Computational Mechanics (1989) 5, 113–118

Computational Mechanics



# Period three may not mean chaos: An example of a piecewise linear map

**R. S. Guttalu** and **F. E. Udwadia**
Dept. of Mechanical Engineering, University of Southern California, Los Angeles, CA 90089, USA

**Abstract.** This paper presents an example of a piecewise linear map used to model the dynamics of certain nonlinear mechanical systems. It is shown that a period three solution exists, and computational study seems to indicate that it is a global attractor except possibly over a set of measure zero. The computations are performed using the cell mapping approach from both the deterministic and probabilistic points of view.

## 1 Introduction

Li and Yorke have shown that one dimensional iterated maps on an interval possessing stable or unstable period three orbits have orbits of all other periods along with an uncountable set of points which show sensitive dependence to initial conditions (Li and Yorke 1975; Straffin 1978; Collet and Eckmann 1983). While few general statements can be made about the stability of the periodic orbits of arbitrary continuous maps, Singer (1978) and Guckenheimer et al. (1977) have shown that for functions which are $C^3$ unimodal and with negative Schwarzian derivative [more precisely, $S$-modal (Collet and Eckmann 1983)], at most one stable, periodic orbit exists and that if such an orbit exists the measure of points not attracted to the orbit is zero.

With few exceptions (Hsu and Yee 1975) only a few investigators have looked at piecewise linear maps, and general results for the stability of their periodic orbits are lacking in the literature. Such maps often arise in the modeling of the dynamics of nonlinear mechanical systems. Often it is convenient to study the dynamics of higher dimensional nonlinear systems by using an appropriate Poincaré section map or return map in one dimension. Such maps which may often contain just a few points can then be approximated by piece-wise linear maps (Yorke and Yorke 1979). In this paper we consider a simple piecewise linear map and show that it has just one stable periodic orbit of period three which is a global attractor except possibly for points of measure zero. The results are computational in nature and use the method of cell mapping to establish the existence of periodic orbits. These computations are followed up by applying probalistic cell mapping to search for non-periodic orbits. Our results show that certain physical systems which can be modeled by piecewise linear maps which may not be topologically conjugate to $S$-modal maps have stable period three orbits. Further this period three orbit may be a strong attractor almost all over the phase space.

## 2 Characteristics of the map and computational results

The dynamics of a one-dimensional map $f$ is described by

$$x(n+1) = f(x(n)), \quad n = 0, 1, 2, \ldots \tag{1}$$

The map $f$ used for the study is shown in Fig. 1 along with its third, fourth and eight iterates. It is of a type that often arises in the study of nonlinear mechanical systems. The interval [0, 1] is mapped into itself. We notice clearly the emergence of the stable period three solution. We denote a period

