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SOME UNIQUENESS RESULTS RELATED TO BUILDING STRUCTURAL IDENTIFICATION*

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Abstract. This paper studies the nature of uniqueness in the identification of building structural systems subjected to strong ground shaking. Characterization of the stiffness distribution in the structure from a knowledge of response of one of the floors to a base excitation is investigated. It is shown that uniqueness of the results, in the inverse problem, can be established by proper sensor location. At sensor locations where nonunique solutions are present, an upper bound on the number of such solutions has been presented. The degree of nonuniqueness is found to monotonically increase with increasing height of sensor in the building system from at most one, for a sensor located at the first floor level, to at most $N!$ for a sensor located at the Nth floor of an N story structure.

Introduction. A logical prelude to the prediction of the dynamic response of a structure to a known set of inputs, is the determination of its dynamic properties. Several investigators [1]–[5] have worked on the problem of identifying structural parameters from dynamic tests of full scale structures where the loading comprised either a sinusoidal force at a particular level of the structure or low level excitations such as wind and microtremors. With the recent accent on the aseismic design of structures, more and more structures built in seismically active regions of the world are being instrumented with strong motion accelerographs nowadays, the aim being to determine their structural properties from records obtained during the high level excitations created by ground shocks, earthquakes, etc. Whereas many researchers have utilized these records to establish parametric structural models (e.g., [6]) of building systems, few, if any, have tried to investigate the uniqueness aspects associated with the inverse problem. From a practical viewpoint, this consideration may become a serious one [7], [8] because even if the identification scheme converged (most such schemes using "input-output" records are iterative), the convergence may not be to the correct parameter values, if the inverse problem is nonunique.

In this paper we treat an N story building structural system as an N degree of freedom, spring-mass system. The identification problem then consists of determining the stiffness constants of the system from a knowledge of the base excitation and the corresponding response of *one* of the floors. A related problem was tackled by Hochstadt [9] in a different context. However, the results presented herein go beyond that. Methods of solving this problem are presented, and conditions under which the identification is unique are established. Circumstances under which nonuniqueness is encountered are studied along with estimates of the degree of nonuniqueness in each case.

Though the work presented here has been motivated by problems in building identification, the results arrived at are applicable to the identification of all systems that can be expressed by the same matrix equations, such as LC ladder

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networks. Physical interpretations of the mathematical results arrived at will be explained as they are encountered.

1. Problem formulation. Figure 1 shows that model of an N story structure represented by the floor masses m_i, $i = 1, 2, \cdots, N$, and the corresponding stiffnesses k_i, $i = 1, 2, \cdots, N$, of an N degree of freedom undamped oscillator. Assuming the masses m_i to be known, the identification problem consists of determining the stiffnesses k_i from a knowledge of the base input $z(t)$, and the response recorded at a particular floor. All time functions in this discussion are assumed to be Laplace transformable.

Denoting by $w_n(t)$ the absolute motion of the nth floor, to the base input $z(t)$, we have the following set of equations:

$$M\ddot{\mathbf{w}} + A\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ k_N z(t) \end{bmatrix} \tag{1}$$

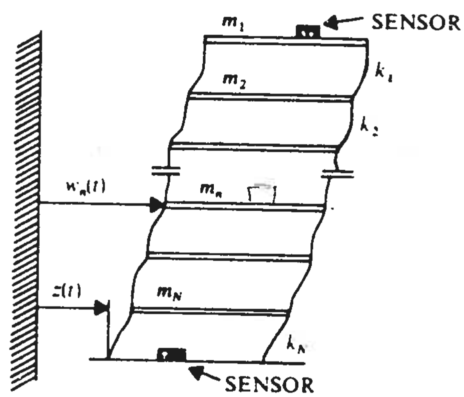

FIG. 1

where

$$M = \begin{bmatrix} m_1 & & & & \\ & m_2 & & & \\ & & m_3 & & \\ & & & \ddots & \\ & & & & m_N \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} k_1 & -k_1 & & & \\ -k_1 & (k_1+k_2) & -k_2 & & \\ & & & & -k_{N-1} \\ & & & -k_{N-1} & (k_{N-1}+k_N) \end{bmatrix}. \tag{2}$$

Since m_i's and k_i's are real and positive for undamped passive physical systems, we can reduce the system equation to

$$\ddot{\mathbf{y}} + K\mathbf{y} = \mathbf{f}z(t) \tag{3}$$

where

$$\mathbf{y}=M^{1/2}\mathbf{w},\quad \mathbf{f}=(0,0,\cdots,k_N/\sqrt{m_n})^T,\quad K=M^{-1/2}AM^{-1/2},$$

and superscript "T" stands for transpose. The matrix K, obtained thus is a symmetric tridiagonal matrix and can be expressed as

$$K=\begin{bmatrix} b_1 & -a_1 & & & \\ -a_1 & b_2 & -a_2 & & \\ & & \ddots & & -a_{N-1} \\ & & & -a_{N-1} & b_N \end{bmatrix}$$

in which

(4) $$a_i=\frac{k_i}{\sqrt{m_i m_{i+1}}},\quad 1\leqq i\leqq N-1,\quad \text{and}\quad b_i=\frac{k_{i-1}+k_i}{m_i},\quad 1\leqq i\leqq N,\quad \text{with } k_0=0.$$

