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Robust Control of Uncertain Systems with Time Varying Delays in Control Input

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Abstract

This paper deals with the robust state feedback control of time invariant dynamic systems with time varying control delays and system uncertainties. We provide here a robust feedback control design for uncertain structural systems with uncertainties in the time delayed control which guarantee that the closed-loop system is stable, despite in the presence of these uncertainties. Numerical simulations are provided to demonstrate the efficiency of the control technique.

Introduction

The problem of designing a state feedback control that guarantees the desired performance of a dynamic system involving uncertain elements has been discussed by Leitmann, Gutman and Corless [2,3]. Also effective solution methods are given by Chen [1]. Often finding controls that guarantee the uniform ultimate boundedness and generalized uniform stability are sufficient.

In previous work by Leitmann, Gutman, Corless and Chen the control system deals with uncertainties which are embedded in the system's structure or which are introduced externally (e.g. measurement noises). We are introducing a new sets of uncertainties in the system in terms of time delays in the control: these have not been looked at by previous researchers. It should be noted that time delays are inherent in many control systems, such as structural control, cold rolling mill, engine speed control and etc. For example in the active control of building structures subjected to strong earthquake shaking, large control forces are required to be generated and it is difficult to deliver these control forces without time delays. The time delay in the control is allowed to be a function of time, but has to be bounded by a constant.

Problem Statement

Consider the system

$$\dot{x}(t) = Ax(t) + \Delta Ax(t) + Bu(t) + \Delta Bu(t) + \delta Bu(t - h(t)) \quad (1)$$

Where $\Delta A, \Delta B, \delta B$ are uncertain elements in their general forms, and h(t) is the time varying element introduced as time delay. In most dynamic systems stiffness and damping matrices usually contain stochastic uncertainties which can be considered here as stochastic uncertain elements, since as it is considered here the uncertain elements in general could have either deterministic or stochastic forms. We use the following assumptions

$$\begin{array}{llll} A & \text{is } n \times n & \Delta A = BD & : \\ & \|D\| = k_2 & & \\ B & \text{is } n \times m & \Delta B = BE & : \\ & \|E\| = k_3,\ k_3 \in [0,1) & & \\ & & \delta B = BF & : \\ & & \|F\| = \hat{k}_3 & \end{array} \quad (2)$$

u is an m-vector

The above assumptions could be interpreted as, the uncertainties in the system are bounded by known constants, k's. Where $k_2, \hat{k}_3 \in [0,\infty)$ and $k_3 \in [0,1)$. The restriction of $k_3 \in [0,1)$ is interpreted as follows: The uncertainty in the control can not be so severe as to reverse the direction of control action, for then one is not able to tell if the control is in the desired direction. (Chen [1]).

Now we assert that if the state feedback control u(t) is chosen so that,

$$u(t) = -\frac{1}{2}\sigma B^T Px(t), \quad (3)$$

where P is the solution of the Lyapunov equation.

$$A^T P + PA = -(Q + H), \quad P, Q, H > 0 \quad (4)$$

then the response of the system (1) is stable, provided that,

$$h(t) < h_0, \ \dot{h}(t) \le a < 1 \text{ and } \rho > 4\eta^2. \quad (5)$$

where,

$$\eta^2 = \frac{\hat{k}_3^2 \|B^T P\|^2}{4(1-a)\lambda_{min}(H)} \text{ and } \rho = \frac{(1-k_3)^2 \lambda_{min}(Q)}{k_2^2} \quad (6)$$

The control gain is chosen as

$$\sigma \le \frac{(1-k_3)}{\eta^2+\theta}, \quad \theta > 0. \tag{7}$$

where θ is a positive constant which has to satisfy the following condition.

$$\frac{\rho-2\eta^2}{2}\left\{1-\sqrt{1-\Omega^2}\right\} < \theta < \frac{\rho-2\eta^2}{2}\left\{1+\sqrt{1-\Omega}\right\} \tag{8}$$

here $\Omega = \dfrac{2\eta^2}{\rho-2\eta^2}$

PROOF

Consider the Lyapunov functional (Hale [4], Lakashmikantham [5])

$$V = x^T P x + \int_{t-h(t)}^{t} x^T(\tau) H x(\tau) d\tau. \tag{9}$$

For convenience denote $x_d = x(t-h(t))$ and $u_d = u(t-h(t))$. [subscript "d" denotes delayed].

