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# Energy control of inhomogeneous nonlinear lattices

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This paper considers energy control of an $n$-d.f. inhomogeneous nonlinear lattice with fixed–fixed and fixed–free ends. The lattice consists of dissimilar masses wherein each mass is connected to its nearest neighbour by a nonlinear or linear memoryless spring element. The potential functions of the nonlinear spring elements are assumed to be qualitatively different. Each potential is described by a twice continuously differentiable, strictly convex function, possessing a global minimum at zero displacement, with zero curvature possibly only at zero displacement. The energy control requirement is viewed from an analytical dynamics perspective and is recast as a constraint on the motion of the dynamical system. No linearizations and/or approximations of the nonlinear dynamical system or the controller are made. Given the set of masses at which control is to be applied, explicit closed form expressions for the nonlinear control forces are obtained. Global asymptotic convergence to any desired non-zero energy state is guaranteed provided that the first mass, or the last mass or, alternatively, any two consecutive masses in the lattice are included in the subset of masses that are controlled. Numerical simulations involving a 101-mass nonlinear lattice demonstrate the simplicity and efficacy of the approach.

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## 1. Introduction

This paper deals with the energy control of an inhomogeneous nonlinear one-dimensional lattice. The inhomogeneous nonlinear lattice is made up of a chain of masses connected together by linear or nonlinear elastic springs, where the *qualitative* nature of the nonlinear spring elements along the chain can be, in general, different from one another. The study of energy distribution in nonlinear lattices consisting of identical masses and identical spring elements, called homogeneous lattices, was initiated by Fermi, Pasta and Ulam (FPU) in 1955 [1]. Contrary to their expectations, the energy in the various modes did not reach a state of equipartition. Instead, the long-term dynamics of the FPU lattice appeared to be periodic with the energy remaining trapped in the small number of modes with which it was initialized. Although many of the puzzling aspects of the FPU phenomenon are well understood now [2,3], the fundamental research that ensued following the seminal work of FPU has opened up many new interesting questions which are still being actively pursued to date [4,5].

The force–displacement curves of the springs studied by FPU included quadratic, cubic and quartic spring potentials. Later, other spring potentials were considered such as the potentials of the Toda lattice [6], the so-called $\phi^4$ lattice [7], and the Klein–Gordon chains [8]. These homogeneous FPU-like lattices are known to display intricate structures in their response such as solitons, phonons, breathers and nanopterons. Nearly all the research done in this field to date has focused on homogeneous lattices. Studies on inhomogeneous lattices in the literature most often either deal with weakly inhomogeneous lattices [9] or are limited to two d.f. systems [10]. However, when dealing with engineering applications, disparities in material stiffness in spatially extended mechanical systems is rather the norm than the exception. Hence, it is important to study inhomogeneous lattices from an engineering standpoint.

The literature on nonlinear lattices with dissimilar masses wherein the nonlinearity of each of the spring elements in the lattice is qualitatively different is, to the best of the authors' knowledge, non-existent. Nevertheless, these type of nonlinear lattices are representative of real life behaviour and arise frequently in engineering models. Besides their applications to engineering situations, the theoretical understanding and control of such nonlinear lattices is of fundamental importance.

In this study, an *n*-d.f. general nonlinear lattice with fixed–fixed and fixed–free boundary conditions is considered, in which

(1) the masses in the lattice can all be chosen to be dissimilar,
(2) different nonlinear and/or linear memoryless spring elements can be chosen, wherein
    (a) the *qualitative* nature of each of the nonlinear spring elements along the lattice can be different, and
    (b) the parameters of the potential functions describing each spring element can also be different.

The spring force associated with each spring element in the lattice is assumed to be derivable from a potential function which (i) is $C^2$, (ii) is strictly convex possessing a global minimum at zero displacement, and (iii) has zero curvature possibly only at zero displacement. Such potential functions lead to a variety of spring forces that include, but are not limited to, the linear spring force, the cubic spring force, the quintic spring force and the Toda spring force. Asymmetries in the potential function would naturally lead to disparities between the magnitudes of the tensile and compressive forces exerted by the springs, as is often the case with many real-life elastic materials. For example, flexible cables in suspension bridges are strong under tensile forces but are weak under compressive forces. Thus, one can potentially model these and many such structural subsystems using nonlinear lattices with asymmetrical potential functions.