Taking the Laplace transforms of both sides of (3), and replacing the transform variables by $i\sqrt{\lambda}$ we get

(5) $$(K-\lambda I)\mathbf{Y}=\mathbf{f}Z(\lambda)$$

where $\mathbf{Y}$ and $Z(\lambda)$ represent transformed quantities, and I is the identity matrix. Solving (5) for $Y_i(\lambda)$ we have

(6) $$Y_i(\lambda)=\frac{\Delta_i}{\Delta}Z(\lambda)$$

where

$$\Delta=\det(K-\lambda I)$$

and Δ_i is the determinant of the matrix obtained from $(K-\lambda I)$ by replacing its ith column by $\mathbf{f}$. For notational convenience let us denote by $P_i(\lambda)$, the determinant of the upper left $i\times i$ submatrix of $(K-\lambda I)$. In this way, $P_N(\lambda)$ is simply $\det(K-\lambda I)$. This allows us to express (6), using (2), as

$$W_i(\lambda)=\frac{1}{\sqrt{m_i}}\frac{k_N}{\sqrt{m_N}}a_{N-1}a_{N-2}\cdots a_i\frac{P_{i-1}(\lambda)}{P_N(\lambda)}Z(\lambda),$$

where $P_0(\lambda)=1$ and $W_i(\lambda)$ is the transform of $w_i(t)$, the response of the $(N+i+1)$th floor. With the use of (4) this becomes

(7) $$\frac{W_i(\lambda)}{Z(\lambda)}=\frac{k_N}{m_N}\frac{k_{N-1}}{m_{N-1}}\cdots\frac{k_i}{m_i}\frac{P_{i-1}(\lambda)}{P_N(\lambda)}.$$

The identification problem we are addressing in this paper can now be restated as follows: Suppose for known masses m_i, $1\leqq i\leqq N$, the function $Z(\lambda)$ is given along with the resulting transformed response $W_\alpha(\lambda)$, for some α between 1 and N. Then, the stiffness constants k_i are to be determined for $1\leqq i\leqq N$.

In the following, we first present some useful properties of the polynomials $P_i(\lambda)$. The first three of these properties (Lemmas 1–3) are well known [10] and have been included briefly, because of their extensive use in our subsequent treatment. Then we consider the identification problem for $\alpha = N-1$, i.e., when the transformed response of the first floor is known. Determination of k_i's in this case is shown to be unique. Thereafter, we consider the problem for other values of α.

2. Properties of the polynomials $P_i(\lambda)$.

LEMMA 1. (a) *The functions $P_i(\lambda)$ defined in the previous section satisfy the recursion relation*

$$P_i(\lambda) = (b_i - \lambda)P_{i-1}(\lambda) - a_{i-1}^2 P_{i-2}(\lambda), \qquad 2 \leqq i \leqq N, \quad \textit{and} \quad P_1(\lambda) = (b_1 - \lambda).$$

(b) *Each $P_i(\lambda)$ mentioned above is a polynomial of degree i with $(-1)^i\lambda^i$ as the leading term, i.e.,*

$$\lim_{\lambda \to \infty} P_i(\lambda)/\lambda^i = (-1)^i.$$

Proof. Part (a) follows directly from the definition of P_i's with use of the notation of (4). Part (b) follows from (a) by induction. □

LEMMA 2. *For $1 \leqq i \leqq N$, $P_i(\lambda)$ and $P_{i-1}(\lambda)$ do not have a common zero if*

$$a_j \neq 0, \qquad 1 \leqq j \leqq N-1.$$

Proof. The proof is by induction. Since

$$P_1(\lambda) = b_1 - \lambda \quad \text{and} \quad P_2(\lambda) = (b_2 - \lambda)(b_1 - \lambda) - a_1^2,$$

the only common zero between them could be $\lambda = b_1$, but that implies $a_1 = 0$, a contradiction. That is, P_1 and P_2 have no zero in common. Now assume that P_i and P_{i-1} have no common zeros for $i \leqq n$, and let

$$P_{n+1}(\alpha) = P_n(\alpha) = 0.$$

Then the recursion relation

$$P_{n+1}(\alpha) = (b_{n+1} - \lambda)P_n(\alpha) - a_n^2 P_{n-1}(\alpha) = 0$$

implies $P_{n-1}(\alpha) = 0$, a contradiction since P_n and P_{n-1} do not have a common zero; hence the result. □

LEMMA 3. *Zeros of all $P_i(\lambda)$'s are simple.*

Proof. Let α be a pth order zero of $P_i(\lambda)$; then it is an eigenvalue of the upper left $i \times i$ submatrix of K called K_i. This K_i, being a real symmetric matrix, can be diagonalized to Λ a diagonal matrix of its eigenvalues by an orthogonal matrix P, i.e.,

$$P^T K_i P = \Lambda$$

where p of the diagonal elements of Λ are α. Then,

$$\text{rank}\,[K_i - \alpha I] = \text{rank}\,[P(\Lambda - \alpha I)P^T] = \text{rank}\,(\Lambda - \alpha I) = i - p,$$

since P is now singular. Thus, $p > 1$ would imply $P_{i-1}(\alpha) = 0$, a contradiction by Lemma 2, hence $p = 1$. □