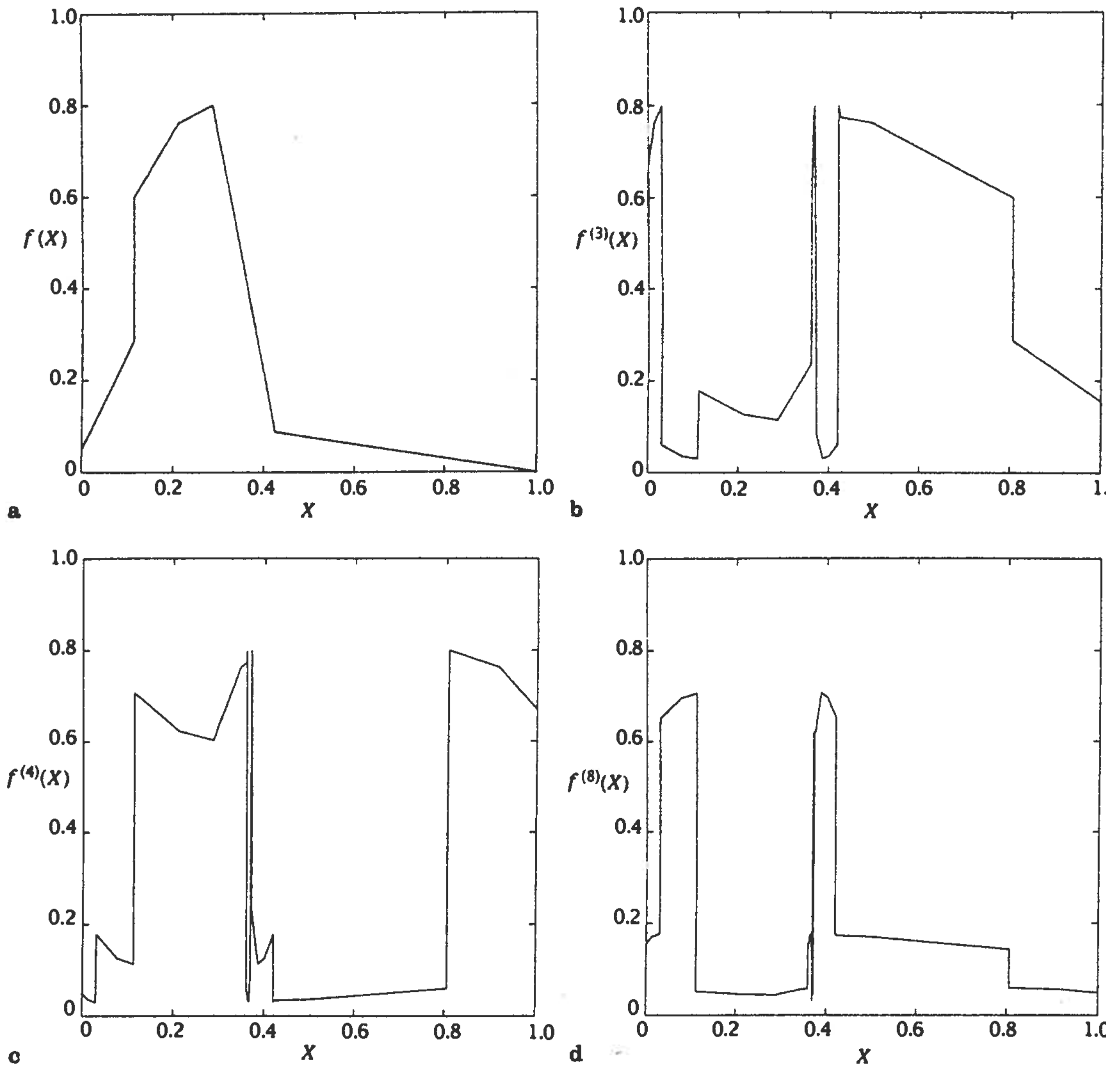



**Fig. 1. a** The piecewise linear map with coordinates (0.0, 0.05), (0.11250, 0.2875), (0.11251, 0.6), (0.2125, 0.7625), (0.2875, 0.8), (0.425, 0.0875), and (1.0, 0.0); **b** the third iterate; **c** the fourth iterate; **d** the eight iterate of the map $f$

three solution by $P$-3. Figure 2 shows results of the iteration process for three different starting locations in [0, 1]. In each case, the iterations converge to the $P$-3 solution located at

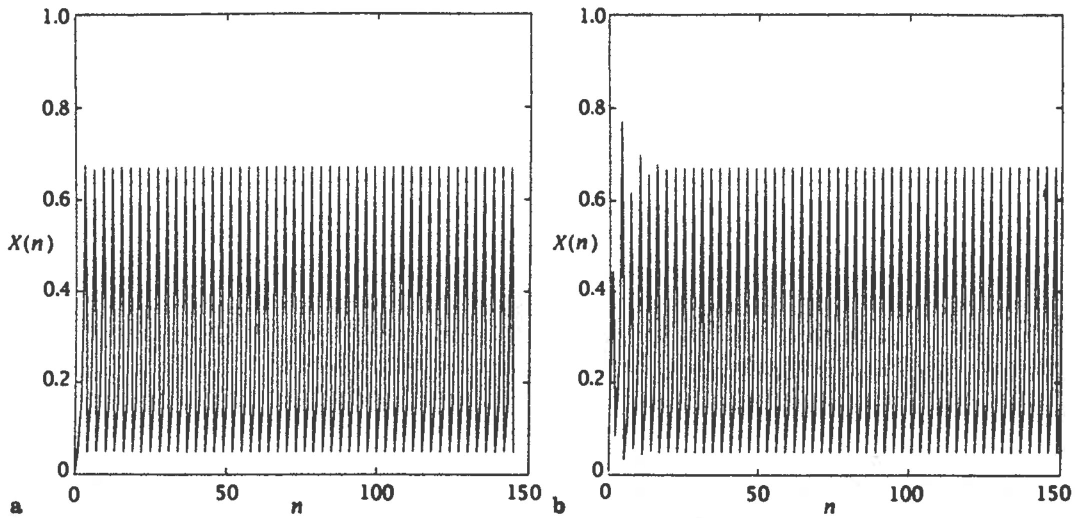
$$x(1) = 5.0147220543433 \times 10^{-2},$$
$$x(2) = 0.15586635448058, \tag{2}$$
$$x(3) = 0.67046112214316,$$

