Stability of Dynamical Systems with Circulatory Forces: Generalization of the Merkin Theorem

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This paper develops a general result on the stability of undamped linear multi-degree-of-freedom systems that are subjected to potential and circulatory forces. The result is a generalization of Merkin's celebrated result on flutter instability in stable potential systems that are subjected to circulatory disturbances. Merkin's result appears as a special case of the generalization obtained herein.

I. Introduction

IN RECENT years, there has been a resurgence of interest in the stability of linear dynamical systems, and especially in dissipation-induced instabilities. The area of dynamical instabilities has had a rich history because it falls at the intersection of physics, mathematics, and engineering. Physicists have long been interested in instabilities induced in the study of natural phenomena, mathematicians and mechanicians have been interested in rigorous stability analysis for idealized models, and engineers have been interested in the analysis and design of engineered systems so as to ensure their safe and stable behavior.

A remarkable aspect of these investigations is the often nonintuitive nature of the results that they provide. Dissipation-induced instabilities have been classified into two categories: those induced by damping and those induced by circulatory forces [1]. The first is exemplified by the celebrated Kelvin–Tait–Chetayev theory [2,3], and the second is exemplified by the equally celebrated Merkin theory [4]. Both theories deal with the loss of stability through the addition of arbitrarily small forces to otherwise stable, or stabilized, systems. Although the Kelvin–Tait–Chetayev theory has a long history, the results obtained by Merkin are relatively recent and date to the 1970s [5]. This paper deals with a generalization of Merkin's result.

Circulatory forces arise in many real-life applications. Some examples are self-oscillations (shimmy) in aircraft wheels, flutter in aerospace systems, control of two-legged walking robots, dynamics of brake squealing, and wear in paper calenders [6–11].

To look at Merkin's result [4], consider a multi-degree-of-freedom system described by the equation

$$\tilde{M}\ddot{z} + \tilde{K}z = 0 \tag{1}$$

where $\tilde{M}$ is an n-by-n positive definite matrix; and z is an n-component real, column vector (n vector). The matrices $\tilde{M}$ and $\tilde{K}$ are constant matrices, and the matrix $\tilde{K}$ is real and nonsymmetric.

Using the transformation $y(t) = \tilde{M}^{1/2}z(t)$, Eq. (1) can be rewritten as

$$\ddot{y} + Ky = 0 \tag{2}$$

where the n-by-n matrix $K = M^{-1/2}\tilde{K}M^{-1/2}$.

The system described by Eq. (2) is time reversible, and it is well known that it is stable if, and only if, each eigenvalue of the matrix K is positive and semisimple (the geometric and algebraic multiplicities are equal) [12]. In particular, using the ansatz $y(t) = w\exp(i\sqrt{\mu}t)$ in Eq. (2), it is easily seen that, for a complex eigenvalue $\mu = a \pm ib$, (where a and b are real numbers) of K, the solution

$$y(t) = w\exp[\mathrm{Im}(\sqrt{\mu})t \pm i\mathrm{Re}(\sqrt{\mu})t] \tag{3}$$

is unbounded and the system exhibits an oscillatory (flutter) instability. In Eq. (3), we have denoted

$$\mathrm{Re}(\sqrt{\mu}) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{|\mu|^2} + a}$$

and

$$\mathrm{Im}(\sqrt{\mu}) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{|\mu|^2} - a}$$

The matrix K can be uniquely split into its (real) symmetric and (real) skew-symmetric parts so that $K = K_s + N$, where the symmetric matrix $K_s = (K + K^T)/2$ and the skew-symmetric matrix $N_{\mathrm{sk}} = (K - K^T)/2$. Thus, Eq. (2) can be rewritten as

$$\ddot{y} + (K_s + N_{\mathrm{sk}})y = 0 \tag{4}$$

The force caused by the presence of the skew-symmetric matrix N_{sk} in Eq. (4) is called a circulatory force or a nonconservative positional force: a term believed to have been first introduced by Ziegler [13,14] in his studies on the stability of rods. The force caused by the presence of the symmetric matrix K_s in Eq. (4) is referred to as a potential force because it comes from a potential, and systems in which there are no circulatory forces ($N_{\mathrm{sk}} = 0$) are referred to as potential systems.