The time derivative of V along the controlled system trajectory is given by

$$\begin{aligned}\dot V &= 2x^T P\dot x + x^T Hx - [1-\dot h]x_d^T Hx_d \\ &= 2x^T P[Ax + Bu + BDx + BEu + BFu_d] + x^T Hx \\ &\quad - [1-\dot h]x_d^T Hx_d \\ &= 2x^T P[A - \tfrac{1}{2}\sigma BB^T P - \tfrac{1}{2}\sigma BEB^T P]x + 2x^T PBDx \\ &\quad - x^T PBFB^T Px_d\sigma + x^T Hx - [1-\dot h]x_d^T Hx_d.\end{aligned} \tag{10}$$

Equation (10) can be simplified to

$$\begin{aligned}\dot V &= x^T[PA + A^T P - \sigma PBB^T P - \sigma PBEB^T P]x + 2x^T D^T B^T Px \\ &\quad - x^T PBFB^T Px_d\sigma + x^T Hx - (1-\dot h)x_d^T Hx_d.\end{aligned} \tag{11}$$

Noticing that $PA + A^T P = -(Q+H)$ and letting $\alpha = \|B^T Px\|$ and $\gamma = \|B^T P\|$, we get

$$\begin{aligned}\dot V &\le x^T[-Q-H]x - \sigma\alpha^2 + \sigma k_3\alpha^2 + 2k_2\|x\|\alpha + \hat k_3\alpha\gamma\|x_d\|\sigma \\ &\quad + x^T Hx - (1-a)\|x_d^T Hx_d\|.\end{aligned} \tag{12}$$

or,

$$\begin{aligned}\dot V &\le -\lambda_{\min}(Q)\|x\|^2 - \sigma(1-k_3)\alpha^2 + 2k_2\|x\|\alpha + \hat k_3\sigma\gamma\|x_d\|\alpha \\ &\quad - (1-a)\lambda_{\min}(H)\|x_d\|^2.\end{aligned} \tag{13}$$

Now, we know $-(\phi-\varphi)^2 \le 0$, $\forall \phi,\varphi$, thus let $\phi = \sqrt{(1-a)\lambda_{\min}(H)}\|x_d\|$ and $\varphi = \dfrac{\hat k_3\sigma\gamma\alpha}{2\sqrt{(1-a)\lambda_{\min}(H)}}$, therefore, we get

$$-(1-a)\lambda_{\min}(H)\|x_d\|^2 + \hat k_3\sigma\gamma\alpha\|x_d\| \le \frac{\hat k_3^2\sigma^2\gamma^2\alpha^2}{4(1-a)\lambda_{\min}(H)}. \tag{14}$$

Recall equation (6)

$$\eta^2 = \frac{\hat k_3^2\gamma^2}{4(1-a)\lambda_{\min}(H)}. \tag{15}$$

By using (14) and (15), equation (13) becomes

$$\dot V \le -\lambda_{\min}(Q)\|x\|^2 - [\sigma(1-k_3) - \eta^2\sigma^2]\alpha^2 + 2k_2\|x\|\alpha. \tag{16}$$

Let's call $r = \sigma(1-k_3) - \eta^2\sigma^2$ and $s = k_2\|x\|$, then

$$\begin{aligned}&-r\alpha^2 + 2s\alpha - \frac{s^2}{r} = -(\sqrt{r}\alpha - \frac{s}{\sqrt{r}})^2 \le 0 \quad \text{,or} \\ &-r\alpha^2 + 2s\alpha \le \frac{s^2}{r}.\end{aligned} \tag{17}$$

which implies that r has to be positive, or equivalently

$$\sigma < \frac{1-k_3}{\eta^2} \quad \text{i.e.} \quad \sigma \le \frac{1-k_3}{\eta^2+\theta}, \quad \text{for some } \theta > 0. \tag{18}$$