LEMMA 4. *If $P_i(\lambda)$ and $P_{i-2}(\lambda)$ have a zero in common then it is at $\lambda = b_i$; its multiplicities as a zero of $P_i(\lambda)$ and $P_{i-2}(\lambda)$ cannot be both greater than one.*

Proof. The first part is obvious by the recursion relation and Lemma 2. For the second. let us assume that r and s are multiplicities of $\lambda = b_i$ as a zero of P_i and P_{i-2} respectively. Then

$$P_i(\lambda) = (b_i - \lambda)^r Q_1(\lambda)$$

and

$$P_{i-2}(\lambda) = (b_i - \lambda)^s Q_2(\lambda)$$

where

$$Q_j(b_i) \neq 0, \qquad j = 1, 2.$$

And, by the recursion relation of Lemma 1(a), we obtain

$$(b_i - \lambda)^{r-1} Q_1(\lambda) = P_{i-1}(\lambda) - a_{i-1}^2 (b_i - \lambda)^{s-1} Q_2(\lambda).$$

Then, $r > 1$ and $s > 1$ would imply $P_{i-1}(b_i) = 0$, a contradiction by Lemma 2. So, r and s cannot be both greater than 1. □

The following lemma about the leading principal minors of K will be useful later on.

LEMMA 5. *Let K_i be the upper left $i \times i$ submatrix of K; then*

$$\det(K_i) = \frac{k_i}{m_i} \frac{k_{i-1}}{m_{i-1}} \cdots \frac{k_1}{m_1}.$$

Proof. Let A_i denote the upper left $i \times i$ submatrix of A, then by (3),

$$\det(K_i) = \det(A_i) \frac{1}{m_i} \frac{1}{m_{i-1}} \cdots \frac{1}{m_1}.$$

Now we show by induction that

$$\det(A_i) = k_i \cdot k_{i-1} \cdots k_1.$$

Clearly

$$\det(A_1) = k_1.$$

Assume $A_i = k_i A_{i-1}$ for $i \leqq n$. Then

$$A_{n+1} = (k_n + k_{n+1}) A_n - k_n^2 A_{n-1} = k_{n+1} A_n$$

and hence the result. □

LEMMA 6. *The submatrices K_i, defined above, are all real symmetric positive definite matrices, and all the roots of $P_i(\lambda) = 0$ are real and positive, for $1 \leqq i \leqq N$.*

Proof. The first part is clear by Lemma 5, since $m_i > 0$, $k_i > 0$ for $1 \leqq i \leqq N$. For the second part note that roots of $P_i(\lambda) = 0$ are eigenvalues of K_i, which are all real since K_i is a real symmetric matrix, and positive since K_i is positive definite. □

LEMMA 7.

$$\frac{P_{i-1}(0)}{P_i(0)} = \frac{m_i}{k_i}, \qquad 2 \leqq i \leqq N.$$

Proof. Using Lemma 5 and the relation

$$P_i(0) = \det(K_i),$$

the result follows. □

3. Unique identification using first floor response. In this section, we consider the identification problem outlined earlier, when $Z(\lambda)$ the ground motion, $W_N(\lambda)$ the response of the first floor, and m_i the masses of all the floors are given. The following theorem ensures unique identification of k_i's.

THEOREM 1. *If there exists a set of k_i's, $1 \leqq i \leqq N$, corresponding to the given functions $Z(\lambda)$ and $W_N(\lambda)$, and m_i, $1 \leqq i \leqq N$, then it is unique.*

Proof. Suppose there are two sets k_i and $\tilde{k}_i$, $1 \leqq i \leqq N$, for which $W_N(\lambda)$ and $\tilde{W}_N(\lambda)$ are the same. We shall denote all the quantities corresponding to $\tilde{k}_i$'s by a tilde sign. Then by (7),

$$\frac{W_N(\lambda)}{Z(\lambda)} = \frac{k_N}{m_N}\frac{P_{N-1}(\lambda)}{P_N(\lambda)} = \frac{\tilde{k}_N}{m_N}\frac{\tilde{P}_{N-1}(\lambda)}{\tilde{P}_N(\lambda)}. \tag{8}$$

From Lemma 1(b) we conclude

$$\lim_{\lambda\to\infty} \frac{\lambda P_{N-1}(\lambda)}{P_N(\lambda)} = -1.$$

Thus multiplying (8) by λ and taking limits as $\lambda \to \infty$, we obtain

$$k_N = \tilde{k}_N.$$

Hence (8) could be rewritten as

$$\frac{P_{N-1}(\lambda)}{P_N(\lambda)} = \frac{\tilde{P}_{N-1}(\lambda)}{\tilde{P}_N(\lambda)},$$

which implies

$$P_{N-1}(\lambda) \equiv \tilde{P}_{N-1}(\lambda) \quad \text{and} \quad P_N(\lambda) \equiv \tilde{P}_N(\lambda),$$

since numerators and denominators on both the sides are polynomials, and there is no pole-zero cancellation by Lemma 2.

