Stability of Gyroscopic Circulatory Systems

Firdaus E. Udwadia
Professor
Aerospace and Mechanical Engineering,
Civil Engineering, Information and Operations
Management, and Mathematics,
430K Olin Hall,
University of Southern California,
Los Angeles, CA 90089-1453
e-mail: feuUSC@gmail.com

This paper presents results related to the stability of gyroscopic systems in the presence of circulatory forces. It is shown that when the potential, gyroscopic, and circulatory matrices commute, the system is unstable. This central result is shown to be a generalization of that obtained by Lakhadanov, which was restricted to potential systems all of whose frequencies of vibration are identical. The generalization is useful in stability analysis of large scale multidegree-of-freedom real life systems, which rarely have all their frequencies identical, thereby expanding the compass of applicability of stability results for such systems. Comparisons with results in the literature on the stability of such systems are made, and the weakness of results that deal with only general statements about stability is exposed. It is shown that the commutation conditions given herein provide definitive stability results in situations where the well-known Bottema–Karapetyan–Lakhadanov result is inapplicable. [DOI: 10.1115/1.4041825]

Introduction

The importance of the investigation of stability and instability of linear dynamical systems is underlined by the fact that it stands at the cross-roads of our understanding of both natural phenomena, like tornados and ocean waves, and engineered structural and mechanical systems such as rotorcraft, spacecraft, and aero-elastic systems. Such linear systems also provide an important inroad into our understanding of the stability (or instability) of general nonlinear dynamical systems, which are ubiquitous in modeling numerous and diverse systems such as those found in the sciences and engineering, economics, sociology, and biology. For, the stability of such nonlinear dynamical systems at hyperbolic fixed points and the capture of their local phase portraits in the vicinity of their fixed points via the Hartman–Grobman result [1,2] rely on our understanding of the stability of their linearized versions. Therefore, in recent years, there has been a resurgence in the study of the stability of linear dynamical systems through the development of new results and the availability of excellent expository texts (e.g., see Ref. [3]).

Instabilities in linear systems can be generated by damping forces, gyroscopic forces, and/or circulatory forces. The first is characterized by the notable Kelvin–Tait–Chetaev theory [4,5]. And the second is exemplified by the stabilization of unstable potential systems through the use of gyroscopic forces [6]. The third is dealt with by the celebrated Merkin's theorem [7], which was generalized in Refs. [8] and [9]. Reference [8] does not deal with the quintessential problem of the severe restriction of Merkin's theorem, which is only applicable to systems whose vibrational frequencies are *all* identical, thereby making the result very narrow in its applicability to many real-life systems. This underlying restriction in Merkin's theory is removed in Ref. [9], and it is shown that for a potential circulatory system, even a single repeated vibrational frequency makes the system unstable. This generalization therefore expands the scope of its applicability to encompass large-scale, real-life, multidegree-of-freedom systems.

While it is easy to visualize the presence of damping forces (caused by dissipative processes in structural or mechanical systems) and also gyroscopic forces (often caused by rotary motion in rotating flexible machinery, spinning elastic systems, and astrodynamics), the presence of circulatory forces is perhaps some-what less intuitive. Yet, such forces arise in many areas of real-life applications. Some examples are control of two-legged walking robots, self-oscillations (shimmy) in aircraft wheels, flutter in aerospace systems, dynamics of brake squealing, and wear in paper calendars [10–16].

We consider a nonlinear system described by the n-vector of generalized coordinates, $q \in R^n$, whose equilibrium point is located, with no loss of generality, at $q = \dot{q} = 0$. The dot represents differentiation with respect to time, t. In the close vicinity of this equilibrium point, we assume that the equation of motion of the system can be written as

$$\tilde{M}\,\ddot{q} + \tilde{G}\dot{q} + (\tilde{K} + \tilde{N})\,q = Q(q, \dot{q}) \tag{1}$$

where $\tilde{M}$ is an n-by-n real (symmetric) positive definite constant matrix and q is an n-component real column vector. The constant matrices $\tilde{G}$ and $\tilde{N}$ are real and skew-symmetric, and the matrix $\tilde{K}$ is constant, real, and symmetric. The nonlinear generalized force n-vector, Q, consists of terms that are quadratic (and/or higher) in $q, \dot{q}$. Linearization about the equilibrium point leads to the equation

$$\tilde{M}\,\ddot{q} + \tilde{G}\dot{q} + (\tilde{K} + \tilde{N})\,q = 0 \tag{2}$$

which would also have resulted were our system linear from the very outset.