Furthermore, because K_s is symmetric, it can be diagonalized by an orthogonal matrix T such that $T^T K_s T = \Lambda$, where Λ is a diagonal matrix containing the eigenvalues λ_i, $i = 1, 2, \ldots, n$ of the matrix K_s, some of which might be identical. Using the transformation $y(t) = Tx(t)$ and premultiplying it by T^T, Eq. (3) becomes

$$\ddot{x} + (\Lambda + N)x = 0 \tag{5}$$

where the skew-symmetric matrix $N = T^T N_{\mathrm{sk}} T$.

With these preliminaries, we are now ready to state Merkin's result [4]. It states the following: "the addition of arbitrarily small circulatory forces to a stable potential system will cause it to become unstable if all the frequencies of vibration of the potential system are equal."

The result is remarkable in its simplicity, and it has initiated a considerable line of research and analysis. Merkin's result [4] deals with instability caused by the addition of circulatory forces to an otherwise (weakly) stable potential system and can be thought of as the counterpart of the Kelvin–Tait–Chetayev result, which deals with

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instability caused by the addition of damping forces to a gyroscopically stabilized system [1].

Using the notation developed, Merkin's result [4] then states the following. Consider the stable potential system described by the equation

$$\ddot{y} + K_s y = 0 \tag{6}$$

in which all the eigenvalues λ_i of the matrix K_s are identical and equal to, say, $\alpha > 0$, so that $K_s = \Lambda = \alpha I$ is a diagonal matrix. Addition of the minutest circulatory force to this system will result in it losing its stability; that is, the system $\ddot{y} + (\alpha I + N_{\mathrm{sk}})y = 0$ will be unstable for arbitrarily small entries in the skew-symmetric matrix N_{sk}.

For systems that can be adequately modeled by a small number of degrees of freedom ($n = 2$ or 3), especially those constrained by considerations of symmetry, this remarkable result has considerable engineering value. However, the restriction that all the eigenvalues of the matrix K_s be identical has considerable consequences in limiting the applicability of this elegant and simple result to many real-life engineering systems. An engineered multi-degree-of-freedom system, except in very rare and exceptional cases, does not usually have all its frequencies of vibration identical. Admittedly, though, in buildings and spacecraft structures, one does often find a couple (or so) of, say, torsional frequencies of vibration that might coincide with some bending frequencies. In fact, in the "space" of symmetric matrices K_s, matrices that have all their eigenvalues equal are not generic because the slightest alteration in the elements of a symmetric matrix can cause its repeated eigenvalues to "separate" and become distinct. Hence, the likelihood of having an engineered multi-degree-of-freedom system with all its frequencies identical would, in general, be exceedingly small (except in systems with considerable symmetry), especially when such systems may typically have hundreds, if not thousands, of degrees of freedom. Thus, practical considerations appear to place considerable restrictions on the applicability of Merkin's result [4].

Such thoughts then lead to the following "what if" questions when trying to analyze the stability of multi-degree-of-freedom stable potential systems encountered in engineering practice that may be subjected to circulatory force perturbations:

1) What if not all but some of the frequencies of vibration provided by the n-by-n matrix K_s were identical? Might the potential system become unstable with arbitrarily small circulatory force additions? If so, what kinds of additions might they be?

2) What if just a single frequency of vibration were repeated with a multiplicity of $k < n$, with all the other frequencies distinct? Would stability ensue?

3) What if just two frequencies were identical (multiplicity of $k = 2$), with all the others distinct? Would the potential system be stable under circulatory perturbations?

The three questions address successively broader classes of dynamical systems: 1) those in which the n-by-n matrix K_s has several distinct eigenvalues $k < n$ (each distinct eigenvalue with a different multiplicity, and each multiplicity greater than one), 2) those with only a single repeated eigenvalue with multiplicity of $k < n$, and 3) those that have just a single repeated eigenvalue with multiplicity of $k = 2$. The three classes of dynamical systems correspond to a progressively widening scope of real-life engineering systems.

It is these questions that are the motivating thoughts behind this paper. In trying to answer them, one is led to a general result from which Merkin's result [4] can be deduced as a special case. Before proving the result, so as not to lose the thread of continuity, we state it as follows:

Result 1: Let $K_s > 0$ be an n-by-n symmetric matrix. An addition to the stable potential system $\ddot{y} + K_s y = 0$ of an arbitrarily small circulatory force [as in Eq. (4)] created by any n-by-n skew-symmetric matrix N_{sk} that commutes with K_s will make the potential system unstable in a flutter instability.