Now, to complete "backward" induction, let us assume that

$$k_i = \tilde{k}_i,$$

$$P_i(\lambda) \equiv \tilde{P}_i(\lambda) \tag{9}$$

and

$$P_{i-1}(\lambda) \equiv \tilde{P}_{i-1}(\lambda) \quad \text{for } n \leqq i \leqq N.$$

Then, to show a similar set of relations for $n-1$, we write, using the recursion relations,

$$P_n(\lambda) = (b_n - \lambda)P_{n-1}(\lambda) - a_{n-1}^2 P_{n-2}(\lambda)$$

and

$$\tilde{P}_n(\lambda) = (\tilde{b}_n - \lambda)\tilde{P}_{n-1}(\lambda) - \tilde{a}_{n-1}^2 \tilde{P}_{n-2}(\lambda).$$

Subtracting the two equations, and using (9) for $i = n$, we get

$$0 = (b_n - \tilde{b}_n)P_{n-1}(\lambda) + a_{n-1}^2 P_{n-2}(\lambda) - \tilde{a}_{n-1}^2 \tilde{P}_{n-2}(\lambda).$$

Now divide by $P_{n-1}(\lambda)$ and let $\lambda \to \infty$. We obtain

$$b_n = \tilde{b}_n,$$

which implies

$$k_{n-1} = \tilde{k}_{n-1}$$

by using (4).

Thus we obtain equations (9) for $2 \leqq i \leqq N$, and $k_1 = \tilde{k}_1$ is obtained easily from $P_1(\lambda) = \tilde{P}_1(\lambda)$ and $k_2 = \tilde{k}_2$. This establishes uniqueness. □

In the following we present an algorithm to determine the values of k_i, $1 \leqq i \leqq N$, when $Z(\lambda)$ and $W_N(\lambda)$ are available in closed form.

ALGORITHM 1. Suppose we are given m_i, for $1 \leqq i \leqq N$, $Z(\lambda)$ and $W_N(\lambda)$. Then we use Step 1 in the following to determine k_N, and Step 2 for k_n, $1 \leqq n \leqq N-1$.

Step 1. We use the relation

$$\frac{W_N(\lambda)}{Z(\lambda)} = \frac{k_N}{m_N} \frac{P_{N-1}(\lambda)}{P_N(\lambda)}$$

to determine

$$k_N = -\lim_{\lambda \to \infty} m_N \lambda \frac{W_N(\lambda)}{Z(\lambda)} \tag{10}$$

using Lemma 1(b). Thus we also obtain

$$\frac{P_N(\lambda)}{P_{N-1}(\lambda)} = \frac{k_N}{m_N} \frac{Z(\lambda)}{W_N(\lambda)}. \tag{11}$$

Step 2. For $1 \leqq n \leqq N-1$, suppose we know $P_{n+1}(\lambda)/P_n(\lambda)$. Then, using the recursion relation we obtain

$$\frac{P_{n+1}(\lambda)}{P_n(\lambda)} = (b_{n+1} - \lambda) - a_n^2 \frac{P_{n-1}(\lambda)}{P_n(\lambda)}.$$

Thus, using Lemma 1(b), we obtain

$$b_{n+1} = \lim_{\lambda \to \infty} \left[\frac{P_{n+1}(\lambda)}{P_n(\lambda)} + \lambda \right]. \tag{12}$$

This allows us to determine k_n as

$$k_n = b_{n+1} m_{n+1} - k_{n+1} \tag{13}$$

and to continue the iterations we determine

$$\frac{P_n(\lambda)}{P_{n-1}(\lambda)} = a_n^2 \left[b_{n+1} - \lambda - \frac{P_{n+1}(\lambda)}{P_n(\lambda)} \right]^{-1}$$

where $a_n^2 = k_n^2 / m_n m_{n+1}$ by eq. (4).

In practice, we start Step 2 determinations from $n = N-1$ by taking $P_N(\lambda)/P_{N-1}(\lambda)$ from Step 1, and then determining b_N, k_{N-1} and $P_{N-1}(\lambda)/P_{N-2}(\lambda)$. This continues until $n = 1$. □

In Step 2 above, sometimes it might be simpler to use long hand division of $P_{n+1}(\lambda)$ by $P_n(\lambda)$ to determine $(b_{n+1} - \lambda)$ which would be left as the quotient in the division process. The remainder would correspond to $-a_n^2 P_{n-1}(\lambda)$. This gives $P_{n-1}(\lambda)$ which divides into $P_n(\lambda)$ for the next lower value of n, and so on. In each step it is trivial to determine k_n from the knowledge of $(b_{n+1} - \lambda)$.

We close this section by summarizing our conclusion: that is, given the response at the first floor of a N story building to a known ground motion, it is possible to determine the stiffness of each floor uniquely, provided the masses of all the floors are also given.

4. Nonuniqueness of identification for known top floor response. In this section, we consider the problem of determining k_i, $1 \leqq i \leqq N$, when $W_1(\lambda)$, the transform of the roof response, is given along with $Z(\lambda)$, the transform of the base ground motion, and when the m_i, $1 \leqq i \leqq N$, are all known. We shall show that no matter how many (ground input-roof response) pairs are looked at, this problem "in general" has nonunique solutions.