Using the transformation $z(t) = \tilde{M}^{-1/2}q(t)$ and premultiplying by $\tilde{M}^{-1/2}$, Eq. (2) can be rewritten as

$$\ddot{z} + G_{\rm sk}\dot{z} + (K_s + N_{\rm sk})z = 0 \tag{3}$$

where $G_{\rm sk} = \tilde{M}^{-1/2}\,\tilde{G}\,\tilde{M}^{-1/2}$ and $N_{\rm sk} = \tilde{M}^{-1/2}\,\tilde{N}\,\tilde{M}^{-1/2}$ are skew-symmetric matrices, and matrix $K_s = \tilde{M}^{-1/2}\,\tilde{K}\,\tilde{M}^{-1/2}$ is symmetric. Being skew-symmetric, the eigenvalues of the matrices $G_{\rm sk}$ and $N_{\rm sk}$ are either zero, or conjugate pairs of pure imaginary numbers. Clearly, the dynamical system described by Eq. (3) is equivalent to the one described by Eq. (2). We will be using Eq. (3) from here on, and we will assume that the system is not the trivial system $\ddot{z} = 0$ in which $G_{\rm sk} = K_s = N_{\rm sk} = 0$.

Forces expressed by the skew-symmetric matrix $N_{\rm sk}$ are called circulatory forces or nonconservative positional forces [15,16]; forces caused by the symmetric matrix K_s are called potential forces; and, forces expressed by the skew-symmetric matrix $G_{\rm sk}$ are referred to as gyroscopic forces. The system described by Eq. (3) may be thought of as a generalization of the one considered in Ref. [8] in which only circulatory forces ($N_{\rm sk}$) are included and no gyroscopic forces ($G_{\rm sk} = 0$).

Contributed by the Applied Mechanics Division of ASME for publication in the Journal of Applied Mechanics. Manuscript received August 26, 2018; final manuscript received October 24, 2018; published online November 16, 2018. Assoc. Editor: George Haller.

Main Result

Our result will show that when the three matrices K_s, G_{sk}, and N_{sk} pairwise commute, then the system described by Eq. (3) is unstable. We begin with four lemmas, which deal with some of the properties of these three matrices. They will be used later on and will also establish the notation.

LEMMA 1. *If the n by n symmetric matrix K_s commutes with the nonzero skew-symmetric matrix $N_{sk}(G_{sk},)$ then the matrix K_s must have multiple eigenvalues [9].*

Remark 1. When Lemma 1 is satisfied, the n by n matrix K_s therefore has $k < n$ distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$ with corresponding multiplicities $i_1, i_2, \cdots, i_k$, respectively. Since we have a total of n eigenvalues, $\sum_{j=1}^{k} i_j = n$. The symmetric matrix K_s can be diagonalized using the orthogonal matrix T, and we have the block diagonal matrix $\Lambda = T^{\mathrm{T}} K_s T$, which can be written as

$$\Lambda = \begin{bmatrix} \lambda_1 I_1 & & & \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ & & & \lambda_k I_k \end{bmatrix} \tag{4}$$

Here, each block along the diagonal is the product of one of the distinct eigenvalues, λ_j, of K_s multiplied by the i_j by i_j identity matrix I_j, where i_j is the (algebraic) multiplicity of λ_j. With no loss of generality, we list these blocks along the diagonal of the matrix Λ so that $i_1 \geq i_2 \geq \cdots \geq i_k$.

LEMMA 2. *If the nonzero n by n symmetric matrix K_s commutes with the nonzero skew-symmetric matrix $N_{sk}(G_{sk})$, there exists an orthogonal matrix T that diagonalizes the matrix K_s such that the diagonal matrix*

$$\Lambda = T^{\mathrm{T}} K_s T \tag{5}$$

and the skew-symmetric matrix

$$N = T^{\mathrm{T}} N_{sk} T \ (G = T^{\mathrm{T}} G_{sk} T) \tag{6}$$

both have the same block diagonal structure [9].

LEMMA 3. *If the matrices K_s, G_{sk}, and N_{sk} commute pairwise, then the matrices Λ, G, and N have the same block diagonal structure.*

Proof. By the last lemma, since K_s and G_{sk} commute, the matrices Λ and G have the same block diagonal structure. Also, since K_s and N_{sk} commute, the matrices Λ and N too have the same block diagonal structure. Hence, the three matrices Λ, G, and N have the same block diagonal structure, dictated by the block structure of the matrix Λ. □

Remark 2: When the matrices K_s, G_{sk}, and N_{sk} commute pairwise, by the above lemma, Λ, G, and N have the same block diagonal structure, and we can write the matrices G and N as

$$G = \begin{bmatrix} G_1 & & & \\ & G_2 & & \\ & & \ddots & \\ & & & G_k \end{bmatrix}, \quad \text{and} \quad N = \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_k \end{bmatrix} \tag{7}$$

whose jth (square) diagonal blocks G_j and N_j have the same size as the corresponding jth square diagonal block of Λ. In other words, G_j and N_j are each i_j by i_j skew-symmetric matrices.

LEMMA 4. *The matrices K_s, G_{sk}, and N_{sk} commute pairwise, if and only if the matrices Λ, G, and N also commute pairwise.* □

We are now ready to prove our main result.