In the following, it will be shown that 1) if K_s commutes with $N_{\mathrm{sk}} \neq 0$, then K_s has at least one repeated eigenvalue (say, α with a multiplicity of two or greater); and 2) if K_s has even a single repeated eigenvalue, there are an uncountable infinity of skew-symmetric matrices N_{sk} that commute with it.

In view of this, the preceding result says that, for a stable potential system in which K_s has repeated eigenvalues, there exist (an uncountable infinity of) circulatory forces for which the minutest addition will cause the system to lose stability in flutter. This is indeed also true for a stable potential system in which K_s may have just a single repeated eigenvalue with a multiplicity of two, irrespective of its other eigenvalues that may, for that matter, all be distinct.

As explained before, although engineering structures with a large number of degrees of freedom rarely have all their frequencies of vibration identical as demanded by Merkin's result [4], the coincidence of two (or a few) frequencies of vibration (say, the second torsional frequency and the fourth bending frequency) can, and do, occasionally occur in structural and mechanical systems, especially spacecraft and building systems. The aforementioned generalization of Merkin's result therefore provides a wider compass of applicability to real-life engineering structures.

To show how the aforementioned result leads to Merkin's theorem [4], consider the special situation when the multiplicity of the eigenvalue $\alpha > 0$ of the n-by-n matrix K_s equals n. Then, $K_s = \Lambda = \alpha I$. Noting that K_s now commutes with any and all n-by-n skew-symmetric matrices N_{sk}, Result 1 states that such a stable potential system (with all its frequencies equal) will become unstable through the addition of an arbitrary circulatory force generated by any skew-symmetric matrix N_{sk}, and this is Merkin's theorem [4].

II. Generalization of Merkin Result

We begin by stating some auxiliary results in a set of lemmas and remarks in order to achieve our final objective.

Lemma 1: The matrices K_s and N_{sk} commute if, and only if, the matrices $\Lambda = T^T K_s T$ and $N = T^T N_{\mathrm{sk}} T$ commute.

Proof: If K_s and N_{sk} commute, $K_s N_{\mathrm{sk}} = N_{\mathrm{sk}} K_s$ or $T \Lambda T^T N_{\mathrm{sk}} = N_{\mathrm{sk}} T \Lambda T^T$. Premultiplying both sides of this relation by T^T and postmultiplying it by T yields $\Lambda N = N \Lambda$.

If Λ and N commute, $\Lambda N = N \Lambda$ or $T^T K_s T N = N T^T K_s T$. Premultiplying both sides of this relation by T and postmultiplying them by T^T yields $K_s N_{\mathrm{sk}} = N_{\mathrm{sk}} K_s$. □

Thus, instead of considering the potential system described by Eq. (6) and analyzing its stability when the circulatory perturbation given by the matrix N_{sk} is added, we could equivalently consider the potential system $\ddot{x} + \Lambda x = 0$ and analyze its stability when the circulatory perturbation given by the matrix $N = T^T N_{\mathrm{sk}} T$ is added [see Eq. (5)].

Lemma 2. If the n-by-n diagonal matrix $\Lambda\,(K_s)$ commutes with the skew-symmetric matrix $N(N_{\mathrm{sk}})$ and $\Lambda\,(K_s)$ has distinct eigenvalues, then the matrix $N(N_{\mathrm{sk}})$ must be the zero matrix.

Proof: The matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda_i \neq \lambda_j$ when $i \neq j$. Because $\Lambda N = N \Lambda$, we obtain

$$(\lambda_i - \lambda_j) N(i, j) = 0, \qquad \forall\ i \quad \text{and} \quad j \tag{7}$$

where we have denoted the (i, j) element of the matrix N by $N(i, j)$. In particular, when $i \neq j$, then $\lambda_i \neq \lambda_j$; and from Eq. (7), it follows that the elements $N(i, j) = 0$, $i \neq j$. Thus, all the offdiagonal elements of the skew-symmetric matrix N are zero, and hence $N = 0$. Using Lemma 1 and the fact that, when $N = 0$, the matrix $N_{\mathrm{sk}} = T N T^T = 0$, the result follows when the diagonal matrix Λ is replaced by K_s and the matrix N by N_{sk}. □

Remark 1: We have shown that, if the skew-symmetric matrix $N(N_{\mathrm{sk}})$ is nonzero and it commutes with the matrix $\Lambda(K_s)$, then $\Lambda(K_s)$ must have repeated eigenvalues.