The main focus of the present study is to control the energy of these nonlinear lattices and bring them to a desired energy state. The control approach adopted in this paper distinguishes itself from related work on the subject in five key ways. First, control of FPU-like lattices is restricted to homogeneous lattices in the literature [11,12]. In this study, general inhomogeneous lattices are
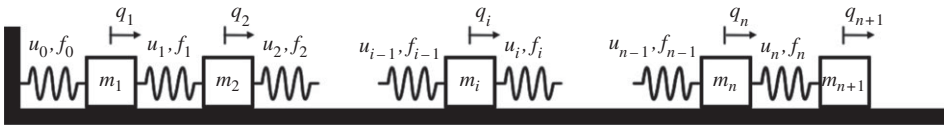
**Figure 1.** An inhomogeneous nonlinear lattice.

considered. Second, the energy control problem is approached using the theory of constrained motion, wherein the key idea is to recast the energy control requirement on the nonlinear lattice as an energy constraint on the system. The fundamental equation of mechanics [13] is employed to determine explicit closed form expressions for the nonlinear control forces. This approach is inspired by recent results in analytical dynamics [13]. Third, the methodology developed herein allows us to explicitly determine control forces needed to be applied to any arbitrarily chosen subset of masses that are designated to be actuators, and still obtain global asymptotic convergence to any desired non-zero energy state provided that the first mass, or the last mass or, alternatively, any two consecutive masses of the lattice are included in this subset. Fourth, once the energy of the system is brought to its desired value, the control forces automatically terminate, and the conservative nature of the ensuing Hamiltonian dynamics is used to maintain it at the desired energy level for all future time. Finally, in spite of the general nature of the nonlinear lattice considered in this study, the control is obtained in closed form with relative ease without the need to make any approximations and/or linearizations of the nonlinear dynamical system.

This paper is organized as follows. The equations of motion of an $n$-d.f. inhomogeneous nonlinear lattice with fixed–fixed and fixed–free boundary conditions are derived in §2. The constrained motion approach is briefly recalled in §3. In §4, the energy control problem is formulated and closed form expressions for the control forces are derived. In §5, the invariance principle [14] is used to derive sufficient conditions for the placement of the actuators so that the control force obtained in §4 gives us global asymptotic convergence to any given non-zero desired energy state. And, finally, in §6, numerical simulations involving a 101-mass nonlinear lattice are presented that illustrate the efficacy and simplicity with which the control approach can be effected. Several of the technical details have been placed in the appendices to maintain the flow of thought.

## 2. Equations of motion

Consider a lattice with $n + 1$ masses wherein each of the masses is connected to its neighbouring mass with the help of a nonlinear memoryless spring element as shown in figure 1. The nonlinear potential of the $i$th spring element is denoted by $u_i(x)$, where $x$ is the displacement of the $i$th spring element from the equilibrium position. The nature of the nonlinearity in each of the spring elements can, in general, be different provided that each of the spring potentials $u_i(x), i = 0, 1, \ldots, n$, satisfies the following three properties:

(1) $u_i(x)$ is a $C^2$ function;
(2) $u_i(x)$ is strictly convex with a global minimum at $x = 0$;
(3) $u_i''(x) = 0$ only at $x = 0$.

By strictly convex, we mean $u_i(\alpha x + (1 - \alpha)y) < \alpha u_i(x) + (1 - \alpha)u_i(y) \, \forall \, \alpha \in (0, 1)$ and $x \neq y$. Without any loss of generality, let us add a suitable constant to $u_i(x)$ such that $u_i(0) = 0$. This along with property (2) implies that the potentials of the spring elements are strictly positive definite (i.e. $u_i(0) = 0, u_i(x) > 0 \, \forall \, x \neq 0$). Properties (2) and (3) also imply that the potentials are strictly radially increasing (i.e. $\alpha > 1 \Rightarrow u_i(\alpha x) > u_i(x) \, \forall \, x \in \Re - \{0\}$). In one dimension, if a function is strictly radially increasing, then it is also radially unbounded and therefore, in our case, each $u_i(x)$ is also

radially unbounded (i.e. $u_i(x) \to \infty$ as $\|x\| \to \infty$). The spring force $f_i(x)$ of the $i$th spring element is also nonlinear in general, and is assumed to be derivable from a potential as

$$F_{\text{spring}}(x) = -F_{\text{restoring}}(x) = f_i(x) = \frac{\partial u_i(x)}{\partial x},$$

where $f_i(0) = 0$, and $x f_i(x) > 0 \, \forall x \neq 0$. Furthermore, it follows from property (2) that $u_i''(x) = f_i'(x) \geq 0$, with $u_i''(x) = f_i'(x) = 0$ possibly only at $x = 0$ according to property (3). Additionally, if the spring potentials are asymmetrical, then the springs exhibit dissimilar tensile and compressive characteristics.