THEOREM. *The system described by equation*

$$\ddot{z} + G_{sk}\dot{z} + (K_s + N_{sk})z = 0 \tag{8}$$

is unstable if the nonzero n by n matrices K_s, G_{sk}, and N_{sk} commute pairwise, i.e.,

$$N_{sk}K_s = K_s N_{sk}, \ K_s G_{sk} = G_{sk} K_s, \quad \text{and} \quad N_{sk} G_{sk} = G_{sk} N_{sk} \tag{9}$$

Proof. Using the transformation $z(t) = T\,y(t)$ and premultiplying by T^{T}, Eq. (3) becomes

$$\ddot{y} + G\,\dot{y} + (\Lambda + N)\,y = 0 \tag{10}$$

where the matrices Λ, G, and N are given in Eqs. (5) and (6).

Note that $N = \mathrm{diag}(N_1, N_2, \ldots, N_k)$ and $G = \mathrm{diag}(G_1, G_2, \ldots, G_k)$ have the same block diagonal structure. The matrices N and G therefore commute if and only if their corresponding blocks commute, i.e., $N_j G_j = G_j N_j$, $j = 1, 2, \ldots, k$. The matrices N_j and G_j are normal, and therefore unitarily diagonalizable; they also commute, and therefore are simultaneously unitarily diagonalizable [17]. We therefore have i_j by i_j unitary matrices U_j such that

$$U_j^* N_j U_j = \Xi_j, \quad \text{and} \quad U_j^* G_j U_j = \Theta_j, \quad j = 1, 2, \ldots, k, \tag{11}$$

where Ξ_j and Θ_j are diagonal matrices containing the eigenvalues of the skew-symmetric blocks N_j and G_j, which are either zero or conjugate pairs of pure imaginary numbers.

Hence, the block diagonal matrix G can be diagonalized by matrix $U = \mathrm{diag}(U_1, U_2, \ldots U_k)$ because

$$\begin{aligned} U^* G U &= \mathrm{diag}(U_1^*, U_2^*, \ldots, U_k^*)\,\mathrm{diag}(G_1, G_2, \ldots, G_k) \\ &\quad \times \mathrm{diag}(U_1, U_2, \ldots, U_k) \\ &= \mathrm{diag}(U_1^* G_1 U_1,\ U_2^* G_2 U_2,\ \ldots,\ U_k^* G_k U_k^*) \\ &= \mathrm{diag}(\Theta_1, \Theta_2, \ldots, \Theta_k) := \Theta \end{aligned} \tag{12}$$

Similarly, the matrix N can be diagonalized by U because

$$\begin{aligned} U^* N U &= \mathrm{diag}(U_1^*, U_2^*, \ldots, U_k^*)\,\mathrm{diag}(N_1, N_2, \ldots, N_k) \\ &\quad \times \mathrm{diag}(U_1, U_2, \ldots, U_k) \\ &= \mathrm{diag}(U_1^* N_1 U_1,\ U_2^* N_2 U_2,\ \ldots,\ U_k^* N_k U_k) \\ &= \mathrm{diag}(\Xi_1,\ \Xi_2, \ldots,\ \Xi_k) := \Xi \end{aligned} \tag{13}$$

Finally, the transformation U leaves the diagonal matrix Λ unchanged because

$$\begin{aligned} U^* \Lambda U &= \mathrm{diag}(U_1^*, U_2^*, \ldots, U_k^*)\,\mathrm{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k) \\ &\quad \times \mathrm{diag}(U_1, U_2, \ldots, U_k) \\ &= \mathrm{diag}(\lambda_1 U_1^* I_1 U_1,\ \lambda_2 U_2^* I_2 U_2,\ \ldots,\ \lambda_k U_k^* I_k U_k) \\ &= \mathrm{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k) = \Lambda \end{aligned} \tag{14}$$

Using the transformation $y(t) = Ux(t)$ and premultiplying by U^*, Eq. (10) can now be decoupled as

$$\ddot{x} + \Theta\,\dot{x} + (\Lambda + \Xi)\,x = 0 \tag{15}$$

where the matrices Θ, Λ, and Ξ are diagonal. We see then that the system described by Eq. (3) has been transformed into Eq. (15) by the transformation $z(t) = TU\,x(t)$.

As $N \neq 0$, there must exist at least one conjugate pair of nonzero pure imaginary numbers $\pm i\xi$, along the diagonal of matrix Ξ. Therefore, there exist at least two decoupled equations in the set of equations given by Eq. (15) that can be written in the form

$$\ddot{x}_j + i\theta \dot{x}_j + (\lambda \pm i\xi)x_j = 0, \quad \xi \neq 0, \ \ \theta \in R \tag{16}$$

Here, x_j is a component of vector x, and $i\theta$, λ, $i\xi$ are eigenvalues of matrices G, K_s, and N, respectively. Using the ansatz $x(t) = w\exp(\mu t)$ in Eq. (16), it is easily seen that, for any nonzero ξ and any real θ, $x_j(t)$ is unbounded since

$$\mathrm{Re}(\mu) = \pm \frac{\sqrt{2}}{4} \sqrt{\sqrt{(\theta^2 + 4\lambda)^2 + (4\xi)^2} - (\theta^2 + 4\lambda)} \neq 0 \quad (17)$$

The system described by Eq. (3) is therefore unstable, since a root of its characteristic equation has a positive real part.