In what is to follow, we will need the following definition.

Definition: Consider an n-by-n block diagonal matrix

$$A = \mathrm{diag}(A_1, A_2, \ldots, A_k) \tag{8}$$

in which the sth (square) diagonal block A_s has dimensions i_s by i_s with $i_1 \geq i_2, \ldots, \geq i_k$, and another n-by-n block diagonal matrix

$$B = \operatorname{diag}(B_1, B_2, \ldots, B_r) \tag{9}$$

for which the pth (square) diagonal block B_p has dimensions j_p by j_p with $j_1 \geq j_2, \ldots, \geq j_r$.

We shall say that matrices A and B have the same block diagonal structure if

$$k = r, \quad \text{and} \quad i_s = j_s, \qquad s = 1, \ldots, k \tag{10}$$

That is, if 1) matrices A and B have the same number of blocks along their diagonals ($k = r$), and 2) the corresponding (square) diagonal blocks of the two matrices have the same dimensions as we go down from the top left to the bottom right along their respective diagonals, then, we say that A and B have the same block diagonal structure. We will need to use this concept as we go along. □

To clarify the concept, we show three numerical examples of matrices with the same block diagonal structure:

Numerical Example 1: Consider the diagonal matrix $\Lambda = \operatorname{diag}(2, 2, 2, 5, 5, 3)$. The matrix Λ can be written as shown in Eq. (11) in block diagonal form. The identity matrix I_1 of the first block has dimension $i_1 = 3$, the identity matrix I_2 has dimension $i_2 = 2$, and the identity matrix I_3 has dimension $i_3 = 1$. Note that the dimensions of the matrices I_1, I_2, and I_3 correspond to the respective multiplicities of the distinct eigenvalues 2, 5, and 3 of the matrix Λ.

The matrix Λ and the skew-symmetric matrix N given by

$$\Lambda = \begin{bmatrix} 2I_1 & & \\ & 5I_2 & \\ & & 3I_3 \end{bmatrix} \tag{11}$$

and

$$N = \begin{bmatrix} N_1 & & \\ & N_2 & \\ & & N_3 \end{bmatrix} = \begin{bmatrix} 0 & a & b & 0 & 0 & 0 \\ -a & 0 & c & 0 & 0 & 0 \\ -b & -c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d & 0 \\ 0 & 0 & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{12}$$

then have the same block diagonal structure because N is block diagonal, and the dimension of the diagonal block $N_1 = 3(= i_1)$, of $N_2 = 2(= i_2)$ and of $N_3 = 1(= i_3)$. In general, arbitrary real values can be chosen for a, b, c, and d in the diagonal blocks of N. However, we note for future use that, if the matrix N is to be nonzero, at least one of the three diagonal blocks of the matrix N must be nonzero; that is, a, b, c, and d, cannot all be zero.

Numerical Example 2: Consider the diagonal matrix $\Lambda = [2, 2, 3, 4, 5]$, which can be written in the form shown in Eq. (13). The identity matrix I_1 in Λ has dimension $i_1 = 2$, and the identity matrices $I_i, i = 2, 3, 4$ are singletons with dimension $i_k = 1$, $k = 2, 3, 4$. The matrix Λ and the skew-symmetric matrix N given by

$$\Lambda = \begin{bmatrix} 2I_1 & & & \\ & 3I_2 & & \\ & & 4I_3 & \\ & & & 5I_4 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

have the same block diagonal structure. Here, the first block of N has dimension $i_1 = 2$, and $i_k = 1$, $k = 2, 3, 4$. The matrix N can be written as $N = \operatorname{diag}(N_1, N_2, N_3, N_4)$, in which N_1 has dimension $i_1 = 2$, and all the other blocks consist of singletons (one-by-one matrices).

Numerical Example 3: Consider the matrix $\Lambda = \operatorname{diag}(2, 2, 0, 0)$. Then, the matrices

$$\Lambda = \begin{bmatrix} 2I_1 & \\ & 0I_2 \end{bmatrix} \quad \text{and}$$

$$N = \begin{bmatrix} N_1 & \\ & N_2 \end{bmatrix} = \begin{bmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{bmatrix} \tag{14}$$

have the same block diagonal structure for arbitrary values of a and b of the skew-symmetric matrix N. The dimension of the first block of Λ is $i_1 = 2$, and the dimension of the second block is $i_2 = 2$. The corresponding diagonal blocks of the matrix N have the same dimensions, and b can be any (real) number. If further one were to require the matrix N to be nonzero, at least one of the two blocks (N_1, N_2) would need to be nonzero. We note that b can be any (real) number.