which is accurate to 14 decimal digits.

The search for periodic orbits of the map $f$ is carried out using the cell mapping approach wherein the interval [0, 1] is discretized into a large number of cells. The cell mapping method is motivated by both the practical and realistic points of view considering the limitations on the accuracy of measuring the physical variables and the round-off errors incurred in numerical computations (Hsu 1980). Here, the continuum of the state space is discretized to obtain a cellularly structured state space where the cell state variable $Z$ assumes integer values according to

$$(Z - \tfrac{1}{2})h \leq x < (Z + \tfrac{1}{2})h, \quad Z = 1, 2, \ldots n. \tag{3}$$

where $h$ is the cell size along the $x$-axis and $n$ the number of cells used. Then, the cell-to-cell mapping dynamical system is described by the following integer map:



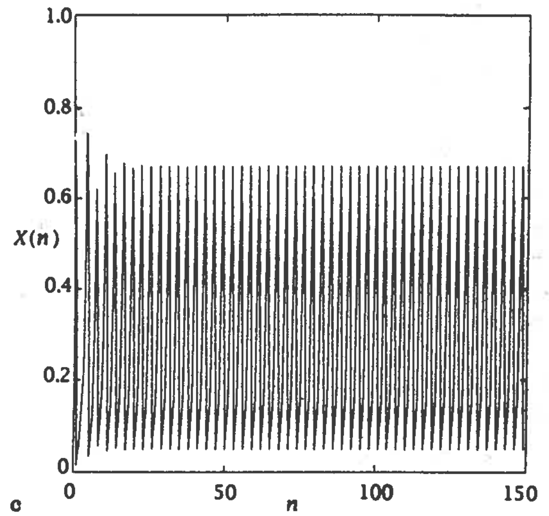




Fig. 2a–c. Iterations of the map $f$ starting from a $x_0 = 0.001$, b $x_0 = 0.112505$, and c $x_0 = 0.9$

$$Z(n+1) = C(Z(n)), \qquad n = 0, 1, 2, \ldots \tag{4}$$

where $C$ is known as the simple cell mapping. In this case a trajectory of (4) starting at an inital cell state $Z_0(0)$ is the set of integer cell sequence $\{Z_0(k)\}$, $k = 0, 1, 2, \ldots$ The crucial step in cell mapping analysis is to unravel the dynamic information of the original system (1) contained in (4) by examining the long time behavior of the cell sequences. In practical applications, one is usually interested in a fixed state space region which contains a finite number of cells even though the number of cells may be huge; the cells in this state space will be referred to as the regular cells. The complement of the state space is referred to as the sink cell. Once a regular cell state of the system maps to the sink cell, its long time behavior is unknown and its motion is eventually locked in the sink cell. An important property of the cell sequences, due to the finite number of cells in the state space, is that all sequences must terminate with a finite number of cell mappings into one of the steady cell states: equilibrium cell, periodic cells, or the sink cell. This is the key to the global cell mapping algorithm described in Hsu and Guttalu (1980) from which the following characteristics of the dynamics of the map (1) can be obtained simultaneously:

(1) Location of the equilibrium states and periodic states in a given cell state space.
(2) Domains of attraction associated with the asymptotically stable equilibrium states and periodic solutions.
(3) Step-by-step evolution of the global behavior of the system starting from any initial state within the cell state space.