Using the results of (17) in (16), we get

$$\dot V \le -\{\lambda_{\min}(Q) - \frac{k_2^2}{\sigma(1-k_3) - \eta^2\sigma^2}\}\|x\|^2 \tag{19}$$

For boundedness in a ball we require that

$$\sigma(1-k_3) - \eta^2\sigma^2 > \frac{k_2^2}{\lambda_{\min}(Q)}. \tag{20}$$

By substituting $\sigma = \dfrac{(1-k_3)}{\eta^2+\theta}$, $\theta > 0$, in (20) we get

$$\frac{\theta}{(\eta^2 + \theta)^2} = \frac{k_2^2}{(1-k_2)^2 \lambda_{min}(Q)} \tag{21}$$

Recalling (6), we observe that the right hand side of (21) is equal to $\frac{1}{\rho}$, thus

$$\theta^2 - \theta(\rho - 2\eta^2) + \eta^4 < 0 . \tag{22}$$

Completing the square for the first two terms in (22), results in

$$\left[\theta - \frac{\rho - 2\eta^2}{2}\right]^2 < \left(\frac{\rho - 2\eta^2}{2}\right)^2 - \eta^4 . \tag{23}$$

Which means that θ has to lie in the following range.

$$\frac{\rho - 2\eta^2}{2}\left\{1 - \sqrt{1 - \Omega^2}\right\} < \theta < \frac{\rho - 2\eta^2}{2}\left\{1 + \sqrt{1 - \Omega^2}\right\} \tag{24}$$

where $\Omega = \frac{2\eta^2}{\rho - 2\eta^2}$.

To make sure that θ will be a real positive constant we require

$$\rho - 2\eta^2 > 0 \quad \text{and} \quad \Omega^2 < 1 . \tag{25}$$

Providing that $\rho > 4\eta^2$ will ensure the satisfaction of conditions given by (25). Therefore, $\dot{V}$ is negative definite and the origin is asymptotically stable.

Simulation

To Check the suggested control action, it has been simulated on a five degree of freedom model with considerably low damping.

The equations of motion for the system are given by,

$$M\ddot{x} + C_d\dot{x} + Kx = u(t) , \tag{26}$$

where,

$$M = \begin{bmatrix} m_1 & 0 & \cdots & \cdots & 0 \\ 0 & m_2 & \ddots & & \vdots \\ \vdots & \ddots & m_3 & \ddots & \vdots \\ \vdots & & \ddots & m_4 & 0 \\ 0 & \cdots & \cdots & 0 & m_5 \end{bmatrix}$$

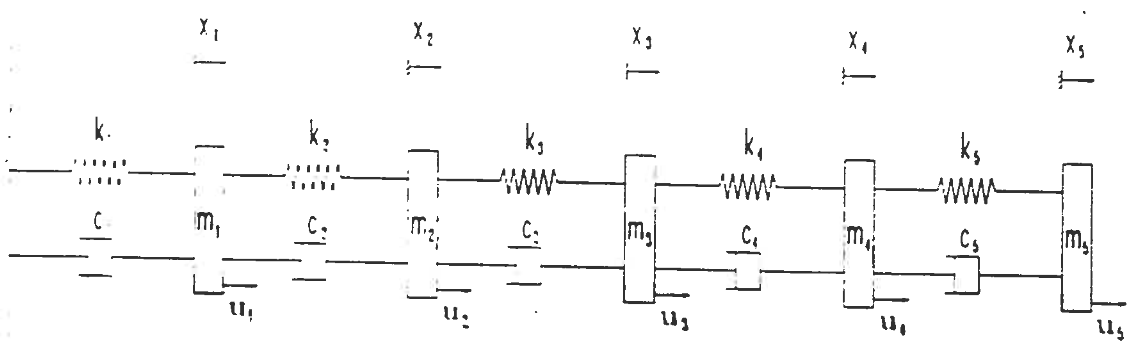
$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \ddots & \vdots \\ 0 & -k_3 & k_3 + k_4 & -k_4 & 0 \\ \vdots & \ddots & -k_4 & k_4 + k_5 & -k_5 \\ 0 & \cdots & 0 & -k_5 & k_5 \end{bmatrix} ,$$

$$C_d = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2 + c_3 & -c_3 & \ddots & \vdots \\ 0 & -c_3 & c_3 + c_4 & -c_4 & 0 \\ \vdots & \ddots & -c_4 & c_4 + c_5 & -c_5 \\ 0 & \cdots & 0 & -c_5 & c_5 \end{bmatrix}$$

and

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} ,$$

with $m_i = 1$, $k_i = 2$ and $c_i = 0.2$ for $i = 1, 2, \ldots, 5$.