Remark 3. It should be noted that the result obtained above does not require the symmetric matrix K_s to be positive definite. It could be indefinite, semidefinite, or negative definite. This is because, as seen from Eq. (17), $\mathrm{Re}(\mu) > 0$ for at least one root of the characteristic equation of the system described in Eq. (8) irrespective of the value of $\lambda \in R$.

Corollary 1. *The system*

$$\ddot{z} + G_{sk}\dot{z} + N_{sk}z = 0, \quad G_{sk},\ N_{sk} \neq 0 \quad (18)$$

is unstable if the matrices G_{sk} and N_{sk} commute.

Proof. We set the matrix $K_s = 0$ in Eq. (8). Now, the first two equalities in Eq. (9) are trivially satisfied. Since $K_s = 0$, K_s has one eigenvalue, which is zero, with multiplicity n. Thus, the eigenvalues of K_s are $\lambda_j = 0$, $j \in \{1, n\}$, and Eq. (17) becomes

$$\mathrm{Re}(\mu) = \pm \frac{\sqrt{2}}{4} \sqrt{\sqrt{\theta^4 + (4\xi)^2} - \theta^2} \neq 0 \quad (19)$$

guaranteeing that we have at least one eigenvalue of the system described in Eq. (18) whose real part is positive, as long as $N_{sk} \neq 0$.

Corollary 2. *The system*

$$\ddot{z} + N_{sk}z = 0, \quad N_{sk} \neq 0 \quad (20)$$

is unstable.

Proof: We set $K_s = 0$ and $G_{sk} = 0$ in Eq. (8). Equation (17) now becomes

$$\mathrm{Re}(\mu) = \pm \frac{1}{\sqrt{2}} \sqrt{|\xi|} \quad (21)$$

again guaranteeing that we have at least one eigenvalue of the system described in Eq. (20) whose real part is positive, as long as $N_{sk} \neq 0$. This of course can be seen more directly by using a unitary transformation that diagonalizes the skew-symmetric matrix N_{sk}. At least two eigenvalues corresponding to the system given in Eq. (20) will then become $\sqrt{\pm i\xi_j}$, and instability follows.

Corollary 3: *For any real number k, the system*

$$\ddot{z} + G_{sk}\dot{z} + (kI + N_{sk})z = 0 \quad (22)$$

is unstable when the matrices G_{sk} and N_{sk} commute.

Proof: The system described by Eq. (22) is a special case of that described in the theorem obtained in this paper. It results from Eq. (8) when the n by n matrix $K_s = kI$, where k is any arbitrary real number. Using the theorem, we see that all the equalities in Eq. (9) are then satisfied by requirements laid out in the corollary. The first two equalities are satisfied because the identity matrix commutes with all matrices. When $k = 0$, Corollary 1 applies. Hence, the system is unstable.

Corollary 4. *A potential system with a circulatory force, described by the equation*

$$\ddot{z} + (K_s + N_{sk})z = 0 \quad (23)$$

is unstable when the matrices K_s and N_{sk} commute.

Proof: We set $G_{sk} = 0$ in Eq. (8). Since the zero matrix commutes with all matrices, the conditions of the theorem then simply require the matrices K_s and N_{sk} to commute. Equation (17) again shows that the system is unstable. This, result, which was first obtained in Ref. [8], arises as a natural consequence of the Theorem. □

Remark 4. Equation (23) is also a generalization of Merkin's theorem [8,9], which we discuss next.

Remark 5. The result given in Corollary 3 for the system described by Eq. (22) was first obtained by Lakhadanov [18]. It is often referred to in the literature as Lakhadanov's theorem. In fact, to the dynamical system

$$\ddot{z} + (kI + N_{sk})z = 0 \quad (24)$$

that was used by Merkin [7]—to establish his celebrated instability theorem—Lakhadanov [18] added gyroscopic forces to obtain the system described by Eq. (22).

Merkin's theorem states that for a stable potential system *whose frequencies of vibration are all identical*, the addition of any (perturbatory) circulatory forces ($N_{sk} \neq 0$) renders the system unstable, i.e., the system described by Eq. (24) is unstable. Lakhadanov's theorem, which we shall call L for short, therefore expands upon Merkin's instability theorem, and states that the addition of gyroscopic forces—the term in G_{sk} in Eq. (22)—to the unstable circulatory system (Eq. (24)) cannot render it stable if the matrices G_{sk} and N_{sk} commute.

As seen from Corollary 3, Lakhadanov's theorem (L) [18] is a special case of the result given in our theorem. The result of L only deals with a potential system all of whose frequencies of vibration are identical, which is then subjected to gyroscopic forces and circulatory perturbations (see Eq. (22)). Large scale, multidegree-of-freedom physical systems that have all their frequencies of vibration identical are indeed very rare, though they could occur in systems with a very small number of degrees-of-freedom, especially those constrained by requirements of symmetry. In large-scale structural and mechanical systems that may have hundreds, if not thousands, of degrees-of-freedom, though one often does find a few repeated frequencies—such as, say, the frequency of the fourth bending mode of vibration being coincident with the second torsional mode—one almost never has a situation in which all the frequencies of vibration are identical. This makes result L, though elegant, limited in its use to actual real-life physical systems.

