when the mode shape information is corrupted by measurement noise, i.e., when we use the measured modal data, φ_i^m, $i=1,2,\ldots,r$. We assume, as is customary, that the measurement errors in determining the r resonant frequencies are negligible when compared to those incurred in the measurement of the mode shapes; hence $\lambda_i=\lambda_i^m$, $i=1,2,\ldots,r$. For convenience we will denote the minimum length-least-squares estimate we have obtained so far by $\tilde{k}^{(0)}$.

This estimate, which is obtained from Eq. (8), gives us, in turn, an estimate, $\tilde{K}^{(1)}$, of the matrix K. Hence a solution of the eigenvalue problem, $\tilde{K}^{(1)}\psi_j=\mu_j\psi_j$, provides: (1) an estimate of all the eigenvalues (frequencies), μ_j, $j=1,2,\ldots,n$ of vibration (estimates of the eigenvalues of K) and (2) an estimate of all the mode shapes, ψ_j, $j=1,2,\ldots,n$ (estimates of the eigenvectors of K). Since $\tilde{K}^{(1)}$ is symmetric, ψ_j are (or can be chosen to be) orthogonal to one another, and can be normalized to have unit length.

In summary, through our estimate of the stiffness parameter vector $\tilde{k}^{(0)}$ at this point we have obtained estimates, μ_j, of the actual eigenvalues, λ_j, of K as well as estimates, ψ_j, of the actual mode shapes, φ_i. We next use this information to iteratively update the estimate $\tilde{k}^{(0)}$.

Since the eigenvectors of $\tilde{K}^{(1)}$ form a basis set in R^n, the "measured" noise-corrupted mode shape φ_i^m can be expanded in terms of the estimated mode shapes ψ_j, $j=1,2,\ldots,n$. We then have

$$\varphi_i^m = \sum_{j=1}^{n} \delta_j^i \psi_j, \quad i = 1,2,\ldots,r \tag{14}$$