Lemma 3: The n-by-n matrix $\Lambda(K_s)$ commutes with a skew-symmetric matrix $N(N_{\text{sk}}) \neq 0$ if, and only if, 1) $\Lambda(K_s)$ has repeated eigenvalues where the diagonal matrix $\Lambda = T^T K_s T$, with T orthogonal; and 2) the diagonal matrix Λ and the skew-symmetric matrix $N = T^T N_{\text{sk}} T$ have the same block diagonal structure: each block along the diagonal of the matrix N is an arbitrary skew-symmetric matrix.

Proof: The first requirement has been proved in Lemma 2; that is, the matrix $\Lambda(K_s)$ has repeated eigenvalues, else the skew-symmetric matrix $N(N_{\text{sk}})$ would have to be the zero matrix.

Let K_s have $k < n$ distinct eigenvalues. Without loss of generality, we can assume that any repeated eigenvalues of K_s lie continuously along the diagonal of Λ and are arranged so that

$$\Lambda = \begin{bmatrix} \lambda_1 I_1 & & & \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ & & & \lambda_k I_k \end{bmatrix} \tag{15}$$

where I_j, $j = 1, \ldots, k$ denote i_j by i_j identity matrices, with $i_1 \geq i_2, \ldots, \geq i_k$. The multiplicity of the repeated eigenvalue λ_j is i_j.

Assume first that K_s and N_{sk} commute. By Lemma 1, this is equivalent to saying that Λ and $N = T^T N_{\text{sk}} T$ commute. Proceeding as in the proof of Lemma 2, because there are k distinct eigenvalues now, we find that the skew-symmetric matrix N must be of a block diagonal matrix that has the form

$$N = \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_k \end{bmatrix} \tag{16}$$

where the matrix N_j is an arbitrary skew-symmetric matrix of dimension i_j by i_j. The corresponding square blocks along the diagonals of the matrices Λ and N have the same dimensions, and hence the matrices Λ and N have the same block diagonal structure. Furthermore, because $N \neq 0$, there is at least one block among the k diagonal blocks of N that is nonzero. Also, if the matrix $\Lambda = \alpha I$, and therefore has only one distinct eigenvalue, then the matrix N that has the same block diagonal structure (so that it commutes with Λ) is any arbitrary n-by-n (real) skew-symmetric matrix.

Conversely, assume that Λ and N have the same block diagonal structure shown in Eqs. (15) and (16), respectively; then, they clearly commute because, on carrying out their multiplication, we find that $\Lambda N = N \Lambda$, or,

$$(T^T K_s T)(T^T N_{\text{sk}} T) = (T^T N_{\text{sk}} T)(T^T K_s T)$$

from which it follows that the matrices K_s and N_{sk} commute because the matrix T is orthogonal. □

Remark 2: It should be noted that the elements of each skew-symmetric block matrix N_i in Eq. (16) are arbitrary.

Remark 3: Given an n-by-n symmetric matrix K_s, for it to commute with a skew-symmetric matrix $N_{\text{sk}} \neq 0$, it must have multiple eigenvalues. Furthermore, in view of the previous remark, there are an uncountably infinite number of skew-symmetric matrices N_{sk} that commute with such a symmetric matrix K_s. All that is required is that the matrix $N = T^T N_{\text{sk}} T$ have the same block diagonal structure as the (diagonal) matrix $\Lambda = T^T K_s T$.

Each of the three numerical examples considered previously showed pairs of matrices Λ and N that commute.

Remark 4: It is important to note that each diagonal block N_i in Eq. (16) has the same dimensions as the multiplicity of the corresponding repeated eigenvalue λ_i [in Eq. (15)] of the n-by-n symmetric matrix K_s. Consequently, if α is a repeated eigenvalue of K_s with multiplicity $n(K_s = \Lambda = \alpha I)$, then the matrix $N_{\text{sk}} = N$ consists of any arbitrary n-by-n skew-symmetric matrix.