Remark 6: There is yet another way of interpreting the result given in the theorem obtained. It shows the effect of adding (perturbatory) circulatory forces to gyroscopically stabilized systems.

The result says then that the addition of a (perturbatory) circulatory force N_{sk} to gyroscopically stabilized potential systems, under the proviso that the commutators $[G_{sk}, N_{sk}] = [K_s, G_{sk}] = [K_s, N_{sk}] = 0$, causes these systems to lose stability.

Remark 7: To place the present result in a broader context, we consider another elegant and useful theorem first apparently obtained, again, by Lakhadanov [18], and independently by Karapetyan [19]. They generalized a result obtained previously by Bottema for 2 and 3 degree-of-freedom systems [20].[1] Their significant and outstanding contribution states that the system described by the equation

$$\ddot{x} + G_{sk}\dot{x} + (K + N_{sk})x = 0 \quad (25)$$

is unstable if $\mathrm{trace}(G_{sk}N_{sk}) \neq 0$. Here, K is a symmetric matrix, while G_{sk} and N_{sk} are skew symmetric. The matrices K, G_{sk}, and N_{sk} are general, noncommuting matrices. We shall call this the Bottema–Lakhadanov–Karapetyan result, or BLK for short.

It should be noted that when $\mathrm{trace}(G_{sk}N_{sk}) = 0$, BLK cannot be applied, and it does not provide any information regarding the stability or instability of the system described by Eq. (25). But when the conditions (see Eq. (9)) of our main theorem are satisfied and $\mathrm{trace}(G_{sk}N_{sk}) = 0$, the system is assuredly unstable. Thus, our main theorem provided herein can yield information about the stability of the system (Eq. (25)) in situations where BLK cannot be applied. However, the condition $\mathrm{trace}(G_{sk}N_{sk}) \neq 0$ is generic (see Remark 8) and therefore is much more widely applicable. This is

[1]The author thanks an anonymous reviewer for pointing out this reference.

because the set of matrices for which the commutation conditions of the theorem are true is far smaller than the set of matrices for which $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$. Yet, as we will see later on, systems with nongeneric properties can contain uncountably infinite members, and a particular system under consideration could have a nongeneric property (see Remarks 8 and 9).

As stated before in Remarks 4 and 5, Merkin's famous theorem has been generalized [8,9]. This Generalized Merkin result says that if to a stable potential system $\ddot{x} + K_s x = 0$ one adds a (perturbatory) circulatory force N_{sk} that commutes with K_s, the new system obtained, namely, $\ddot{x} + (K_s + N_{\mathrm{sk}})x = 0$, will be rendered unstable in a flutter instability. A critical aspect that connects the recent generalization given in Ref. [9] to Merkin's theorem is that for such commutation to occur for a nonzero skew-symmetric matrix N_{sk}, the symmetric matrix K_s must have at least one eigenvalue with multiplicity greater than 1. One can therefore place in context the contribution of this paper through the following observation: just as the result of L given in Eq. (22) provides instability results that expand Merkin's result through the addition of gyroscopic forces to a circulatory system (see Remark 5), the theorem in this paper similarly provides new instability results that expand the Generalized Merkin result, also through the addition of gyroscopic forces to a circulatory system. In other words, L stands in relation to Merkin's theorem in a manner analogous to the way the present theorem stands in relation to the Generalized Merkin result. But unlike Merkin's theorem and L, both the Generalized Merkin result (as pointed out in Ref. [9]) and the present theorem do away with the highly restrictive requirement that all the frequencies of vibration of the system be identical, making these newer results more applicable to real-life systems in various areas of application in physics and engineering. In other words, our theorem states that given an unstable potential circulatory system in which the matrices K_s and $N_{\mathrm{sk}}(\neq 0)$ commute, the system cannot be stabilized by the addition of a gyroscopic matrix G_{sk} that commutes with K_s and N_{sk}.

Remark 8. As mentioned before, the BLK result is indeed more widely applicable to the system described in Eq. (25) since the property that $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) \neq 0$ is, mathematically speaking, generic (or in common parlance, typical) among members of the set of skew symmetric matrices. The commutation conditions that are required to be satisfied in the Theorem herein require a much larger set of equations to be satisfied than the single condition that is required to be satisfied by BLK, and are therefore far less widely applicable.

More specifically, what is meant by a generic property is the following [21]. Consider a given set S of objects (on which a topology has been defined so that we can speak of one element being "close" to another). Let there be some property P that each of the elements, s, of S may or may not have. Then property P is generic relative to S under the following circumstances: (a) if $s \in S$ possesses property P, then there exists a (open) neighborhood A of s such that every $s \in A$ also possesses property P, and (b) every neighborhood in S contains an element possessing property P. If either one of these conditions is not satisfied, the property is called nongeneric (mathematically, nontypical). Clearly then, the property $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) \neq 0$ is mathematically generic relative to the set of skew-symmetric matrices.