Let $m_i$ denote the mass of the $i$th particle in the lattice, where $i = 1, 2, \ldots, n+1$. The coordinate describing the motion of mass $m_i$ measured from its equilibrium position in an inertial frame of reference is denoted by $q_i$ (figure 1). The velocity of mass $m_i$ is denoted by $\dot{q}_i$. The total energy $H$ of the lattice is given by

$$H(q, \dot{q}) = T(\dot{q}) + U(q) = \sum_{i=1}^{n+1} \frac{1}{2} m_i \dot{q}_i^2 + \sum_{i=0}^{n} u_i(q_{i+1} - q_i), \tag{2.1}$$

where $q$ and $\dot{q}$ denote the vector of displacements and velocities of the lattice, respectively. Furthermore, $q_0 \equiv 0$ because the left end of the lattice is fixed for all time $t$ (figure 1). The energy $H$ is a smooth and continuously differentiable function. It is also strictly positive definite and radially unbounded (see appendix C). Using Newton's laws of motion, the equation of motion of an $n$-d.f. nonlinear lattice can be written down in matrix form as

$$\underbrace{\begin{bmatrix} m_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & m_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & m_n \end{bmatrix}}_{M_{n \times n}} \underbrace{\begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_i \\ \vdots \\ \ddot{q}_n \end{bmatrix}}_{\ddot{q}_{n \times 1}} = \underbrace{\begin{bmatrix} f_1(q_2 - q_1) - f_o(q_1) \\ \vdots \\ f_i(q_{i+1} - q_i) - f_{i-1}(q_i - q_{i-1}) \\ \vdots \\ f_n(q_{n+1} - q_n) - f_{n-1}(q_n - q_{n-1}) \end{bmatrix}}_{F_{n \times 1}(q)}, \tag{2.2}$$

where $q_{n+1} \equiv 0$ for a fixed–fixed lattice, and $f_n \equiv 0$ for a fixed–free lattice.

# 3. Constrained motion approach and the fundamental equation of mechanics

In this paper, the fundamental equation of mechanics [13] is used to derive the equations of motion of the controlled (constrained) $n$-d.f. nonlinear lattice and thus to obtain the explicit nonlinear control forces required to achieve the desired energy stabilization. The fundamental equation is known for the relative ease with which the constrained equations of motion of a complex multibody system can be derived in comparison to other classical methods.

Consider an unconstrained, discrete dynamic system of $n$ particles [15] similar to our $n$-d.f. nonlinear lattice with appropriate boundary conditions (as described in §2, equation (2.2)). The equations of motion of this unconstrained (uncontrolled) system at a certain instant of time $t$ can be written down using Newton's laws or Lagrange's method as

$$M(q, t)\ddot{q} = F(q, \dot{q}, t), \quad q(0) = q_o, \quad \dot{q}(0) = \dot{q}_o, \tag{3.1}$$

where $M$ is the $n$-by-$n$ symmetric, positive definite mass matrix, $q$ is the $n$ vector of generalized coordinates of the system, and $F$ is the $n$ vector of generalized 'given' forces acting on the

unconstrained system. The acceleration $a$ of the unconstrained system is given by

$$a(q, \dot{q}, t) = [M(q, t)]^{-1} F(q, \dot{q}, t). \tag{3.2}$$

Consider now that we impose a set of $m$ constraints on the unconstrained system, all of which may or may not be independent, i.e. some of the constraints may be a combination of others [16]

$$\phi_i(q, \dot{q}, t) = 0, \quad i = 1, 2, 3, \ldots, m. \tag{3.3}$$

The initial conditions stated in equation (3.1) are assumed to satisfy these constraint equations. However, in some cases, it may not be possible to initialize the unconstrained system from points in the phase space where the constraints are satisfied. Thus, instead of considering the existing set of $m$ constraints described by equation (3.3), we modify the constraint equations as follows [17]