The cell mapping method we have employed above is not capable of allowing multiple image cells and requires extremely fine cells to delineate the boundaries of the global regions of attraction. To overcome these deficiencies, a generalized cell mapping theory as proposed by Hsu (1981) is used. This allows each cell to have several image cells with each image cell having a definite fraction of the total probability. This formulation leads naturally to a description of the dynamical system (1) in terms of finite Markov chains:

$$\zeta(n+1) = \mathbf{P}\zeta(n), \qquad n = 0, 1, 2, \ldots \tag{6}$$

where $\zeta(n)$ is the cell probability vector and $\mathbf{P}$ is the transition probability matrix. Again the nonlinear problem is cast in the form of a linear system (6) and involves computation of $\mathbf{P}$ from (1) and the classification of cell states into persistent cells or transient cells to arrive at a probabilistic description of the global behavior of the system. The generalized cell mapping algorithm described by Hsu et al. (1982) can be used to obtain the following information:
(1) Location of acyclic groups and periodic groups (persistent groups),
(2) Limiting probability distribution of persistent groups,
(3) Absorption probability of transient cells into persistent groups.
For our map $f$, $\mathbf{P}$ may be computed from (1) using the sampling technique by taking a number of points in each cell state and finding the distribution of image cells to determine the probability of mapping to a particular cell. For computation we choose 100,000 cells in the interval $[0,1]$ with $h = 10^{-5}$. The probability transition matrix $\mathbf{P}$ is obtained by sampling 1000 subintervals in each cell to compute the probability of mapping from one cell to its image cells. For the analysis, we use the matrix $\mathbf{P}^3$ and iteratively obtain the long term probability distribution. The analysis of the results shows that there exists only one cyclic group consisting of the $P$-3 orbit and the rest of the cells turn out to be transient cells. The $P$-3 cyclic group is made up of three subgroups which are given below with their associated limiting probabilities $p$:

| | | |
|---|---|---|
| Subgroup 1: | $Z(1) = [5.01400 \times 10^{-2}, 5.01500 \times 10^{-2})$, | $p = 0.110126$ |
| | $Z(2) = [5.01500 \times 10^{-2}, 5.01600 \times 10^{-2})$, | $p = 2.10341 \times 10^{-2}$ |
| Subgroup 2: | $Z(3) = [0.155860, 0.155870)$, | $p = 0.444637$ |
| Subgroup 3: | $Z(4) = [0.670450, 0.670460)$, | $p = 0.168186$ |
| | $Z(5) = [0.670460, 0.670470)$, | $p = 0.255990$ |

The sum of the probabilities is 0.9999731 which is accurate to four decimal places. Figure 5a shows the long time probability distribution of the above orbit which is obtained using the Markov chain approach. This shows up as three distinct peaks in the probability density distribution indicating