However, it should be pointed out that those properties P that may not be mathematically generic relative to a given set S can be, and very often are, important in the physical world, and especially so for our understanding of physical phenomena. For example, an orbit that has the property P of being isolated and periodic—a limit cycle—of a nonlinear dynamical system is nongeneric (and nontypical) relative to the set S of all orbits of the dynamical system in phase space. Yet, it is of great importance in describing and understanding the behavior and physics of the nonlinear system. More generally, smooth curves drawn in 3D phase space that have the property P of being closed, are also nontypical relative to the set of smooth curves that can be drawn; yet they are of significant importance in physics and engineering since they represent periodic orbits—often a key to our understanding of physical phenomena. Another way of saying this is that though the chance that a (random) smooth curve drawn in 3D phase space would exactly close on itself is infinitesimally small, such closed curves represent closed periodic orbits that are quintessential to our understanding of physical phenomena. From a theoretical standpoint, one can think of the property of a square matrix being singular. This is a mathematically nongeneric property relative to the set of square matrices, and matrices with this property belong to a nongeneric (nontypical) set. In common parlance, one might assert that: "in general," square matrices are not singular. However, this "nongeneral" or nontypical property has profound theoretical importance in our understanding and prediction of the behavior of almost every dynamical system. For example, the eigenvalue problem, among others, rests on setting the determinant of a matrix to zero, thereby utilizing and basing our understanding of dynamical systems on a matrix that has a nongeneric property, which we might describe as belonging to a very special not-at-all-commonly-found set of matrices.

In like manner, the set of skew symmetric matrices G_{sk} and N_{sk} that have the nongeneric (nontypical) property $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$ could arise in the modeling of physical applications, and can constitute sets whose members are, in fact, uncountably infinite, as shown below.

Remark 9. The theorem obtained herein opens up the possibility of being useful in cases where $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$. It shows that if $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$ and the conditions of the theorem are satisfied, the system is unstable. In this sense, the theorem complements the BLK result. The BLK result and the present Theorem can together be thought of, in a sense, as the "circulatory counterpart" of the celebrated Kelvin–Tait–Chetaev theorem [4,5], which states that the addition of a (perturbatory) positive definite damping matrix (force) D to gyroscopically stabilized systems will cause them to lose stability.

It should be noted that while the statement of BLK (see Remark 7, Eq. (25)) is equivalent to the statement that if the system described by Eq. (25) is stable then $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$, it does not state that the converse is also true. And it is the acceptance of the converse, namely, that $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$ implies stability, which is sometimes erroneously invoked in engineering practice.

The cause of this misunderstanding is often the observation that the property $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) \neq 0$ is generic, and loosely speaking, typical for the set of skew symmetric matrices G_{sk} and N_{sk}. So, given any two skew symmetric matrices, this property is typically satisfied, and by the BLK theorem (see Remark 7), the system described by Eq. (25) is unstable. Therefore, when this property is not satisfied and $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$—and this is the erroneous step—the system must be stable.

To illustrate some of these ideas, we consider the following two examples.

Example 1. Consider the system described by Eq. (25) with the following matrices:

$$K_s = \begin{bmatrix} k_1 I_2 & 0 \\ 0 & k_2 I_2 \end{bmatrix}, \quad G_{\mathrm{sk}} = \begin{bmatrix} 0 & -g & 0 & 0 \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & -g \\ 0 & 0 & g & 0 \end{bmatrix}, \quad \text{and}$$
$$N_{\mathrm{sk}} = \begin{bmatrix} 0 & n & 0 & 0 \\ -n & 0 & 0 & 0 \\ 0 & 0 & 0 & -n \\ 0 & 0 & n & 0 \end{bmatrix} \tag{26}$$

where I_2 is the 2×2 identity matrix, and k_1, k_2, g, and n are arbitrary real constants. We find that the conditions of our theorem are satisfied since the three matrices commute pairwise, and therefore the gyroscopic system is unstable for arbitrary (infinitesimal) perturbations (given by the magnitude of n). In fact, the real parts of four of the eigenvalues of this system, μ_1, μ_2, μ_3, and μ_4, can be explicitly obtained, and are

$$\mathrm{Re}(\mu_1 \text{ and } \mu_2) = \frac{1}{4}\sqrt{2\sqrt{(g^2+4k_1)^2+16n^2}-2(g^2+4k_1)} \quad (27)$$

$$\mathrm{Re}(\mu_3 \text{ and } \mu_4) = \frac{1}{4}\sqrt{2\sqrt{(g^2+4k_2)^2+16n^2}-2(g^2+4k_2)} \quad (28)$$

These roots are seen to be positive as long as $n \neq 0$, no matter how small its value, confirming the result given by the theorem. The real parts of the other four roots have the same absolute values as those given in Eqs. (27) and (28), except that they are negative.

Yet, we find that $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$. This makes the BLK result inapplicable. It cannot be used to answer whether the system is stable or unstable, and the question of the stability of the system would remain unresolved were the theorem developed herein not available. Furthermore, this example points out that the condition $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$ does not, in general, imply stability.