where $\delta_j^i=\psi_j^T\varphi_i^m$. We then obtain,

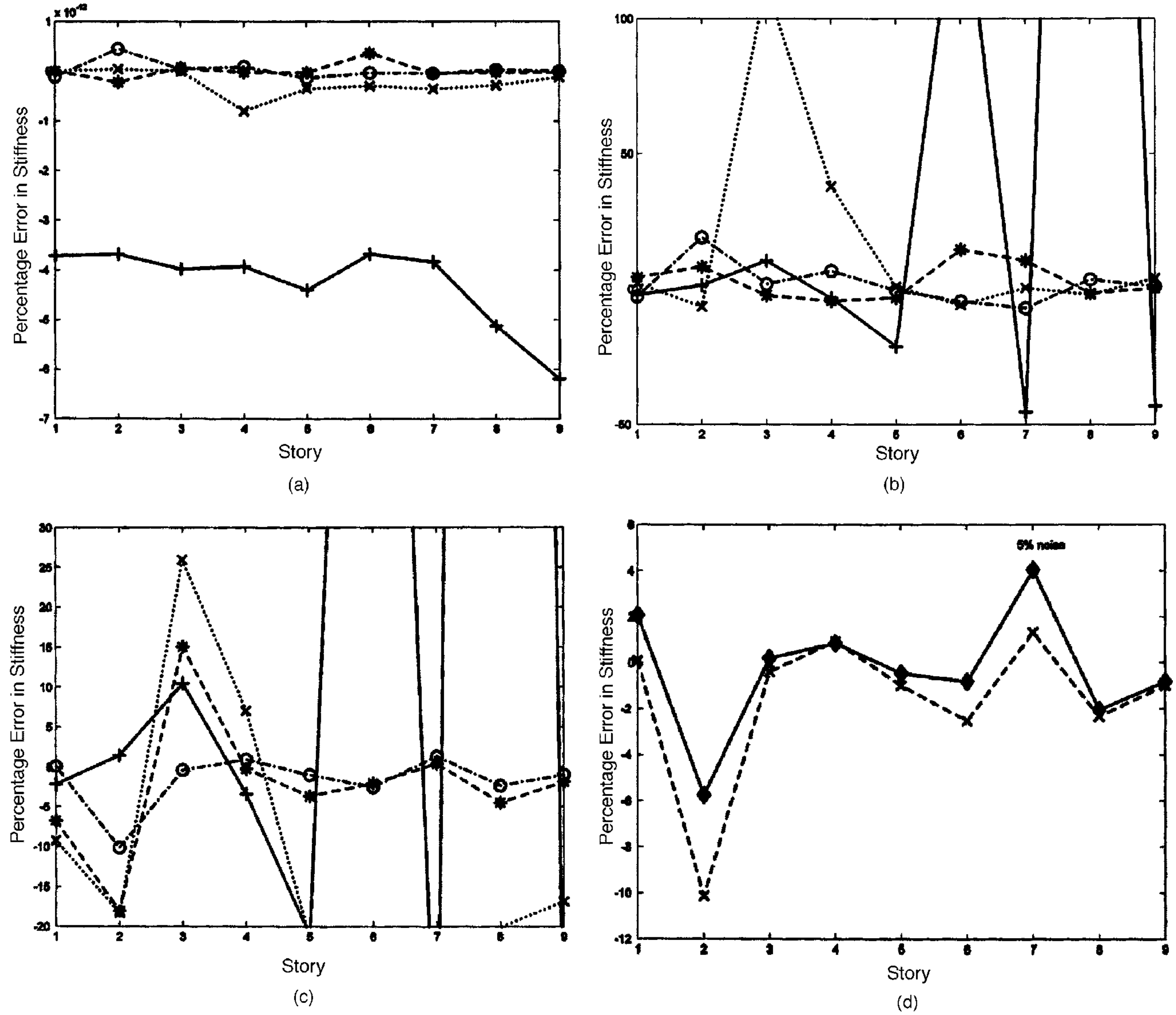

Fig. 2. (a) Identification (ID) with noise-free data; + shows ID from first mode; × shows ID from second mode; * shows ID from third mode; and ○ shows ID from fourth mode. (b) Identification (ID) with 5% noise in modal amplitude data; + shows ID from first mode; × shows ID from second mode; * shows ID from third mode; and ○ shows ID from fourth mode. (c) Identification (ID) with 5% noise in modal amplitude data using generalized inverses; + shows ID using $r=1$; × shows ID using $r=2$; * shows ID using $r=3$; and ○ shows ID using $r=4$. (d) Improvement in estimation caused by iterative scheme.

$$\tilde{K}^{(1)}\varphi_i^m = \tilde{K}^{(1)}\sum_{j=1}^{n} \delta_j^i\psi_j = \sum_{j=1} \delta_j^i\tilde{K}^{(1)}\psi_j = \sum_{j=1}^{n} \delta_j^i\mu_j\psi_j, \quad i=1,2,\ldots,r \qquad (15)$$

As before, we can now express $\tilde{K}^{(1)}\varphi_i^m$ as $\Phi_i^m\tilde{k}^{(0)}$, and Eq. (15) yields

$$\Phi_i^m\tilde{k}^{(0)} = \sum_{j=1}^{n} \delta_j^i\mu_j\psi_j, \quad i=1,2,\ldots,r \qquad (16)$$

However, the measured r lowest frequencies are assumed to be accurately known and we can use this information in Eq. (16) by replacing the μ_j's by the known (i.e., accurately measured) eigenvalues λ_j for $j=1,2,\ldots,r$. This additional information when injected into Eq. (16) provides us with the opportunity to update ourestimate of the stiffness parameter vector from $\tilde{k}^{(0)}$ to $\tilde{k}^{(1)}$, and we obtain

$$\Phi_i^m\tilde{k}^{(1)} = \sum_{j=1}^{r} \delta_j^i\lambda_j\psi_j + \sum_{j=r+1}^{n} \delta_j^i\mu_j\psi_j, \quad i=1,2,\ldots,r \qquad (17)$$

This forms the basis of our iterative improvement of the stiffness parameter vector $\tilde{k}^{(0)}$. We note that we have used: (1) our information from the r measured eigenvalues (frequencies of vibration) of the system, which we assume are accurate, (2) our best hereto available estimates, μ_j, of the remaining $(n-r)$ unmeasured eigenvalues (frequencies of vibration) of the system, and (3) our best hereto available estimates, ψ_j, $j=1,2,\ldots,n$ derived from the measured eigenvectors φ_j^m, $j=1,2,\ldots,r$, which are corrupted by measurement noise.