Remark 5: If the n-by-n matrix Λ (K_s) with $k < n$ distinct eigenvalues commutes with the skew-symmetric matrix $N(N_{\text{sk}}) \neq 0$, then, because both $\Lambda\,(K_s)$ and $N(N_{\text{sk}})$ are diagonalizable, they can be simultaneously diagonalized [15]. In fact, each diagonal block N_i of the matrix N is skew symmetric, and is therefore a normal matrix that can be diagonalized by a unitary transformation. Hence, there exist unitary matrices U_i, $i = 1, 2, \ldots, k$ such that

$$\bar{U}_i^T N_i U_i = \Xi_i, \qquad j = 1, 2, \ldots, k \tag{17}$$

where $U_i^{-1} = \bar{U}_i^T$ (the overbar denotes the complex conjugate). Each diagonal matrix Ξ_i contains the eigenvalues of the skew-symmetric block N_i, which are either zero or pure imaginary, with each imaginary eigenvalue partnered by its complex conjugate.

The block diagonal matrix N can therefore be diagonalized by the block diagonal matrix

$$U = \text{diag}(U_1, U_2, \ldots, U_k) \tag{18}$$

so that

$$\begin{aligned} U^{-1} N U &= \text{diag}(\bar{U}_1^T, \bar{U}_2^T, \ldots, \bar{U}_k^T)\text{diag}(N_1, N_2, \ldots, N_k) \\ &\quad \times \text{diag}(U_1, U_2, \ldots, U_k) \\ &= \text{diag}(\bar{U}_1^T N_1 U_1, \bar{U}_2^T N_2 U_2, \ldots, \bar{U}_k^T N_k U_k) \\ &= \text{diag}(\Xi_1, \Xi_2, \ldots, \Xi_k) = \Xi \end{aligned} \tag{19}$$

The diagonal matrix Ξ has elements that are either zero or pure imaginary numbers (along with their conjugates). Because $N \neq 0$, there must be at least one pair of pure imaginary numbers along the diagonal of Ξ.

Lemma 4: If K_s and $N_{\text{sk}} \neq 0$ commute, then they can be simultaneously diagonalized by the matrix TU, where $\Lambda = T^T K_s T$; and the matrix U is given by Eqs. (17) and (18). The eigenvalues of the matrix $K_s + N_{\text{sk}}$ are given by the diagonal entries of $\Lambda + \Xi$, at least one of which is a complex conjugate pair.

Proof: We have shown in Lemma 2 that, if K_s and $N_{\text{sk}} \neq 0$ commute, then K_s must have repeated eigenvalues; hence, the matrix Λ must be expressible in the form given in Eq. (15). In Lemma 3, it has been shown that commutation implies that the matrix must have the same block diagonal structure as Λ.

We then have

$$\begin{aligned} U^{-1} T^T K_s T U &= U^{-1} \Lambda U \\ &= \text{diag}(\bar{U}_1^T, \bar{U}_2^T, \ldots, \bar{U}_k^T)\text{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k) \\ &\quad \times \text{diag}(U_1, U_2, \ldots, U_k) \\ &= \text{diag}(\lambda_1 \bar{U}_1^T I_1 U_1, \lambda_2 \bar{U}_2^T I_2 U_2, \ldots, \lambda_k \bar{U}_k^T I_k U_k) \\ &= \text{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k) = \Lambda \end{aligned} \tag{20}$$

which is a diagonal matrix. Also, using Eq. (19), we have

$$U^{-1} T^T N_{\text{sk}} T U = U^{-1} N U = \Xi \tag{21}$$

which is a diagonal matrix that has at least one pair of pure imaginary numbers. Since

$$U^{-1} T^T (K_s + N_{\text{sk}}) T U = \Lambda + \Xi,$$

the eigenvalues of the matrix $K_s + N_{\text{sk}}$ are given by the sum of the diagonal elements of the matrices Λ and Ξ, therefore, there is at least one pair of complex eigenvalues (which would be pure imaginary). □

Remark 6: When $K_s > 0$, then $\Lambda > 0$ and $K_s + N_{\text{sk}}$ has at least one pair eigenvalues of the form $a \pm ib$, with $a, b > 0$. In fact, all the complex eigenvalues of $K_s + N_{\text{sk}}$ have this form. □

This leads to our final result.

Theorem 1: If to the stable potential system $\ddot{y} + K_s y = 0$ one adds any arbitrary (nonzero) circulatory term given by the skew-symmetric matrix N_{sk} that commutes with K_s, then, the potential system will be rendered unstable in a flutter instability.