Example 2. We next consider a system again described by Eq. (25) but now with the pairwise noncommuting matrices given by

$$K_s = \mathrm{diag}(k_1, k_2, k_3, k_4), \quad G_{\mathrm{sk}} = \begin{bmatrix} 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & -b \\ 0 & 0 & b & 0 \end{bmatrix}, \quad \text{and}$$

$$N_{\mathrm{sk}} = \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{bmatrix} \quad (29)$$

where a and b are arbitrary constants at least one of which is different from zero, and the constants k_i, $i = 1, 2, 3, 4$, are all unequal from one another. Since K_s now does not have the same block diagonal structure as $G_{\mathrm{sk}}(N_{\mathrm{sk}})$, the matrices K_s and G_{sk} (K_s and N_{sk}) do not commute, and therefore the present theorem is inapplicable, since the equalities given in Eq. (9) are not satisfied. However, $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 2(a^2+b^2) > 0$, and hence by BLK [18–20], the system described by Eq. (25) with the set of matrices given in Eq. (29) is assuredly unstable.

The two examples shown above can be extended to $4n$ by $4n$ matrices in a straightforward manner by adding 4×4 diagonal blocks that have structures shown in Eqs. (26) and (29). We therefore notice that, in general, the present theorem as well as the BLK result may be useful, depending on the application at hand, though of course the genericity of the BLK result leads to its far wider applicability (see Remark 8).

Finally, going back to Example 1 above, we point out that the property $\mathrm{trace}(G_{\mathrm{sk}}N_{\mathrm{sk}}) = 0$ is shared by all 4×4 matrices G_{sk} and N_{sk} with structures shown in Eq. (26) (and more generally, by all their $4n$ by $4n$ matrix counterparts mentioned above) irrespective of the actual values of g and n. They therefore constitute an uncountably infinite set of dynamical systems that have this property. In what follows, we shall call systems described by Eq. (3) as G–K–N systems, for short.

Remark 10. More generally, the characteristic polynomial of any n degree-of-freedom G–K–N system has degree $2n$, and can be written as

$$\mu^{2n} + a_1\mu^{2n-1} + a_2\mu^{2n-2} + a_3\mu^{2n-3} \cdots + a_{2n-1}\mu + a_{2n} = 0 \quad (30)$$

Since $G = -G^{\mathrm{T}}$, the coefficient of the term μ^{2n-1}, namely, $a_1 = \mathrm{trace}(G) \equiv 0$. When $a_1 = 0$, if the system described by Eq. (3) is stable, then $a_{2i-1} = 0$, $i = 2, 3, \ldots, n$ (see Theorem 1, in Ref. [19]). Hence, when even a single coefficient of an odd power of μ in the characteristic polynomial is not zero, the system is unstable. It is a mathematically nongeneric property (see Remark 8) for all the coefficients of the odd powers of μ in a $2n$-degree polynomial to be zero, relative to the set of all $2n$-degree polynomials (see Remark 8 for the definition of nongeneric). Alternatively put, it is a generic property that at least one coefficient of an odd power of μ in the polynomial in Eq. (30) is nonzero, relative to G–K–N systems. And this is what is exactly meant, in a more precise manner, when Ref. [22] states that:

$$\text{in general, G-K-N systems are unstable} \quad (31)$$

That is, such systems possess the mathematically generic property that not all the coefficients of odd powers in μ of their characteristic polynomials are zero, and therefore they are in general unstable, relative to the set of G–K–N systems. Exceptions though, however infrequent, may arise when a particular system does not satisfy the genericity property underlying such a statement.

Use of a statement like (31) without explicit specification and/or knowledge of the cause of the genericity has some important drawbacks. It provides no direction to the stability analyst on whether a particular system under consideration falls in that general category to which such a statement refers. One would need to further explore whether the underlying property that defines the genericity is satisfied to make progress on assessing its stability, because a particular system under consideration may not satisfy this genericity property. From a practical engineering standpoint, this last consideration is paramount, and it rules out the sole use of such statements in definitively ascertaining the stability or instability of a given particular dynamical system unless considerably more information about that particular system is obtained and incorporated.

Remark 11. Though the gyroscopic circulatory system considered in this paper is seemingly simple and has been worked on for over 60 years now, its stability analysis still remains incomplete since there can be, it appears, an uncountably infinite number of nongeneric G–K–N systems to which the BLK result and the present theorem are both inapplicable. This is an important area for future research and will not be discussed further here. It will be the topic of a later communication.

Conclusions

This paper presents a set of conditions that assure the instability of gyroscopic circulatory systems. It is shown that when the potential, gyroscopic, and circulatory matrices pairwise commute, the system is unstable. Consequences of the commutation on the multiplicity of the eigenvalues of the potential system are developed. The well-known Lakhadanov instability theorem, which is restricted to systems in which the potential part of the dynamical system has all its frequencies of vibration identical, appears as a special case of the result obtained herein. The removal of this stringent restriction on the potential system renders the present result applicable to a wider class of real-life applications that deal with multidegree-of-freedom systems found in physics, engineering, and other fields of application.