For clarity we restate the problem in the following manner: Suppose we are given a physical N degree of freedom system with m_i, k_i, $1 \leqq i \leqq N$, real and positive. We shall denote this system by the $2N$ element set $\{k_i, m_i\}$. We excite this system with the base ground motion $z(t)$ and determine the corresponding roof response $w_1(t)$. Repeating this procedure, we obtain an ensemble of roof responses $w_1^n(t)$, $n = 1, 2, \cdots$, corresponding to an ensemble of base ground motions $z^n(t)$, $n = 1, 2, \cdots$. With these $[z^n(t), w_1^n(t)]$ pairs in hand, in this section, we will attempt to answer the following questions.

a) In general, does another system $\{\tilde{k}_i, m_i\}$ exist such that it produces the same pairs $[z^n(t), w_1^n(t)]$ as those given by the system $\{k_i, m_i\}$?

b) In general, how many such systems $\{\tilde{k}_i, m_i\}$ exist?

c) Among these systems are there any for which the $\tilde{k}_i$, $1 \leqq i \leqq N$ are all real and positive; i.e., is the system physically realizable?

First we state a well known theorem called Bezout's theorem, which will be used in the following discussion. Bezout's theorem states that [11] if $f_1, f_2, \cdots, f_m$ be hypersurfaces in m-dimensional projective space which intersect in a finite set $\{M_j\}$ of points, and if d_i be the degree of f_i, there may then be assigned multiplicities σ_j to the M_j independent of the coordinate system, such that counted with these multiplicities the number of intersections is $d = d_1 \cdot d_2 \cdots d_m$.

THEOREM 2. *If the system $\{k_i, m_i\}$, $m_i, k_i > 0$ yields the input-output pairs $[z^n(t), w_1^n(t)]$ then there exist, in general, $N! - 1$ other systems $\{\tilde{k}_i, m_i\}$ which yield the same input-output pairs.*

Proof. From (7) we obtain

$$\frac{W_1(\lambda)}{Z(\lambda)} = \frac{k_n}{m_N} \frac{k_{n-1}}{m_{N-1}} \cdots \frac{k_1}{m_1} \frac{1}{P_N(\lambda)}$$

which could be rewritten as

$$\frac{W_1(\lambda)}{Z(\lambda)} = \det(K) \frac{1}{P_N(\lambda)}.$$

by Lemma 5. Now let us assume that in addition to the system $\{k_i, m_i\}$ there exists another system $\{\tilde{k}_i, m_i\}$ such that, for both systems the ratio $W_1(\lambda)/Z(\lambda)$ is the same, for all values of λ. Using tildes to denote all the quantities relating to the $\{\tilde{k}_i, m_i\}$ system, we then need

$$\frac{W_1(\lambda)}{Z(\lambda)} = \det(K)\frac{1}{P_N(\lambda)} = \det(\tilde{K})\frac{1}{\tilde{P}_N(\lambda)}. \tag{14}$$

Since, by Lemma 1, $P_N(\lambda)$ is a polynomial of degree N with leading coefficient $(-1)^N$, a knowledge of $W_1(\lambda)$ and $Z(\lambda)$ amounts to knowing $\det(K)$ and $P_N(\lambda)$ separately. Furthermore, we have

$$P_N(\lambda) = \det(K - \lambda I). \tag{15}$$

This yields, by Lemma 8, the condition

$$\tilde{P}_N(\lambda) = P_N(\lambda) = \det(K - \lambda I). \tag{16}$$

Thus, in order to determine the $\tilde{k}_i$'s we equate the coefficients of various powers of λ on both sides of (16). This leads to N nonlinear algebraic equations in the $\tilde{k}_i$'s which have the following form as shown in Lemma 9:

$$\sum_{i_1=1}^{N} \alpha_{1i_1}\tilde{k}_{i_1} = \sum_{i_1=1}^{N} \alpha_{1i_1}k_{i_1} = a_1. \tag{17-1}$$

$$\sum_{i_2>i_1}\sum_{i_1=1}^{N} \alpha_{2i_1i_2}\tilde{k}_{i_1}\tilde{k}_{i_2} = \sum_{i_2>i_1}\sum_{i_1=1}^{N} \alpha_{2i_1i_2}k_{i_1}k_{i_2} = a_2, \tag{17-2}$$

$$\begin{aligned} &\sum_{i_n>i_{n-1}} \cdots \sum_{i_2>i_1}\sum_{i_1=1}^{N} \alpha_{ni_1i_2\cdots i_n}\tilde{k}_{i_1}\tilde{k}_{i_2}\cdots\tilde{k}_{i_n} \\ &\qquad = \sum_{i_n>i_{n-1}} \cdots \sum_{i_2>i_1}\sum_{i_1=1}^{N} \alpha_{ni_1i_2\cdots i_n}k_{i_1}k_{i_2}\cdots k_{i_n} = a_n, \end{aligned} \tag{17-n}$$

$$\tilde{k}_1\tilde{k}_2\cdots\tilde{k}_N = k_1k_2\cdots k_N = a_N. \tag{17-N}$$

In this set, the a_i's are all known from the left hand side of (15) and can be expressed in terms of the roots, λ_i, of the equation $P_N(\lambda) = 0$ as follows:

$$\begin{aligned} a_1 &= \sum_{i=1}^{N} \lambda_i, \\ a_2 &= \frac{1}{2}\sum_{\substack{i=1 \\ j=1}}^{N} \lambda_i\lambda_j, \\ &\vdots \\ a_N &= \lambda_1\lambda_2\cdots\lambda_N. \end{aligned} \tag{18}$$

By Lemma 6, then we observe that $a_i > 0$, $1 \leq i \leq N$. Also, all the α's are known (since they involve the known masses m_i) and are positive by Lemma 10.