Eq. (17) can be rewritten for convenience as

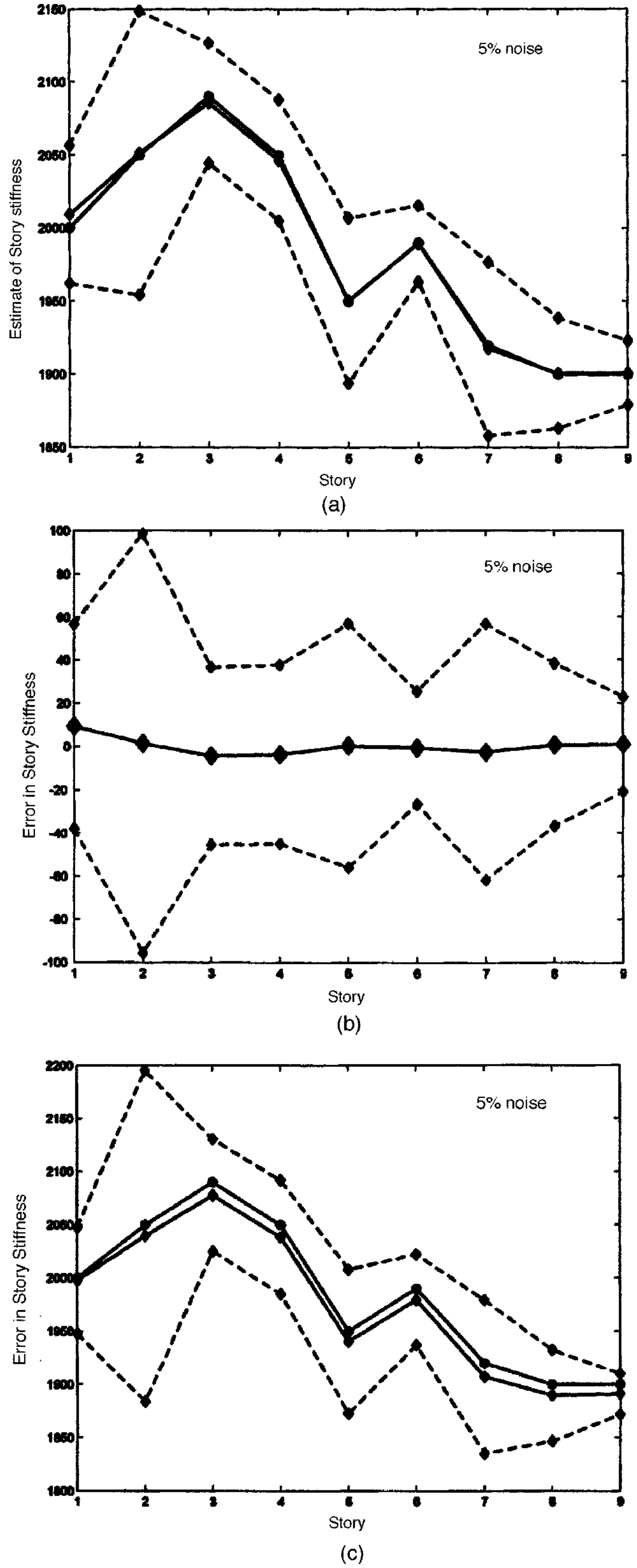

Fig. 3. (a) Expected value of the identified estimate (solid line with diamonds), true stiffness (solid line with circles), and the expected estimate ± standard deviation (dashed lines with diamonds). (b) Expected value of the error in the stiffness at each story (solid line) and the ±1-sigma band around it (dashed lines). (c) Results with no iterative improvement. Expected value of stiffness at each story (solid line with diamonds), true stiffness (solid line with circles), and expected value ±1-sigma band (dashed line with diamonds).

$$\Phi_i^m \tilde{k}^{(1)} = [\psi_1 \quad \psi_2 \quad \dots \quad \psi_n] \begin{bmatrix} \delta_1^i \lambda_1 \\ \delta_2^i \lambda_2 \\ . \\ . \\ \delta_r^i \lambda_r \\ \delta_{r+1}^i \mu_{r+1} \\ . \\ . \\ \delta_n^i \mu_n \end{bmatrix} := \Psi h_i, \quad i = 1, 2, \dots, r \tag{18}$$

Here Ψ is the n by n orthogonal matrix of eigenvectors of $\tilde{K}^{(1)}$. Lastly, since we have r measured mode shapes, we can, as before, stack the information again as

$$B\tilde{k}_r^{(1)} := \begin{bmatrix} \Phi_1^m \\ \Phi_2^m \\ . \\ . \\ \Phi_r^m \end{bmatrix} \tilde{k}_r^{(1)} = \begin{bmatrix} \Psi h_1 \\ \Psi h_2 \\ . \\ . \\ \Psi h_r \end{bmatrix} = b_r^{(1)} \tag{19}$$

so that we again obtain the minimum-length-least-squares solution for $\tilde{k}_r^{(1)}$ in Eq. (19) as

$$\tilde{k}_r^{(1)} = B^+ b_r^{(1)} \tag{20}$$

It should be pointed out that the matrix B in Eq. (19) is the same as it was in Eq. (7); it is obtained from the r measured mode shapes.