$$\psi_i(q, \dot{q}, \ddot{q}, t) = \dot{\phi}_i + \beta \phi_i = 0, \quad i = 1, 2, 3, \ldots, m, \tag{3.4}$$

where $\beta(q, \dot{q}) > 0$ is chosen so that the system of equations (3.4) has an equilibrium point described by equation (3.3) and that this equilibrium point is stable. This set of $m$ modified constraints can now be expressed in the general constraint matrix form as

$$A(q, \dot{q}, t) \ddot{q} = b(q, \dot{q}, t), \tag{3.5}$$

where $A$ is an $m$-by-$n$ constraint matrix of rank $r$ (i.e. $r$ out of the $m$ constraint equations are independent) while $b$ is a column vector with $m$ entries. The presence of constraints causes the acceleration of the constrained system to deviate from its unconstrained acceleration at every instant of time $t$. This deviation in the acceleration of the constrained (controlled) system is brought about by a force, $F^C$, called the constraint (control) force, which is exerted on the system by virtue of the fact that the unconstrained system must now further satisfy an additional set of constraints. The equation of motion of the constrained system can now be written down as

$$M(q, t) \ddot{q} = F(q, \dot{q}, t) + F^C(q, \dot{q}, t), \tag{3.6}$$

where $F^C$ is the set of additional forces that arise by virtue of the application of the $m$ constraints. One can also envision $F^C$ to be the set of control forces that are required to be applied to the uncontrolled open loop system to obtain the controlled closed loop system. Udwadia & Kalaba [13] proposed the following closed form expression for the constraint force (or the control force)

$$F^C(q, \dot{q}, t) = M^{1/2}(AM^{-1/2})^+(b - Aa), \tag{3.7}$$

where $(AM^{-1/2})^+$ denotes the Moore–Penrose inverse of the matrix $(AM^{-1/2})$. Equations (3.6) and (3.7), referred to as the 'fundamental equation of mechanics', provide us with the optimal set of control forces such that the constraints are exactly satisfied at every instant of time $t$ [18]. They are optimal in the sense that they minimize the control cost given by $J(t) = [F^C]^T M^{-1} [F^C]$ at each instant of time. The generality of the formulation makes it applicable in many diverse areas of mechanics. Applications of this formulation to problems of motion synchronization of multiple uncoupled/coupled chaotic gyroscopes, rotational dynamics, and spacecraft formation-keeping are provided in [19–22].

## 4. Problem formulation and constraint equations

Consider an $n$-d.f. nonlinear lattice with appropriate boundary conditions (see equation (2.2)). The energy control problem for this unconstrained system is formulated as follows.

*Given a set of k masses selected from among n masses of the n-mass lattice, find the explicit control forces applicable to this set of k masses such that the total energy of the nonlinear lattice approaches a 'given' positive value H\* as $t \to \infty$.*

$$H(q(t), \dot{q}(t)) \to H^* \quad \text{as } t \to \infty, \ H^* > 0. \tag{4.1}$$

Although we assume at this stage that the locations of these $k$ actuators (where $1 \le k \le n$) can be arbitrarily selected from among the $n$ masses in the lattice, we will later show that in order to have global asymptotic convergence, the set of actuator locations need to satisfy certain conditions when $k < n$ and the system is underactuated (see §5).