In order to clarify the structure of (17) we present the set for $N=3$:

(19-1) $$\tilde{k}_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+\tilde{k}_2\left(\frac{1}{m_2}+\frac{1}{m_2}\right)+\frac{\tilde{k}_3}{m_3}=a_1.$$

(19-2) $$\tilde{k}_1\tilde{k}_2(m_1+m_2+m_3)+\tilde{k}_1\tilde{k}_3(m_1+m_2)+\tilde{k}_2\tilde{k}_3m_1=a_2.$$

(19-3) $$\tilde{k}_1\tilde{k}_2\tilde{k}_3=a_3.$$

Thus, we see that the nth equation in the above set (17) is of degree n. Also noting that the set has at least one solution set, $\tilde{k}_i=k_i$, $1\leqq i\leqq N$, we can use Bezout's theorem to establish that if the number of solutions is finite, then there are $N(N-1)\cdots 3\cdot 2\cdot 1$, i.e., $N!$ solution sets counted with their multiplicities, of which one set is constituted by $\tilde{k}_i=k_i$, $1\leqq i\leqq N$. □

LEMMA 8. *Given two sets $k_1, k_2, \cdots, k_N$ and $\tilde{k}_1, \tilde{k}_2, \cdots, \tilde{k}_n$,*

(20) $$\left(\prod_{e=i}^{N}\frac{k_e}{m_e}\right)\frac{P_{i-1}(\lambda)}{P_N(\lambda)}=\left(\prod_{e=i}^{N}\frac{\tilde{k}_e}{m_e}\right)\frac{\tilde{P}_{i-1}(\lambda)}{\tilde{P}_N(\lambda)}$$

if and only if

$$\frac{P_{i-1}(\lambda)}{P_N(\lambda)}=\frac{\tilde{P}_{i-1}(\lambda)}{\tilde{P}_N(\lambda)}.$$

Proof. If

$$\left(\prod_{e=i}^{N}\frac{k_e}{m_e}\right)\frac{P_{i-1}(\lambda)}{P_N(\lambda)}=\left(\prod_{e=i}^{N}\frac{\tilde{k}_e}{m_e}\right)\frac{\tilde{P}_{i-1}(\lambda)}{\tilde{P}_N(\lambda)},$$

cross multiplying and equating the coefficients of λ^{N+i-1} on both sides of the equation, by Lemma 1(b), we have

$$\prod_{e=i}^{N}\frac{k_e}{m_e}=\prod_{e=i}^{N}\frac{\tilde{k}_e}{m_e}.$$

With the use of this relation in (20) the result follows. Furthermore, if

$$\frac{P_{i-1}(\lambda)}{P_N(\lambda)}=\frac{\tilde{P}_{i-1}(\lambda)}{\tilde{P}_N(\lambda)}$$

by Lemma 1(b) we must have

$$P_{i-1}(\lambda)=\tilde{P}_{i-1}(\lambda)\quad\text{and}\quad P_N(\lambda)=\tilde{P}_N(\lambda).$$

This implies

$$\det(K_{i-1})=\det(\tilde{K}_{i-1})\quad\text{and}\quad\det(K_N)=\det(\tilde{K}_N).$$

Using Lemma 5 and dividing the above two equations we have

$$\prod_{e=i}^{N}\frac{k_e}{m_e}=\prod_{e=i}^{N}\frac{\tilde{k}_e}{m_e}$$

and hence the result. □

LEMMA 9. *In all the terms that appear in the expansion of $P_N(\lambda)$, the k_i's appear with power at most one.*

Proof.

$$P_N(\lambda)=\det(K-\lambda I)=\sum(-1)^{t(p)}u_{1i_1}\cdots u_{Ni_N} \tag{21}$$

where $p=(i_1, i_2, \cdots, i_N)$ is a permutation of $(1, 2, \cdots, N)$, and $t(p)$ is the degree of the permutation p [12]. Since u_{ij}'s are elements of the tridiagonal matrix $(K-\lambda I)$, many terms in the above summation are zero. It is also seen that the highest power of any k_i that can occur is two.

From the tridiagonal structure of the matrix it follows that if a term in the above sum contains an off-diagonal element $u_{\alpha\alpha-1}$, it must contain $u_{\alpha-1\alpha}$ also. For each such term containing the product $u_{\alpha\alpha+1}\cdot u_{\alpha+1\alpha}$, there is another one with opposite sign in which the product is replaced by $u_{\alpha\alpha}u_{\alpha+1\alpha+1}$. Using

$$u_{\alpha\alpha}=\frac{k_{\alpha-1}+k_\alpha}{m_\alpha}-\lambda,$$

$$u_{\alpha+1\alpha}=u_{\alpha\alpha+1}=\frac{k_\alpha}{\sqrt{m_\alpha m_{\alpha-1}}}$$

we conclude that the term containing k_α^2 cancels out. □

LEMMA 10. *The coefficients α in the equation set* (17) *are all positive.*

Proof. It suffices to prove that all the terms multiplying $(-\lambda)^i$ in the expansion of $\det(K-\lambda I)$ are positive.