The estimate $\tilde{k}_r^{(1)}$ is next used to create an estimate $\tilde{K}^{(2)}$ of the stiffness matrix, which in turn yields the new estimate $\tilde{k}_r^{(2)}$, and the iteration continues until the improvement in the vector $\tilde{k}_r^{(i)}$ is deemed to be negligible, or for a prefixed number of iterations. The algorithm that we have developed can be summarized as follows:

1. Use the measured mode shape φ_i^m to obtain the n by s matrices Φ_i^m, $i=1,2,\dots,r$;
2. Stack the r matrices Φ_i^m's to obtain B;
3. Stack the vectors $\lambda_i^m \varphi_i^m$, $i=1,2,\dots,r$, to obtain the vector $b_r^{(0)}$;
4. Calculate $\tilde{k}_r^{(0)} = B^+ b_r^{(0)}$ [Eq. (8)]; and
5. For $i=1$ to N_iterations,
   do:

   Obtain $\tilde{K}^{(i)}$, its eigenvalues μ_j, and its eigenvectors ψ_j, $j=1,2,\dots,n$;

   Calculate $\delta_j^p = \psi_j^T \varphi_p^m$, $j=1,2,\dots,n$; $p=1,2,\dots,r$;

   Calculate the vectors Ψh_p [Eq. (18)] using measured λ_p^m, $p=1,2,\dots,r$, and μ_p, $p=r+1,\dots,n$;

   Stack the vectors Ψh_p, $p=1,2,\dots,r$, to get $b_r^{(i)}$; [Eq. (19)]

   Calculate $\tilde{k}_r^{(i)} = B^+ b_r^{(i)}$;

   If $\|\tilde{k}_r^{(i)} - \tilde{k}_r^{(i-1)}\| < \Delta$, exit do loop;

   end do loop.

Numerical Example (Continued)

To illustrate the iterative improvement of the estimate of the stiffness parameter vector, we show in Fig. 2(d) the results from the same modal data shown in Figs. 2(b and c) by using the same