Proof: Consider the system $\ddot{y} + (K_s + N_{\text{sk}})y = 0$. If the n-by-n matrix $K_s > 0$ commutes with the skew-symmetric matrix $N_{\text{sk}} \neq 0$, then at least one pair of eigenvalues of the matrix $K_s + N_{\text{sk}}$ has the form $a \pm ib$, with $a, b > 0$ (Remark 6). Hence the system is unstable with flutter instability, as described in Eq. (3). □

We now deduce in a more formal manner Merkin's Theorem from this result.

Corollary 1 (Merkin's theorem): If the n-by-n matrix $K_s > 0$ has all its eigenvalues identical then the eigenvalues of the matrix $K_s + N_{\text{sk}}$ are complex for any arbitrary skew-symmetric matrix $N_{\text{sk}} \neq 0$.

Proof: Since $K_s = \alpha I$, it commutes with every n-by-n skew-symmetric matrix N_{sk}. By Theorem 1, the matrix $K_s + N_{\text{sk}}$ has at least one pair of complex eigenvalues for any (nonzero) skew-symmetric matrix N_{sk}. Hence arbitrarily minute circulatory perturbations to the stable potential system $\ddot{y} + K_s y = 0$, will cause flutter instability. □

We have shown that if the skew-symmetric matrix N_{sk} is nonzero and it commutes with the matrix K_s, then K_s must have repeated eigenvalues (see Remark 1). The explicit nature of the matrices N_{sk} that commute with the n-by-n matrix K_s that has $k < n$ distinct eigenvalues can be found as follows. This leads us to a second result.

Theorem 2: Let T be the orthogonal transformation such that the diagonal matrix $\Lambda = T^T K_s T$ has repeated eigenvalues. The matrix $N = \text{diag}(N_1, N_2, \ldots N_k)$ that has the same block diagonal structure as Λ, and where N_i, $i = 1, 2, \ldots, k$ are arbitrary skew-symmetric matrices, will always commute with Λ, and the matrix $N_{\text{sk}} = TNT^T$ will always commute with K_s. Arbitrarily small elements in the matrices N_i, will lead to a flutter instability of the system $\ddot{y} + (K_s + N_{\text{sk}})y = 0$.

Proof: This is a direct consequence of Lemma 4. If

$$\Lambda = \begin{bmatrix} \lambda_1 I_1 & & & \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ & & & \lambda_k I_k \end{bmatrix} \tag{22}$$

in which the dimensions of the identity matrices I_j, $j = 1, 2, \ldots, k$ (which we have denoted by i_j) equal the corresponding multiplicities of the eigenvalue, then the matrix

$$N = \begin{bmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_k \end{bmatrix} \tag{23}$$

that has the same block diagonal structure as Λ for arbitrary skew-symmetric matrices N_i, $i = 1, 2, \ldots, k$ will commute with Λ. By Lemma 1, the matrices K_s and N_{sk} will therefore commute. □

We now deduce, in a more formal manner, Merkin's theorem from this result [4].

Corollary 2: For a stable potential system that has a single repeated eigenvalue, with a multiplicity of two or greater, there exist arbitrary circulatory perturbations that will cause the system to become unstable in a flutter instability.

Proof: Say the system has a repeated eigenvalue of α. Hence, the matrix Λ has a block αI_j along its diagonal where the dimension of the identity matrix I_j equals the multiplicity $m = 2$ (or greater) of the eigenvalue. The matrix Λ commutes with a skew-symmetric matrix N for which the jth diagonal block N_j is skew-symmetric and contains arbitrary elements. This guarantees that there are complex eigenvalues of the matrix $\Lambda + N$ of the form $a \pm ib$, with a, $b > 0$; hence, the system is unstable in flutter. □

Remark 7: We note that the corollary is applicable to stable potential systems in which the n-by-n matrix K_s has $k < n$ distinct eigenvalues (each with its own multiplicity) because such systems will have at least one eigenvalue with a multiplicity of two or higher, making the corollary applicable.

Remark 8: As mentioned earlier, Merkin's theorem [4] has led to considerable research in the area of circulatory systems. Recently, a sufficient condition for flutter was developed by Bulatovic [16] that states that, for the circulatory system $\ddot{y} + (K_s + N_{\mathrm{sk}})y = 0$, where K_s is an n-by-n symmetric matrix and N_{sk} is an n-by-n skew-symmetric matrix, if

$$\|N_{\mathrm{sk}}\|^2 > \|K_s\|^2 - \frac{1}{n}[\mathrm{Trace}(K_s)]^2 \tag{24}$$

then the system is unstable. In Eq. (24), $\|\cdot\|$ denotes the Frobenius norm. This flutter criterion thus says that a sufficiently "large" circulatory force will make the stable potential system lose stability.