In (21) there is one term of the form

$$(b_1-\lambda)(b_2-\lambda)\cdots(b_N-\lambda). \tag{22}$$

All the other terms contain off-diagonal elements of $(K-\lambda I)$ in pairs as was mentioned in Lemma 9. These terms must cancel out in the final expression of $\det(K-\lambda I)$ since they contain k_i^2, etc. (Lemma 9). Thus all the remaining terms come from the product (22), in which the coefficients of $(-\lambda)^i$ are all positive. □

For the case $N=3$, it can be shown by algebraic substitutions that k_3 is a root of a sixth degree equation while k_1 and k_2 are determined uniquely from (19-1) and (19-2). The six solutions so obtained are in accordance with the result of Theorem 2. It is easily seen that for $N=2$, such a procedure would lead to solving a quadratic equation which would yield two solutions.

THEOREM 3. *If the equation set* (17) *has a finite number of solutions, and if there exists one system* $\{k_i, m_i\}$ *with the* k_i *all real which satisfies* (17), *then there exists, in general, at least one other system* $\{\tilde{k}_i, m_i\}$ *such that the* $\tilde{k}_i$ *are all real.*

Proof. We note that the coefficients of the equation set (17) are all real so that if s_i is a solution set of the equations, then its complex conjugate s_i^* is also a solution set. Also by Theorem 2 the total number of possible solution sets is $N!$ which is an even number. Hence since the complex solutions of the equation set (17) occur in pairs and since one set of k_i's is given to be real there must exist another set $\tilde{k}_i$ which is also real. □

THEOREM 4. *If a real solution set* $\{\tilde{k}_i, m_i\}$ *of the equations* (17) *exists then* $\tilde{k}_i>0$, $1\leq i\leq N$.

Proof. By (16) the system $\{k_i, m_i\}$ is such that

$$\tilde{P}_N(\lambda)=P_N(\lambda)=\det(K-\lambda I)=\det(\tilde{K}-\lambda I).$$

Hence the eigenvalues of the matrices K and $\tilde{K}$ are identical. But the eigenvalues of K are all positive (Lemma 6), and hence the eigenvalues of $\tilde{K}$ are all positive with $\det(\tilde{K}_i)>0,\ 1\leqq i\leqq N$.

But by Lemma 5,

$$\det(\tilde{K}_i)=\frac{\tilde{k}_i}{m_i}\frac{\tilde{k}_{i-1}}{m_{i-1}}\cdots\frac{\tilde{k}_1}{m_1},$$

hence

$$\det(\tilde{K}_1)>0 \quad \text{implies} \quad \tilde{k}_1>0.$$

Also,

$$\det(\tilde{K}_2)>0$$

and the above imply

$$\tilde{k}_2>0.$$

Proceeding as above, it can thus be shown that

$$\tilde{k}_i>0, \quad 1\leqq i\leqq N. \quad \square$$

5. Some general remarks on the degree of nonuniqueness. In this section we first present the case when the response of the second floor is known. Thereafter, some general observations about other cases will be made.

Using (7) for the second floor we obtain

$$\frac{Z(\lambda)}{W_{N-1}(\lambda)}=\frac{m_N m_{N-1}}{k_N k_{N-1}}\frac{P_N(\lambda)}{P_{N-2}(\lambda)}.$$

Using the recursion relation of Lemma 1(a) twice, we obtain

$$\text{(23)}\qquad \begin{aligned}\frac{Z(\lambda)}{W_{N-1}(\lambda)}=\frac{m_N m_{N-1}}{k_N k_{N-1}}\Big[&\lambda^2-(b_N+b_{N-1})\lambda+(b_N b_{N-1}-a_{N-1}^2)\\ &+a_{N-2}^2\lambda\frac{P_{N-3}(\lambda)}{P_{N-2}(\lambda)}-b_N a_{N-2}^2\frac{P_{N-3}(\lambda)}{P_{N-2}(\lambda)}\Big].\end{aligned}$$

Then by (4) and Lemma 1(b) we obtain the following three equations for k_N, k_{N-1} and k_{N-2}:

$$\text{(24)}\qquad \begin{aligned}&k_{N-1}k_N=\alpha_1,\\ &k_N\frac{1}{m_N}+k_{N-1}\left(\frac{1}{m_N}+\frac{1}{m_{N-1}}\right)+k_{N-2}\frac{1}{m_{N-1}}=\alpha_2,\\ &(k_N+k_{N-1})k_{N-2}-k_{N-2}^2\frac{m_N}{m_{N-2}}=\alpha_3.\end{aligned}$$

It can be shown that these equations can have at most four solutions. Once we know one set of values of k_N, k_{N-1}, and k_{N-2}, we find $P_{N-3}(\lambda)/P_{N-2}(\lambda)$ from (23). This enables us to determine $k_{N-3}, k_{N-4},\cdots,k_1$ by Algorithm 1. Thus, there are at most four sets of values of $k_1,\cdots,k_N$ which yield the same second floor response for a given ground motion. In this context, we point out that a special case arises for $N=3$. For this value of N, we know $P_1(\lambda)$ and $P_3(\lambda)$. Here, $P_1(\lambda)$ gives us directly the value of k_1 by (4). Then (24) can be solved uniquely for k_2 and k_3 implying that the identification problem has a unique solution. For $N=4$, similar considerations allow us to conclude that there are at most two solutions to the identification problem.

The following theorem gives an upper bound on the number of solutions to the identification problem when the response of the nth floor is known.