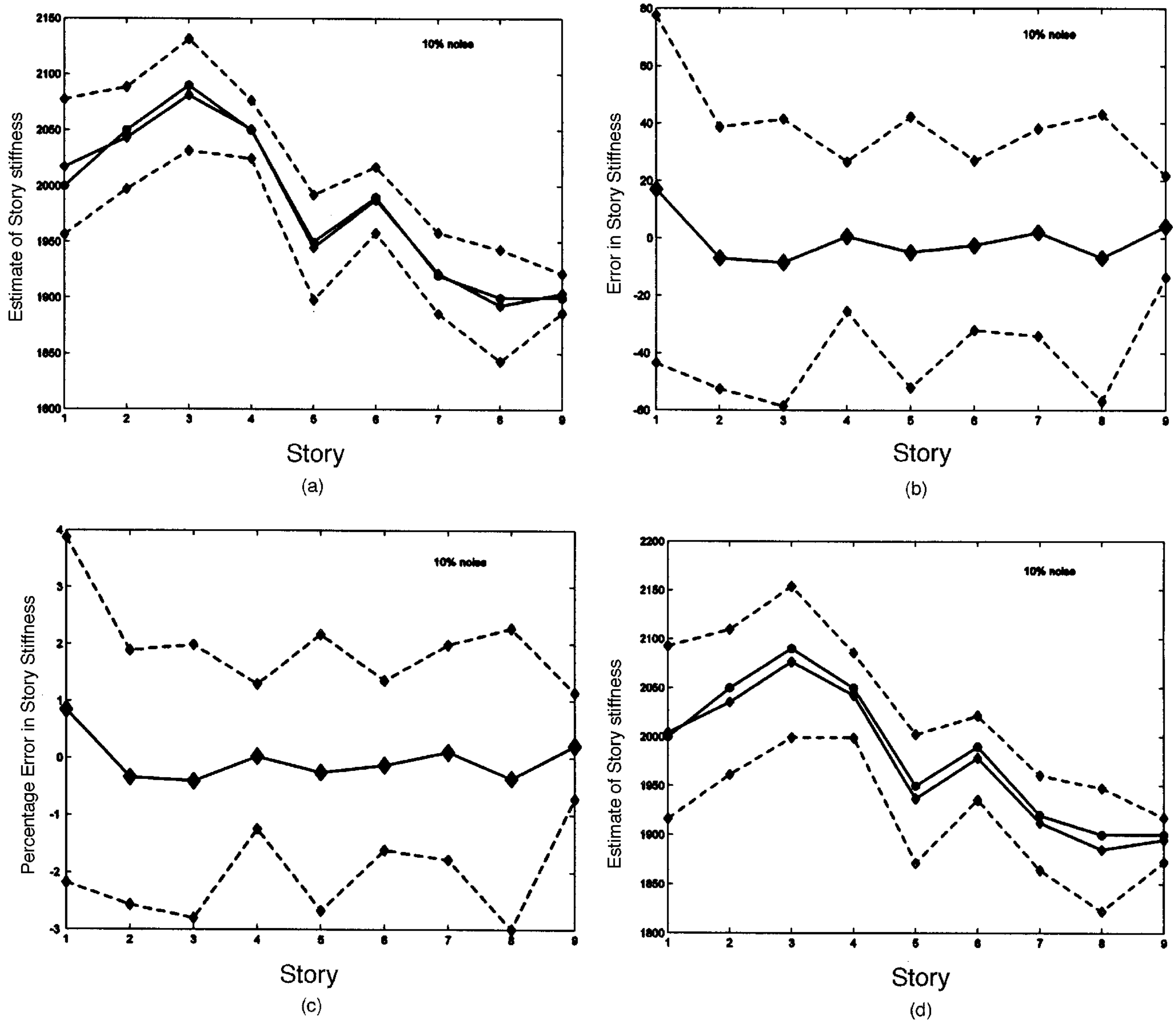

Fig. 4. (a) Expected value of the stiffness (solid line with diamonds), true stiffness (solid line with circles), and the ±1-sigma band using the iterative improvement (dashed lines). (b) Expected error in estimation of stiffness (solid line) and ±1-sigma error band using the iterative improvement (dashed lines). (c) Expected percentage error (solid line) and ±1-sigma error band using iterative improvements (dashed lines). (d) Expected value of the stiffness true stiffness and the ±1-sigma band *without* iterative improvement with the same notation of lines as in (a).

realization as before of the random noise that corrupts the mode shape data. The solid line shows the identification results obtained after 1,000 iterations, the dashed line is the result generated by direct use of the generalized inverse [i.e., Eq. (8)]. We note in passing that negligible changes in the stiffness occur after 150 iterations.

Since the only experimental information that we assume here is that each component of the measured mode shape (corresponding to each natural frequency of vibration) is corrupted with zero mean, uniformly distributed white noise, what is of relevance is the expected values of the stiffness estimate obtained and its standard deviation. To find the expected value of the estimate, the simulation was carried out 1,000 times with $\alpha_{noise}=5\%$, each time using a different set of randomly generated noise to corrupt the mode shape measurements. Fig. 3(a) shows the expected value of the estimate of the stiffness at each story along with a 2-sigma band shown by the dashed lines to indicate the variance in the expected estimate. The true (exact) stiffness is shown by the solid line with circles, and the expected value of the stiffness is shown by the solid line with diamonds. As seen from the plot, the identification scheme appears to work remarkably well, and the expected estimate closely tracks the true stiffness.

To get a feeling for the expected error in the estimation of the stiffness at each story, i.e., $E\{\tilde{k}-k\}$, we show it in Fig. 3(b) along with its ±1-sigma band. As seen from the figure, the estimation is quite accurate, and the expected error in the estimation of the stiffness is very small, the maximum of {mean error +1 sigma} being 5% at story number 2.

Lastly, we show in Fig. 3(c) the expected value (with a sample size of 1,000) of the estimate along with the ±1-sigma bands when only the generalized inverse is used *without* the iterative improvement in the estimates discussed in this section. Again the solid line with circles is the true stiffness, the dark solid line with diamonds is the expected value of the stiffness, and dashed lines indicate the mean ±1-sigma band. Comparing Fig. 3(a) with 3(c) we observe that the iteration discussed in this section [Eqs. (17)–(20)] has a definite advantage, both in terms of the expected value getting closer to the true stiffness, and in reducing the size of the ±1-sigma band.

We observe that the estimates that we have obtained use information from only four modes of vibration of the structure. This is the usual situation, for complete information on all the modes and all the resonant frequencies of structures is seldom experimentally decipherable from measurement data. This is due to limitations in experimental testing. Also, higher modes "sample" smaller spatial domains and our models may be inadequate in representing these spatial heterogeneities.

We show next the influence of using more modal information on the identification results. We next use measured modal data from six modes instead of four. Clearly, the additional information should improve the estimates of the stiffness distribution and reduce the uncertainty in the expected value of the estimates that are obtained. However, noise in the measurements of the modal amplitudes usually increases with increasing natural frequency, and so to simulate the noise-corrupted mode shape amplitude data we shall use $\alpha_{noise}=10\%$ in Eq. (13) instead of 5% as we did when we used only four modes. We exhibit our results in Fig. 4.