## (a) Formulation of the energy constraints

Consider an unconstrained $n$-d.f. nonlinear lattice as described in §2. The unconstrained acceleration $a(q)$ of this lattice can be computed using equations (3.2) and (2.2). Suppose now that out of these $n$ masses, we apply control inputs to $k$ arbitrarily selected masses, where $1 \le k \le n$. The locations of these $k$ masses where a control input is applied is denoted by the ordered set $S_C = \{i_1, i_2, i_3, \ldots, i_k\}$ where, with no loss of generality, we order these locations along the lattice such that $i_1 < i_2 < \cdots < i_k$. Similarly, the set of $(n - k)$ masses at which no control is applied is given by the complement of the set $S_C$, which we denote by $S_N = S_C^c = \{1, 2, 3, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_k\} = \{j_1, j_2, \ldots, j_{n-k}\}$ where again $j_1 < j_2 < \cdots < j_{n-k}$. It is also convenient to represent this information in terms of matrices. The following matrices are defined to simplify the notation. A $k$-by-$n$ 'control selection matrix', $C$, is defined such that every element of its $g$th row ($1 \le g \le k$) is zero except for the $i_g$th element (where $i_g \in S_C$), which is unity. Similarly, we define an $(n - k)$-by-$n$ 'no-control selection matrix', $N$, such that every element of its $h$th row ($1 \le h \le n - k$) is zero except for the $j_h$th element (where $j_h \in S_N$), which is unity. The mass matrices associated with the set of controlled and uncontrolled masses are represented by $M_C = \text{diag}(m_{i_1}, m_{i_2}, \ldots, m_{i_k})$ and $M_N = \text{diag}(m_{j_1}, m_{j_2}, \ldots, m_{j_{n-k}})$, respectively, and the corresponding displacements are represented by the column vectors $q_C = [q_{i_1} \ q_{i_2} \ \cdots \ q_{i_k}]^T$ and $q_N = [q_{j_1} \ q_{j_2} \ \cdots \ q_{j_{n-k}}]^T$, respectively.

While dealing with the energy control problem (4.1), we interpret the energy requirement as an energy constraint on the unconstrained $n$-d.f. nonlinear lattice.

*1. Constraint of 'energy stabilization'.* Using equation (2.1), the energy stabilization constraint is given by

$$\phi(q, \dot{q}) = H(q, \dot{q}) - H^* = (\tfrac{1}{2}\dot{q}^T M \dot{q} + U(q)) - H^* = 0, \tag{4.2}$$

where $H(q, \dot{q})$ is rewritten in matrix–vector notation. The constraint (4.2) resembles equation (3.3) and therefore needs to be differentiated once with respect to time so that it can be expressed in the general form of equation (3.5). Furthermore, we modify the constraint by introducing $\beta > 0$ as in equation (3.4) so that the nonlinear lattice can be initiated from any arbitrary non-zero initial energy state. The modified energy stabilization constraint is now given by

$$\begin{aligned}
\psi(q, \dot{q}, \ddot{q}) &= \frac{d}{dt}(\phi) + \beta\phi = \frac{d}{dt}\left[\left(\frac{1}{2}\dot{q}^T M \dot{q} + U(q)\right) - H^*\right] + \beta(H - H^*) = 0 \\
&= \frac{1}{2}\dot{q}^T(M + M^T)\frac{d\dot{q}}{dt} + \left(\frac{dq}{dt}\right)^T\left(\frac{\partial U}{\partial q}\right) + \beta(H - H^*) = 0 \\
&= \dot{q}^T M \ddot{q} - \dot{q}^T F + \beta(H - H^*) = 0
\end{aligned} \tag{4.3}$$

*2. Constraint of 'No Control'.* In addition to the energy stabilization constraint, a constraint of 'no control' is imposed on all the masses that belong to the set $S_N$ that are left unactuated. Since no control is applied to these masses, the prevailing unconstrained motion of these masses (2.2) can themselves be considered as constraints. Thus, this set of $(n - k)$ 'no control' constraints can

be described in matrix form as

$$N(M\ddot{q} - F) = 0, \tag{4.4}$$

When the constraints described by equations (4.3) and (4.4) are expressed in the general constraint matrix form (see equation (3.5)), this leads to an $(n - k + 1)$-by-$n$ constraint matrix $A$ and an $(n - k + 1)$ sized column vector $b$ given by

$$A = \begin{bmatrix} \dot{q}^T M \\ NM \end{bmatrix} = \begin{bmatrix} \dot{q}^T \\ N \end{bmatrix} M, \quad b = \begin{bmatrix} \dot{q}^T F - \beta(H - H^*) \\ NF \end{bmatrix}. \tag{4.5}$$