THEOREM 5. *The finite number of solutions, when the response of the n-th floor is known for* $2<n<N$, *are* at most $(N-n)!(n-1)!(n-1)$.

Proof. Using (7) for the nth floor we can obtain $P_N(\lambda)$ and $P_{N-n}(\lambda)$. This yields two sets of equations similar to the set (17) having N and $N-n$ equations respectively. The latter set involves $N-n$ unknowns $k_1, \cdots, k_{N-n}$ and has at most $(N-n)!$ solutions. Each one of these solution sets can be substituted in the former set having N equations out of which only n are to be used. This is so because there are only n unknowns remaining, namely $k_{N-n+1}, \cdots, k_N$. On substitution, the last $N-n+1$ equations in the set (17) reduce in degree so that they all become of degree $n-1$. The degrees of the first $n-1$ equations remain unchanged. Therefore, the n unknowns can be determined using these $n-1$ equations along with one equation of degree $n-1$ from the remaining $N-n+1$ equations. This leads to at most $(n-1)!(n-1)$ solution sets of the n unknowns.

Therefore, for all the N unknowns, there are at most $(N-n)!(n-1)!(n-1)$ solutions. □

The above results have been summarized in Table 1 for $N \leqq 6$.

TABLE 1
Maximum number of solutions of the identification problem for an N story structure given the response at the n-th floor

N \ n	6	5	4	3	2	1
6	6!	96	36	24	4	1
5		5!	18	8	4	1
4			4!	4	2	1
3				3!	1	1
2					2!	1

6. Illustrative example.

(a) *Nonuniqueness of roof response*. Consider a three degree of freedom system $\{k, m\}$ with $m_1=1$, $m_2=1$, $m_3=2$ and $k_1=1$, $k_2=1$, $k_3=2$. For another three degree of freedom system $\{\bar{k}, m\}$ we must have

$$(19^*) \qquad \begin{aligned} &4\bar{k}_1+3\bar{k}_2+\bar{k}_3=9,\\ &4\bar{k}_1\bar{k}_2+2\bar{k}_1\bar{k}_3+\bar{k}_2\bar{k}_3=10,\\ &\bar{k}_1\bar{k}_2\bar{k}_3=2. \end{aligned}$$

$\bar{k}_1=\frac{1}{2}$, $\bar{k}_2=1$, $\bar{k}_3=4$ satisfies the equation set (19*) and therefore represents a system different from $\{k, m\}$. The two systems are, however, indistinguishable as far as their roof response to any base excitation is concerned. We observe that both sets $\{k, m\}$ and $\{\bar{k}, m\}$ represent physically reasonable building models with the stiffnesses gradually decreasing with increasing height.

It may be further proved that for the general three degree-of-freedom system the following hold:

a) if $k_1=\bar{k}_1$ then $k_2=\bar{k}_2$ and $k_3=\bar{k}_3$,
b) if $k_3=\bar{k}_3$ then $k_1=\bar{k}_1$ and $k_2=\bar{k}_2$,
and c) if $k_2=\bar{k}_2$ then $k_1\neq\bar{k}_1$ and $k_2\neq\bar{k}_2$
only if $m_2^2=m_1(m_3-m_2)$.

(b) *Uniqueness of first and second story response.* As seen from Table 1, the response history matching at the second or first story leads in this case to a unique identification of the system $\{k, m\}$.

7. Discussion and conclusions. In this paper, we have modeled an N story structure by an N degree of freedom lumped-mass-spring system. Identification of the stiffnesses has been investigated in detail for output responses measured at two sensor locations: a) a sensor at the roof level and b) a sensor at the first story level. It has been shown that the use of input-output records yields a unique determination of the stiffnesses in the latter case. Identification for the former sensor location leads to nonunique stiffness estimates, there being, in general, $N!$ different systems that could yield the same input-output pairs. Some general comments on the identification problem for other sensor locations have been made. For a sensor located at the nth floor level, it has been shown that a maximum of $(N-n)!(n-1)!(n-1)$ different systems could yield identical "base-nth floor response" pairs. As given in Table 1, however, tighter estimates on the maximum number of solutions can be obtained for small values of N, by inspecting the system of equations in specific cases.

Though the first story sensor location case does yield unique estimates, it should be pointed out that the analysis here deals with noise free data. The presence of noise in the measurements would in general lead to low signal-to-noise ratios for measurements at the first story level so that though the identification problem has a unique solution, in actual practice, the variance of the estimates may become large. Measurements at the roof on the other hand, would be larger in amplitude so that the noise related estimation errors would be smaller if roof records are used. However, one faces here the nonuniqueness problem. Furthermore, the analysis herein assumes that either a complete knowledge of the input-output time histories or their transforms is available. This again is not possible from a practical standpoint.

Though the structural system in this paper has been modeled in a very simple manner and may be far from realistic in several instances, the analysis presented here is indicative of the type of problems that could arise in the testing and identification of building structures. The analysis of damped structural models has specifically been excluded here for the sake of simplicity and work along these lines is continuing.

The results illustrated here also find application in the identification of soil properties from records obtained at "rock" level and at the surface of a layered soil system which is modeled by a shear beam subjected to vertically propagating shear waves. Even granting a complete knowledge of the rock motions, as shown above, for an N layered soil system the number of possible models that could be arrived at from a knowledge of "rock input-surface response" type studies may be as high as $N!$.

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