We observe that even with the sizably larger measurement noise of 10% in the mode shape measurements, the estimates of the 1-sigma band in the percentage error in the stiffness estimates covers only ±4%.

Localized Damage Detection: Case Study

We now assume that the structural (or mechanical) system has, perhaps after being subjected to horizontal ground shaking, suffered a localized deterioration along its height; and the stiffness at the third story has reduced from its previous value by about 30%. Our aim is to estimate the new stiffness distribution and investigate how well this localized drop in stiffness can be captured by modal testing, using, again, only four modes of vibration. We assume that the true stiffness of the damaged structure is given by

$$k_1 = 1{,}950; \quad k_2 = 1{,}900; \quad k_3 = 1{,}460; \quad k_4 = 1{,}910; \quad k_5 = 1{,}910;$$

$$k_6 = 1{,}930; \quad k_7 = 1{,}920; \quad k_8 = 1{,}900; \quad \text{and } k_9 = 1{,}900$$

Except for k_3, the other stiffness parameters are *purposely* taken to have values close to one another to see if our identification scheme can *simultaneously* track both large and small changes in the stiffness distribution.

Fig. 5(a) shows that the weakened structure can be identified well, and the location of the drop in the stiffness is quite evident and accurately determined using noise corrupted modal information from only four modes. The solid line is the true stiffness distribution, the dashed line is the stiffness after 1,000 iterations, and the dash-dot line is the result of Eq. (8), with no iterative improvement. These estimates are obtained using one realization of the noisy modal data. The solid line in Fig. 5(b) shows that the percentage error in estimating the stiffness with iterative improvements is less than 4%. The dashed line in Fig. 5(b) shows results without iterative improvements.

The expected value of the stiffness estimates are again estimated using 1,000 realizations of noise corrupted "measured" mode shapes. The solid line in Fig. 6(a) shows the expected value of the stiffness estimates at each story, the shaded line shows the true stiffnesses, and the dashed lines show the ±1-sigma bands.

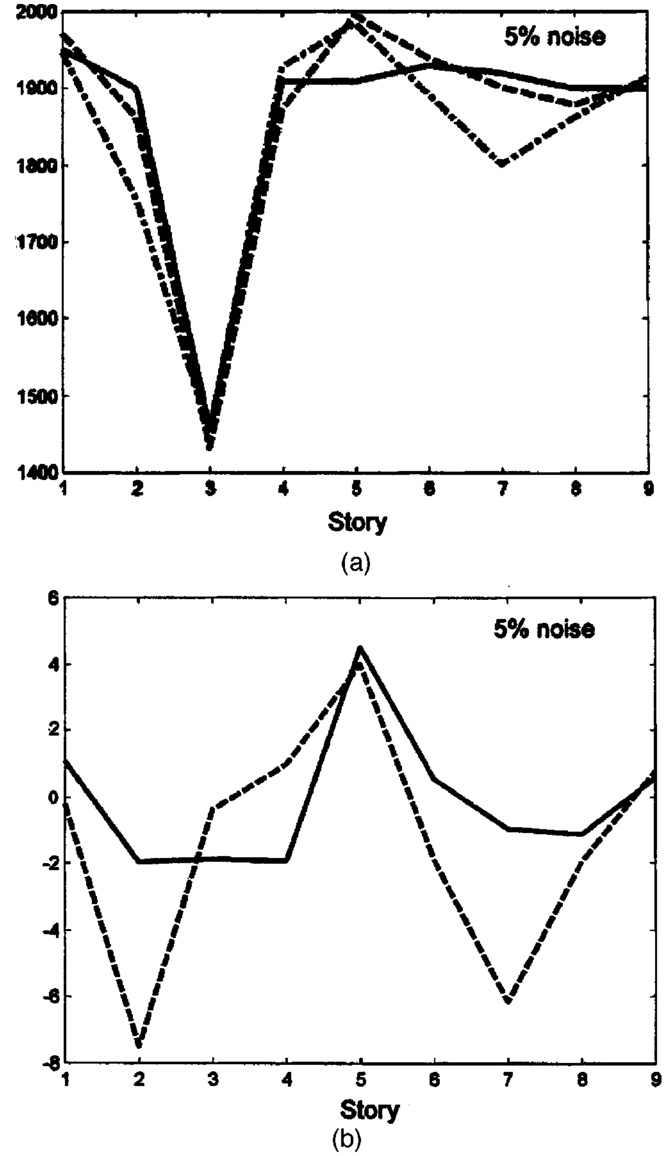

Fig. 5. (a) Estimate of stiffness using one realization of noisy modal measurements and (b) Percentage error in stiffness estimates.