It should be noted that [16] considered symmetric matrices K_s that did not necessarily commute with N_{sk} (and therefore, K_s may have distinct eigenvalues), and it provided only a sufficient condition for instability. Being a generalization of Merkin's result [4], this paper considers matrices Λ that may have one or more repeated eigenvalues, and which therefore always commute with skew-symmetric matrices N that have the same block diagonal structure as Λ (see Theorem 2).

To illustrate the flutter criterion given in Eq. (24) and the manner in which it differs from the results obtained in this paper, consider the circulatory system described by $\ddot{x} + (\Lambda + N)x = 0$ with the diagonal matrix Λ given by

$$\Lambda = \mathrm{diag}(1, 1, 1, 4) = \mathrm{diag}(I_1, 4I_2) \tag{25}$$

where I_1 is the three-by-three block diagonal identity matrix ($i_1 = 3$), and I_2 is a singleton (one-by-one identity matrix with $i_2 = 1$).

Because Trace $(\Lambda) = 7$, and $\|\Lambda\|^2 = 19$, the flutter criterion given in relation (24) says that a flutter instability will occur in this circulatory system when the elements of the skew-symmetric matrix N are large enough to satisfy the relation

$$\|N\|^2 > \|\Lambda\|^2 - (1/4)[\mathrm{Trace}(\Lambda)]^2 = 6.75 \tag{26}$$

But, the matrix Λ commutes with every skew-symmetric matrix

$$N(a, b, c) = \begin{bmatrix} 0 & a & b & 0 \\ -a & 0 & c & 0 \\ -b & -c & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

for arbitrary values of the real numbers a, b, and c because Λ and N have the same block diagonal structure. And, for every such matrix N given in Eq. (27), by Theorem 2, the circulatory system will be unstable in flutter.

Consider, for example, the matrix $N(1/2, 0, 0)$ given in Eq. (27), so that $\|N\|^2 = 1/2$. Thus, the stable potential system $\ddot{x} + \Lambda x = 0$ could lose stability with the addition of a circulatory force given by a matrix N that has a much smaller Frobenius norm than that prescribed by the flutter condition [Eq. (24)]. Being only a sufficient condition for flutter, relation (24) may not, in general, be satisfied; and the system can still have a flutter instability. From an engineering perspective, condition (24) may therefore not always be useful for potential systems, especially those where K_s has repeated frequencies. In fact, as Theorem 1 points out, the system will be driven into a flutter instability with the addition of a circulatory force given by the matrix $N_\varepsilon = \varepsilon N(a, b, c)$ for arbitrarily small values of ε, and therefore by skew-symmetric matrices N_ε with arbitrarily small (Frobenius) norms.

Lastly, we note that using the matrices Λ and N given in Eqs. (25) and (27), respectively, the stable potential system $\ddot{y} + K_s y = 0$ with $K_s = T\Lambda T^T$ will become unstable through the addition of a circulatory contribution given by the matrix $N_{\mathrm{sk}} = \varepsilon TNT^T$ for arbitrarily small values of ε and for arbitrary orthogonal matrices T.

III. Conclusions

Merkin's celebrated theorem [4] states that a stable potential system loses its stability through the addition of an arbitrarily small circulator perturbation if the potential system has all its frequencies the same. Most real-life multi-degree-of-freedom systems are modeled by numerous degrees of freedom, and it is generally extremely rare to find a real-life system that has all its frequencies identical, except in very special situations that are strongly constrained by considerations of symmetry. This severely limits the practical applicability of Merkin's elegant result.

This paper generalizes Merkin's theorem [4] by relaxing the requirement that the stable potential system must have all its frequencies coincident. The new generalization thus brings real-life multi-degree-of-freedom systems encountered in engineering practice within the compass of its applicability. It is shown that, for a stable potential system that has one or more frequencies that are repeated, there are arbitrarily small circulatory forces (perturbations) whose addition will make the system unstable in a flutter instability. The exact character of these arbitrarily small circulatory forces has been explicitly provided in the paper.

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