Fgure 1: Five degree of freedom system.

The uncertainties in the system appear in the matrices K and C_d as we can not exactly measure the system's parameters k_i and c_i's. Let's assume there are no uncertainties in the m_i's, but ten percent in the k_i's, and twenty percent in the c_i's, (i.e., $\Delta k_i = random[-0.1,0.1]k_i$, $\Delta c_i = random[-0.2,0.2]c_i$, where "random" means random number with uniform distribution.). The matrices K and C_d in the decomposed form can be written as $K+\Delta K$ and $C_d + \Delta C_d$, where ΔK and ΔC_d have the same structure as the matrices K and C_d with the components consisting of Δk_i and Δc_i's, respectively. Now we can write (26) as follow

$$M\ddot{x} + (C_d + \Delta C_d)\dot{x} + (K + \Delta K)x = u(t) \tag{27}$$

To write the system's equations in the form of equation (1), we can convert the (27) to the following state space form

$$\dot{y} = \left[\begin{array}{c|c} 0 & I_{5\times 5} \\ \hline -M^{-1}(K+\Delta K) & -M^{-1}(C_d + \Delta C_d) \end{array}\right] y + \left[\begin{array}{c} 0 \\ \hline M^{-1} \end{array}\right] u \tag{28}$$

where,
$$y = [x_1, x_2, \dots, x_5, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_5]^T. \tag{29}$$

By assuming that at the most two percent of the control input is coming into the system as the result of the time delay, equation (28) is written as

$$\dot{y} = \underbrace{\left[\begin{array}{c|c} 0 & I_{5\times 5} \\ \hline -M^{-1}K & -M^{-1}C_d \end{array}\right]}_{A} y + \underbrace{\left[\begin{array}{c|c} 0 & 0 \\ \hline -M^{-1}\Delta K & -M^{-1}\Delta C_d \end{array}\right]}_{\Delta A} y + \underbrace{\left[\begin{array}{c} 0 \\ \hline M^{-1} \end{array}\right]}_{B} u + \underbrace{\left[\begin{array}{c} 0 \\ \hline qM^{-1} \end{array}\right]}_{\delta B} u_d \tag{30}$$

where "q" is a uniformly distributed random number in the range of (-0.02,.002), and the delay function "h(t)" is chosen as $(1.2 - e^{-t})$.

Now by looking at equation (30) we can clearly see the matrices D,E,F and by taking the appropriate norms of these matrices can find the uncertainty bounds. By using the uncertainty bounds and checking the condition given in the equation (5) the control gain has been calculated and the integration of the system's equations has been done using MATLAB. The results of the simulation are shown in the following figures.

Figure 2 shows the simulation results for the system without the control action. The oscillatory behavior is due to the underdamped characteristics of the system. (i.e. very small damping. Figure 3 shows the results for the controlled system. By comparing Figure 2 and Figure 3, we see that both the displacement and the velocity of each mass have been considerably decreased and died out after a short period of time, which shows the asymptotic stability of the response of the system under proposed control input given by equation (3).