Fig. 6(b) shows the expected error in the stiffness estimates and the ±1-sigma error bands. We see that the identification results are rather accurate and the ±1-sigma bands are far smaller than the drop in stiffness at the third story level, indicating that accurate damage detection can be accomplished in an efficient manner using our simple system identification approach. Fig. 6(c) shows the expected percentage error in estimation of the stiffness. The ±1-sigma bands show that changes of more than ±5% in the stiffness parameters can be rapidly identified with reasonable reliability. Lastly, Fig. 6(d) shows the expected values of the identification results without any iterative improvements. We note that though not as good as those obtained after iteration, these results too are considerably superior to those that are obtainable by other system identification techniques when using noise-corrupted data.

Conclusions and Discussion

We have proposed here a simple and efficient way of estimating the stiffness distribution in a structure from *incomplete* modal

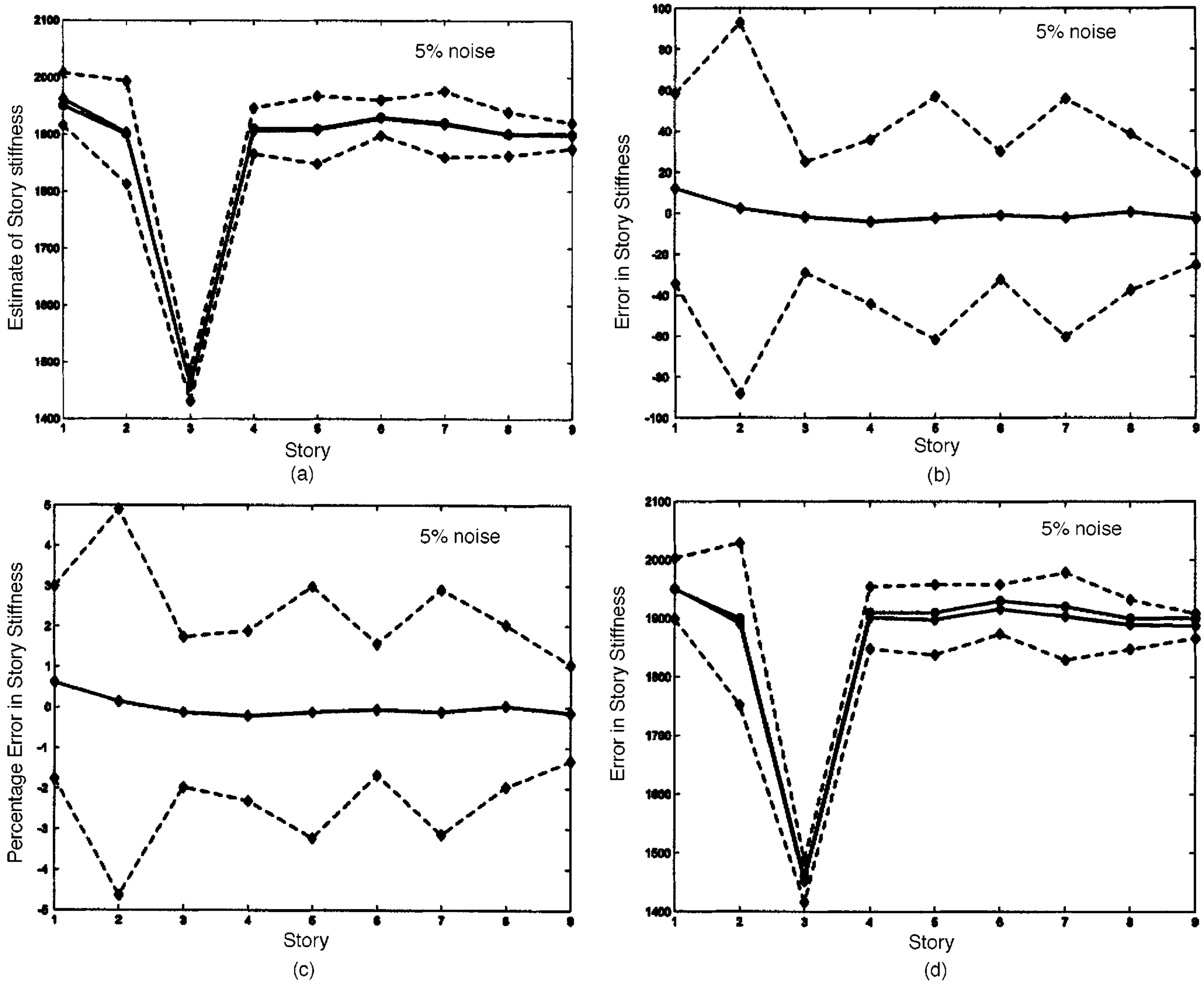

Fig. 6. Case study of damage detection. The notation for the lines in (a)–(d) is the same as that for the corresponding Figs. 4(a–d).