## (b) Constrained equations of motion of the $n$-d.f. nonlinear lattice

Once the matrices $M, F, a, A$ and $b$ are known for the nonlinear lattice, the explicit nonlinear control force $F^C$ can be computed in closed form (see appendix A for a detailed derivation) as

$$F^C(q, \dot{q}) = \frac{-\beta(H(q, \dot{q}) - H^*)}{\dot{q}_C^T M_C \dot{q}_C} C^T C M \dot{q}, \tag{4.6}$$

where $\dot{q}_C^T M_C \dot{q}_C = \sum_{g=1}^k (m_{i_g} \dot{q}_{i_g}^2)$. The control force possesses a singularity when the velocities of the set of masses that are controlled are all simultaneously zero. To avoid this, we choose $\beta$ as

$$\beta(q, \dot{q}) = (\dot{q}_C^T M_C \dot{q}_C) \lambda(q, \dot{q}), \quad \text{where } \lambda(q, \dot{q}) > 0. \tag{4.7}$$

Moreover, for simplicity, we choose $\lambda(q, \dot{q}) = \lambda_o$, where $\lambda_o$ is a positive constant that can be suitably altered to control the rate at which the system converges to the desired energy state $H^*$. The explicit control force is now given by

$$F^C = -\lambda_o(H(q, \dot{q}) - H^*) C^T C M \dot{q} = -g(q, \dot{q}) C^T C M \dot{q}. \tag{4.8}$$

Although it might appear that the control force, which depends linearly on the momentum of the controlled masses, resembles a velocity feedback type of control, the nonlinear gain $g(q, \dot{q})$ changes the nature of the feedback. The equation of motion of the constrained (controlled) $n$-d.f. nonlinear lattice (with appropriate boundary conditions) can be written down using equation (3.6) as

$$M\ddot{q} = F + F^C = F - \lambda_o(H(q, \dot{q}) - H^*) C^T C M \dot{q} \tag{4.9}$$

or alternatively as

$$M\ddot{q} + \lambda_o(H(q, \dot{q}) - H^*) C^T C M \dot{q} - F(q) = 0, \tag{4.10}$$

where the 'given' force $F$ is obtained from the unconstrained system (2.2) and the constraint force $F^C$ is computed using (4.8). Equation (4.10) resembles the familiar form of a self-excited oscillator with nonlinear damping, akin to a Van der Pol-type system. When $H < H^*$, the damping is negative and the energy of the system is raised. Conversely, when $H > H^*$, the damping is positive and the energy of the system is lowered. When $H = H^*$ is attained, the control force terminates and the conservative nature of the lattice is used to remain at $H^*$ for all future time.

## 5. Global asymptotic convergence to any non-zero desired energy state $H^*$

In this section, our aim is to show that (i) the control force $F^C$ (equation (4.8)) gives us global asymptotic convergence to any non-zero desired energy state $H^*$ provided that the first mass, or the last mass, or alternatively, any two consecutive masses of the $n$-d.f. lattice are included in the set of masses that are controlled, and (ii) the origin O in phase space is an unstable fixed point in the controlled system. The controlled $n$-d.f. system (described by equation (4.9)) possesses a single isolated equilibrium point at the origin (see derivation in appendix B). Our aim is to prove that this fixed point at the origin O is unstable. LaSalle's invariance principle [14] helps us in establishing both these results.

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*Invariance principle*: The invariance principle in $\Re^n$ is postulated as follows:

Let $\Omega$ be a compact set ($\Omega \subset D \subset \Re^n$) that is positively invariant. Let $V : D \to \Re$ be a continuously differentiable function such that $\dot{V}(x) \le 0$ in $\Omega$. Let $E$ be the set of all points in $\Omega$ where $\dot{V}(x) = 0$. Let $P$ be the largest invariant set in $E$. Then, every solution $x(t)$ starting in $\Omega$ approaches $P$ as $t \to \infty$.

Consider a continuously differentiable scalar function $V$ as

$$V(q, \dot{q}) = \tfrac{1}{2}(H(q, \dot{q}) - H^*)^2, \quad H^* > 0 \tag{5.1}$$

defined on the set $\Omega$ described by

$$\Omega = \{(q, \dot{q}) \in \Re^{2n} \mid \epsilon \le H(q, \dot{q}) \le c\}, \tag{5.2}$$

where $0 < \epsilon < H^* < c$. By choosing $\epsilon > 0$, an open region around the origin O (prescribed by $q \equiv \dot{q} \equiv 0$) is excluded from the set $\Omega$. Our basic motive in choosing $\Omega$ as in equation (5.2) is to establish that the origin O is an unstable fixed point and all trajectories in $\Re^{2n} - \{O\}$ asymptotically converge to the compact and invariant set defined by $H(q, \dot{q}) = H^*$. Now, to apply the invariance principle, we need to first establish that the set $\Omega$ is compact and positively invariant in $2n$-dimensional phase space.