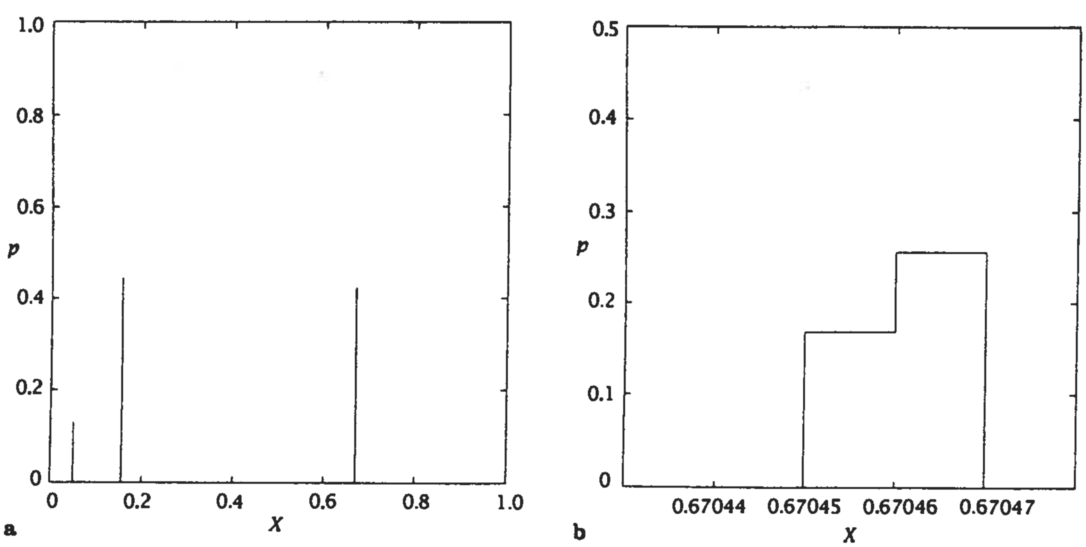



**Fig. 5a and b.** Long time probability distribution **a** of the map $f$ indicating a globally stable $P$-3 orbit, and **b** of the third subgroup of the $P$-3 orbit

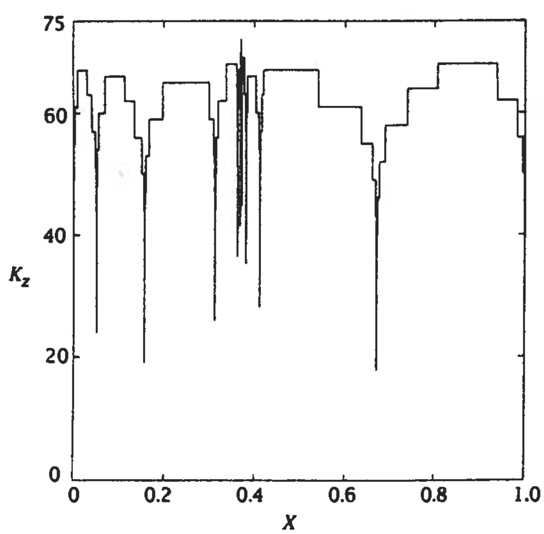
Since the cell mapping $C$ is an approximation of the original system, the accuracy of the results can be improved by simply increasing the number of cells in the state space. The cell mapping technique is applied in two steps. The first step is a computational step, in which a cell mapping $C$ is obtained from the nonlinear map $f$ by computing the image cell using the center point of each cell in the cell space. Since $f$ maps the interval $[0, 1]$ to itself, no cells in the cell space will be mapped to the sink cell. In the second step, global properties of each cell are obtained by using the simple cell mapping algorithm which essentially consists of a simple sorting procedure, and no computations are involved here. Overall, the cell mapping method avoids repetitive time consuming calculations.

We observe that the nonlinear map $f$ is replaced by an equivalent integer map $C$ which essentially contains the dynamics of the map. We apply the cell mapping method to the map $f$ by dividing the interval $[0, 1]$ into $n = 10^7$ cells with cell size $h = 1.0 \times 10^{-7}$. The results indicate that there exists a group of three cells of period three ($P$-3 cells) located in the intervals given by

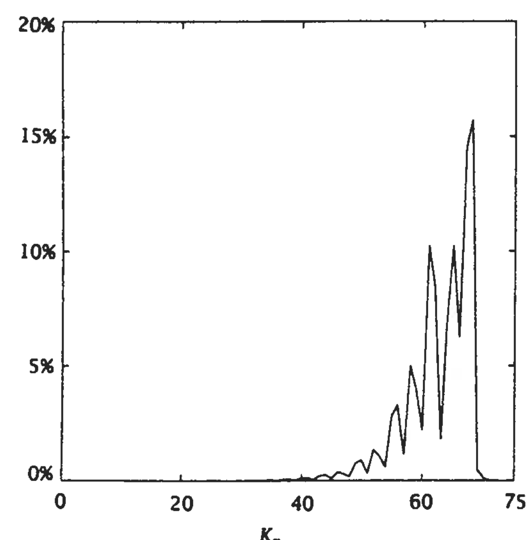
$$\begin{aligned} Z(1) &= [5.014720 \times 10^{-2}, 5.014730 \times 10^{-2}), \\ Z(2) &= [0.15586640, 0.1558665), \\ Z(3) &= [0.6704612, 0.6704613). \end{aligned} \tag{5}$$