information. It relies on two key ideas: the use of the Moore-Penrose generalized inverse of a rectangular matrix, which gives the minimum length, least-squares solution, and which can be computed rapidly and efficiently; and a subsequent iterative improvement of the estimates obtained by expanding, at each iteration, the noise-corrupted measured mode shapes in terms of the *best available estimates* of the mode shapes and frequencies of vibration, as well as the *measured* frequencies of vibration. These iterative adjustments use the already computed Moore-Penrose inverse.

Our numerical computations show that the iterative scheme reduces the variance of the estimates and brings their expected values closer to the true (exact) values. The method appears to work remarkably well for estimating the stiffness distribution of building structures. Most importantly, we find that the method gives good results in the presence of noise-corrupted mode shape data, and is capable of tracking very small changes in the stiffness parameters. Its accuracy in the presence of noise appears to well surpass most other identification methods, which are far more computationally intensive and which are in common use today [e.g., compared to results in Udwadia and Ghodsi (1984); Kalaba and Udwadia (1993); and Udwadia and Kalaba (1996)]. As pointed out in Udwadia and Ghodsi (1984), even 2% noise in measured data could substantially degrade the identification results thereby making them questionable, or at least highly unreliable.

Though in this paper we have used information from the lowest r mode shapes and frequencies of vibration, we could have chosen *any* r of the measured mode shapes and r corresponding measured frequencies of vibration for the identification. The identification method presented here can be extended to this situation in a straightforward and obvious manner.

It would be useful to compare the method proposed herein with those commonly in use—the so-called updating methods. We note that in the method proposed herein:

1. The "structure" of the stiffness matrix is assumed to be known, and in addition, the way in which the unknown stiffness parameters enter each element of the stiffness matrix are assumed to be known.
2. The elements of the stiffness matrix are assumed to be linear functions of the parameters to be estimated; this is not as much of a restriction as may appear at first sight, for the "assemblage" of the stiffness matrix using, say, finite element method models provides this sort of information in a natural way.
3. We use the Moore-Penrose generalized inverse to obtain the

actual stiffness estimates, and *no* a priori stiffness estimates are required as in all the stiffness updating methods. The computation of this inverse is efficient and is available in most computation environments, like *MATLAB*.

4. Unlike the usual updating methods that try to estimate each element of the n by n stiffness matrix, thereby increasing the number of unknowns [usually to $n(n+1)/2$], the method proposed here uses the knowledge of the way in which the unknown parameters enter the stiffness matrix; this reduces the number of unknowns significantly. The reduction in dimensionality causes improved efficiency and less sensitivity to measurement noise.
5. Our approach averts “connectivity” problems that require addressing by the use of additional constraints when using updating methods that employ nonlinear minimization techniques. Current methods to handle such connectivity problems cause large increases in computational budgets and the performance of such methods under noisy measurement conditions appears uncertain.
6. Even when there is no measurement noise, the accuracy of the available updating methods is known to often rely on the proper choice of “weighting matrices,” and there appears to be no systematic method of choosing these matrices.
7. Updating methods are highly susceptible to measurement errors; we have shown that the method developed here can provide significant, high-quality information about the stiffness parameters even when the measurements of the mode shapes are incomplete and corrupted by noise.
8. Unlike most updating methods, no attempt is made to orthonormalize the measured mode shapes.
9. The iterative procedure used herein provides a significant advantage both in terms of reducing the variance of the stiffness estimates and in getting the expected estimates closer to the true values.

The method provided in this paper for estimating the stiffness distribution in structures from modal data is conceptually simple; yet it appears to be extremely powerful in identification of the stiffness distribution in structures. We have demonstrated its ability to detect *simultaneously* both large and small variations in structural properties from noisy modal data. *Its capability to identify parameter variations at multiple scales from noisy data appears to be uncommon among the identification methods available to date*. The method is computationally efficient. It appears to be superior to the usual, and much more complex, updating methods so far developed in system identification. In addition, the method does *not* require: a priori estimates of the parameters that are to be identified; the use of weighting matrices that are required in updating methods; and the use of constraint relations (which could be many) to handle connectivity issues between the parameters. The method is shown to provide a reliable way of detecting localized damage in building structures following large dynamics loads, such as those created by high winds and strong earthquake ground shaking. The simplicity and computational efficiency of the method makes it valuable for rapid and reliable on-line damage detection in structures.

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