1. *$\Omega$ is a compact set*: A detailed derivation of this result is presented in appendix C.

2. *$\Omega$ is positively invariant*: A set $W$ is said to be positively invariant if $x(0) \in W$ implies $x(t) \in W$ for all $t \ge 0$ [23]. Let us compute $\dot{V}$ along the trajectories of the controlled $n$-d.f. nonlinear lattice (4.9) as shown below.

$$\begin{aligned}
\dot{V}(q, \dot{q}) &= (H - H^*)\frac{\mathrm{d}H}{\mathrm{d}t} = (H - H^*)\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{q}^{\mathrm{T}}M\dot{q} + U(q)\right)\right] \\
&= (H - H^*)[\dot{q}^{\mathrm{T}}(M\ddot{q}) + \dot{q}^{\mathrm{T}}(-F)] \\
&= (H - H^*)[\dot{q}^{\mathrm{T}}(F - \lambda_o(H - H^*)C^{\mathrm{T}}CM\dot{q}) + \dot{q}^{\mathrm{T}}(-F)] \\
&= -\lambda_o(H - H^*)^2\dot{q}^{\mathrm{T}}C^{\mathrm{T}}CM\dot{q} \\
&= -\lambda_o(H - H^*)^2\dot{q}_C^{\mathrm{T}}M_C\dot{q}_C \le 0 \,\forall\, \Re^{2n}.
\end{aligned} \tag{5.3}$$

Since $V \ge 0$ (equation (5.1)) and $\dot{V} \le 0$ (equation (5.3)) at all points that lie in the set $\Omega$, we deduce that the set $\Omega$ is positively invariant. Note that this result also holds true if $\beta$ were to be given by equation (4.7) instead.

3. *Set $E$*: The set $E$ is defined as consisting of all points in the set $\Omega$ where $\dot{V} = 0$. From equation (5.3)), we deduce that $\dot{V}$ is zero in the set $\Omega$ when

$$E = \{(q, \dot{q}) \in \Re^{2n} \mid \dot{q}_C \equiv 0 \cup H(q, \dot{q}) \equiv H^*\}. \tag{5.4}$$

4. *Set $P$*: The set $P$ is defined to be the union of all invariant sets within $E$ [14]. The set of all points satisfying $H(q, \dot{q}) = H^*$ is positively invariant because when $H(q, \dot{q}) = H^*$ is substituted into the equations of motion of the controlled lattice (equation (4.9)), the control force is zero and we obtain our uncontrolled system (2.2), which is conservative and for which the energy remains constant (which in this case is $H^*$) for all time $t$. Next, we need to ensure that the only invariant set in $E(\subseteq \Omega)$ is the set defined by $H(q, \dot{q}) = H^*$ so that all trajectories in $\Omega$ are globally attracted to this set. Consequently, we require the invariant set(s) satisfying $\dot{q}_C \equiv 0$ to lie outside $\Omega$. But, by actuating an arbitrary set of $k$ masses out of $n$ masses in the lattice, one cannot always guarantee that this holds true as is shown by the following example.

Consider a three-mass, homogeneous, nonlinear lattice with fixed ends. The controlled (constrained) equations of motion of the lattice with a single actuator placed at the second mass of the three-mass lattice is given by

$$m\ddot{q}_1 = f(q_2 - q_1) - f(q_1), \tag{5.5}$$

$$m\ddot{q}_2 = f(q_3 - q_2) - f(q_2 - q_1) - \lambda_o(H - H^*)m\dot{q}_2 \tag{5.6}$$

and

$$m\ddot{q}_3 = f(-q_3) - f(q_3 - q_2), \tag{5.7}$$

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where $f$ meets all the criteria of a spring force function as discussed in §2. Since the second mass alone is controlled, when $\dot{q}_C \equiv 0$, we have $\dot{q}_2 \equiv 0$. Consequently, $\ddot{q}_2 \equiv 0$. This reduces equation (5.6) to