The results also show that none of the regular cells is mapped to the sink cell and that all the transient cells are mapped to the $P$-3 cells. The number of cell mappings $K_z$ required for a cell at a particular location to map into the $P$-3 orbit (5) is shown in Fig. 3. Since we have chosen a large number of cells for this computation, it is impractical to show every cell location on the $x$-axis in this figure. We average the number of maps $K_z$ over 200 cells continuously at a time throughout the unit interval. The maximum number of maps required for a cell to map to $P$-3 orbit (5) is 75. It is interesting to note that on the average about 150 iterations are needed for an initial state to map to $P$-3 orbit (2) obtained by double precision computations. Figure 4 gives the histogram of the percentage of cells requiring the number of maps to reach the $P$-3 orbit (5). As seen the maximum number of maps required, starting from any cell located in $[0, 1]$, to reach the stable $P$-3 orbit is only 75 which indicate that the $P$-3 solution may be a strong attractor for points almost over the interval $[0, 1]$.

A more refined cell mapping analysis of $f$ with $n = 5 \times 10^7$ cells and $h = 2.0 \times 10^{-8}$ again shows that there exists only one $P$-3 orbit located precisely at (5). In fact the $P$-3 orbit was found to be remarkably stable even under small perturbations of the slopes of the different segments of the map $f$.




3




4

**Figs. 3 and 4.** 3 Number of maps (or steps) required for a cell to reach the $P$-3 orbit with $n = 10^7$, and $h = 1.0 \times 10^{-7}$; the data is averaged continuously over 200 cells at a time. 4. A histogram of percentage of cells and number of mappings required for them to reach $P$-3 orbit the presence of a strongly attracting $P$-3 orbit. Figure 5 b shows the long time probability distribution of the third subgroup on an expanded scale; this is represented as one peak with a total probability of 0.612823 in Fig. 5a. The probabilistic analysis of the piecewise linear map again indicates that the $P$-3 orbit is almost globally stable.

## 3 Conclusions

We have shown here through computer experimentation that piecewise linear functions may lead to iterated maps which have stable period three solutions which may be almost global attractors, the Lebesgue measure of points not attracted to the orbit being zero. Thus certain piecewise linear maps may have similar characteristics to $S$-modal systems.

## Acknowledgements

The research was supported by the USC Faculty Research and Innovation Fund and the National Science Foundation under grant No. EET-8709267.

## References

Collet, P.; Eckmann, J. P. (1983): Iterated maps on the intervals as dynamical systems. Birkhäuser: Boston

Guckenheimer, J.; Oster, G.; Ipaktchi, A. (1977): The dynamics of density dependent population models. J. Math. Biol. 4, 101–147

Hsu, C. S. (1980): A theory of cell-to-cell mapping dynamical systems. J. Appl. Mech. 47, 931–939

Hsu, C. S. (1981): A generalized theory of cell-to-cell mapping for nonlinear dynamical systems. J. Appl. Mech. 48, 634–642

Hsu, C. S.; Guttalu, R. S. (1980): An unravelling algorithm for global analysis of dynamical systems: An application of cell-to-cell mappings. J. Appl. Mech. 47, 940–948

Hsu, C. S.; Yee, H. C. (1975): Behavior of dynamical systems governed by a simple nonlinear difference equation. J. Appl. Mech. 42, 870–876

Hsu, C. S.; Guttalu, R. S.; Zhu, W. H. (1982): A method of analyzing generalized cell mappings. J. Appl. Mech. 49, 895–904

Li, T.; Yorke, J. A. (1975): Period three implies chaos. Am. Math. Month. 82, 985–992

Singer, D. (1978): Stable orbits and bifurcation of maps of the interval. SIAM J. Appl. Math. 35, 260–267

Straffin, P. D. (1978): Periodic points of continuous functions. Math. Mag. 51, 99–105

Yorke, J. A.; Yorke, E. D. (1979): Metastable chaos: The transition to sustained chaotic behavior in the Lorenz model. J. Stat. Phys. 21, 263–277

## Note added in proof

The methods of simple cell mapping and generalized cell mapping employed in this paper are described in detail in a recent research monograph by C. S. Hsu (1987): Cell-to-Cell Mapping: a method of global analysis of nonlinear systems. Springer-Verlag: New York

*Communicated by S. N. Atluri, February 18, 1988*