The contribution of the main result in this paper is shown in the context of the present literature. The result can be construed as an extension of the Generalized Merkin theorem when gyroscopic forces are added to potential circulatory systems. The relationship of the result to the Generalized Merkin theorem is analogous to the relationship of Lakhadanov's result to Merkin's theorem. It is shown that general statements like (31) about sets of dynamical systems that simply state they are in general (un)stable need to be used with some caution. The use of such statements without proper knowledge and/or attribution of the reason for their genericity (or nongenericity) do not guarantee either stability or instability when applied to a given, particular dynamical system of interest, because they then lack the definitive knowledge required to unequivocally ascertain the system's stability or instability. Equally relevant is the fact that the set of systems that may have a mathematically nongeneric property could be: (1) uncountably infinite in number, and (2) very important in building theoretical constructs. Besides being useful in modeling the physical world, systems with nongeneric properties can often provide a framework for understanding observed phenomena. It should be noted that despite the simplicity of the system considered in this paper and the fact that its stability has been a subject of interest for more than 60 years, we do not yet have a general theory that covers all the nongeneric systems that may arise. This leaves the general stability analysis of gyroscopic circulatory systems incomplete as of now.

The new result obtained in this paper is also compared with the elegant and long-established BLK instability theorem. Though far less widely applicable than the BLK theorem, it is shown to be complementary to it in that it could be useful in providing information on the nature of the stability of dynamical systems for which the elegant, older theorem might be inapplicable.

References

[1] Arrowsmith, D. K., and Place, C. M., 1998, *Dynamical Systems, Differential Equations, Maps and Chaotic Behavior*, Chapman and Hall, New York, pp. 77–81.

[2] Perko, L., 1991, *Differential Equations and Dynamical Systems*, Springer Verlag, New York, pp. 120–127.

[3] Kirillov, O. N., 2013, *Nonconservative Stability Problems of Modern Physics*, Walter de Gruyter, Berlin.

[4] Kelvin, B. W. T., and Tait, P. G., 1879, *Treatise on Natural Philosophy*, Cambridge University Press, New York.

[5] Zajac, E. E., 1964, "The Kelvin-Tait-Chetaev Theorem and Extensions," J. Astronaut. Sci., **11**(2), pp. 46–49.

[6] Huseyin, K., Hagedorn, P., and Teschner, W., 1983, "On the Stability of Linear Conservative Gyroscopic Systems," J. Appl. Math. Phys., **34**(6), pp. 807–815.

[7] Merkin, D. R., 1974, *Gyroscopic Systems*, Nauka, Moscow, Russia.

[8] Bulatovic, R. M., 1999, "On the Stability of Linear Circulatory Systems," Z. Angew. Math. Phys., **50**, pp. 669–674.

[9] Udwadia, F. E., 2017, "Stability of Dynamical Systems With Circulatory Forces: Generalization of the Merkin Theorem," Am. Inst. Aeronaut. Astronaut., **55**(9), pp. 2853–2858.

[10] Beletsky, V. V., 1995, "Some Stability Problems in Applied Mechanics," Appl. Math. Comput., **70**(2–3), pp. 117–141.

[11] Zhuravlev, V. P., and Klimov, D. M., 2010, "Theory of the Shimmy Phenomenon," Mech. Solids, **45**(3), pp. 324–330.

[12] Hagedorn, P., Eckstein, M., Heffel, E., and Wagner, A., 2014, "Self-Excited Vibrations and Damping in Circulatory Systems," ASME J. Appl. Mech., **81**(10), p. 101008.

[13] Spelsberg-Korspeter, G., Hochlenert, D., and Hagedorn, P., 2011, "Self-Excitation Mechanisms in Paper Calendars Formulated as a Stability Problem," Tech. Mech., **31**(1), pp. 15–24.

[14] Dowell, E. H., 2011, "Can Solar Sails Flutter?," AIAA J., **49**(6), pp. 1305–1307.

[15] Ziegler, H., 1953, "Linear Elastic Stability, a Critical Analysis of Methods—Part 1," Z. Angew. Math. Phys., **4**(2), pp. 89–121.

[16] Ziegler, H., 1953, "Linear Elastic Stability, a Critical Analysis of Methods—Part 2," Z. Angew. Math. Phys., **4**(3), pp. 167–185.

[17] Horn, R. A., and Johnson, C. R., 1991, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, p. 103.

[18] Lakhadanov, V. M., 1975, "On Stabilization of Potential Systems," J. Appl. Math. Mech., **39**(1), pp. 45–50.

[19] Karapetyan, A. V., 1975, "About the Stability of Nonconservative Systems," Vestn. Mosk. Univ. Ser. 1: Mat. Mekh., **4**, pp. 109–113.

[20] Bottema, O., 1955, "On the Stability of the Equilibrium of a Linear Mechanical System," Z. Angew. Math. Phys., **6**(2), pp. 97–104.

[21] Casti, J., 1992, *Reality Rules: I*, John Wiley, New York, pp. 63–64.

[22] Hagedorn, P., and Hochlenert, D., 2015, *Technische Schwingungslehre*, Verlag-Europa-Lehrmittel, Vollmer GmbH & Co. KG, Haan, Germany, pp. 155–157.