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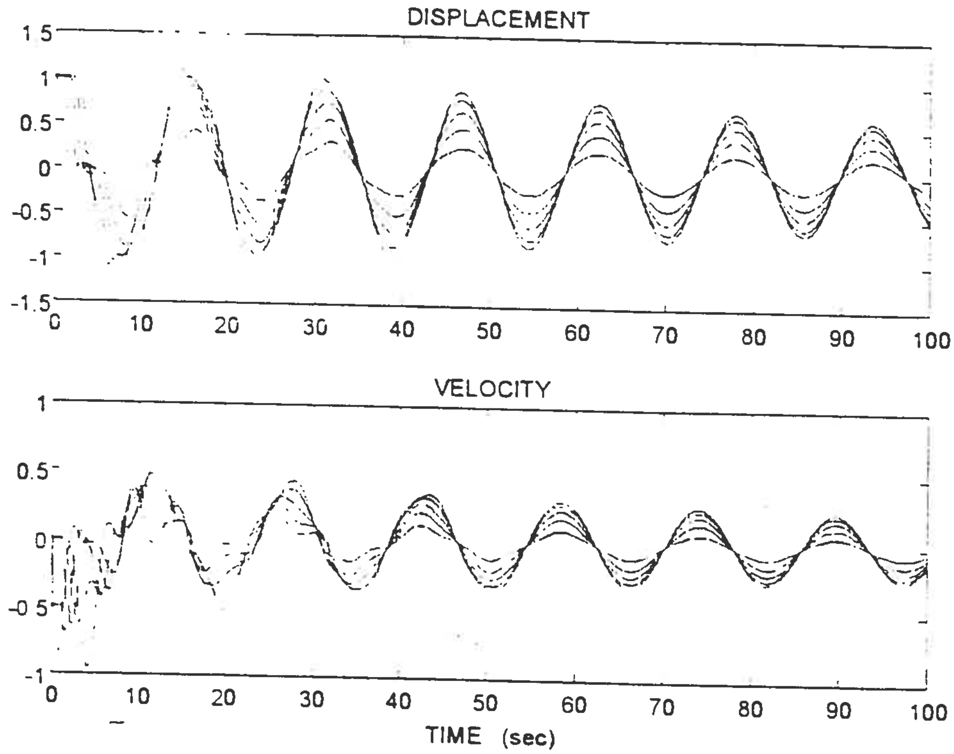

Figure 2: Uncontrolled system.

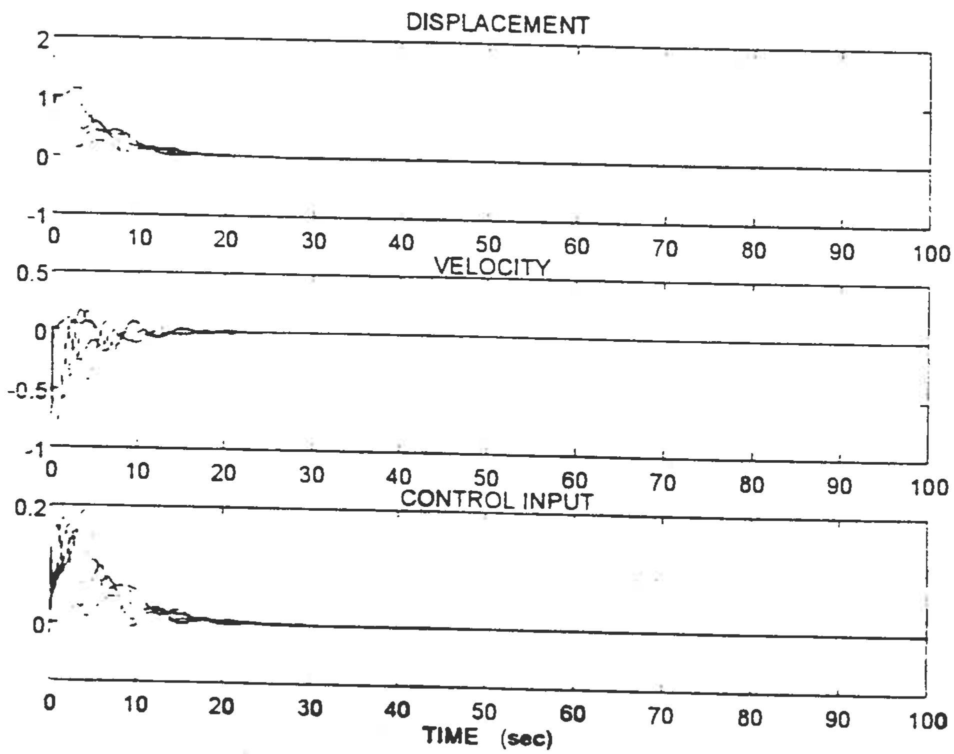

Figure 3: System under proposed control input.