$$q_3(t) - q_2(t) = q_2(t) - q_1(t). \tag{5.8}$$

Differentiating equation (5.8) with respect to time $t$ and noting that $\dot{q}_2 \equiv 0$, we obtain $\dot{q}_3 = -\dot{q}_1$. Consequently, $\ddot{q}_3 = -\ddot{q}_1$. Solving this along with equations (5.5) and (5.7) yields $q_2 \equiv 0$ and $q_1 \equiv -q_3$. Hence, there exist sets of invariant orbits described by $Q = \{(q_1(t), \dot{q}_1(t), 0, 0, -q_1(t), -\dot{q}_1(t))\}$ that satisfy $\dot{q}_2 \equiv 0$ and that lie inside $\Omega$. And therefore besides $H(q, \dot{q}) = H^*$, there are additional invariant sets in $E$ (and hence in $\Omega$) that satisfy $\dot{q}_2 \equiv 0$ and to which the trajectories are confined. Thus, in such a case, one cannot guarantee that the set $H(q, \dot{q}) = H^*$ is globally attracting in $\Omega$. To ensure that the set(s) of invariant orbits satisfying $\dot{q}_C \equiv 0$ lie outside $\Omega$, the actuators must be placed appropriately so that $\dot{q}_C \equiv 0$ only yields the set $q \equiv \dot{q} \equiv 0$ (origin O), which lies outside $\Omega$. Next, we state and prove the conditions on the actuator locations under which this can be guaranteed.

*Result*: *For a fixed–fixed (or fixed–free) inhomogeneous nonlinear lattice, when the set of locations of the actuators includes at least one of the following configurations:*

(i) *a single actuator is placed on the first mass $m_1$, or the last mass $m_n$, of the lattice, and*
(ii) *two actuators are placed on two consecutive masses located anywhere in the lattice, i.e. $i_x, i_y \in S_C$ such that $|i_x - i_y| = 1$,*

*then the only invariant set satisfying $\dot{q}_C \equiv 0$ is the origin O ($q \equiv \dot{q} \equiv 0$).*

*Proof.* Consider that the $i$th mass of the lattice is actuated. The constrained equation of motion of the $i$th mass of the lattice is given by

$$m_i \ddot{q}_i = f_i(q_{i+1} - q_i) - f_{i-1}(q_i - q_{i-1}) - \lambda_o(H - H^*)m_i\dot{q}_i. \tag{5.9}$$

Since the $i$th mass is controlled, $i \in S_C$, and when $\dot{q}_C \equiv 0$, we have $\dot{q}_i \equiv 0$. Consequently, $\ddot{q}_i \equiv 0$. This reduces equation (5.9) to

$$f_i(q_{i+1} - q_i) = f_{i-1}(q_i - q_{i-1}). \tag{5.10}$$

Differentiating equation (5.10) with respect to time $t$, we obtain

$$f_i'(q_{i+1} - q_i)(\dot{q}_{i+1} - \dot{q}_i) = f_{i-1}'(q_i - q_{i-1})(\dot{q}_i - \dot{q}_{i-1}), \tag{5.11}$$

which simplifies to

$$f_i'(q_{i+1} - q_i)(\dot{q}_{i+1}) = f_{i-1}'(q_i - q_{i-1})(-\dot{q}_{i-1}). \tag{5.12}$$

Now, if we additionally have either $\dot{q}_{i-1} \equiv 0$ or $\dot{q}_{i+1} \equiv 0$, then we obtain certain simplifying results. We derive our results assuming $\dot{q}_{i-1} \equiv 0$ but a similar derivation follows if $\dot{q}_{i+1} \equiv 0$ instead. In equation (5.12), if $\dot{q}_{i-1} \equiv 0$, then we obtain $f_i'(q_{i+1} - q_i)(\dot{q}_{i+1}) \equiv 0$ which implies that either $\dot{q}_{i+1} \equiv 0$ or $f_i'(q_{i+1} - q_i) = 0$. If $f_i'(q_{i+1} - q_i) = 0$, then $(q_{i+1} - q_i) \equiv 0$ from property (3) in §2, which once again after differentiation with respect to time $t$ yields $\dot{q}_{i+1} \equiv 0$. Consequently, $\ddot{q}_{i+1} \equiv 0$. Substituting these results into the constrained equation of motion of the $(i + 1)$th mass of the lattice gives us

$$f_{i+1}(q_{i+2} - q_{i+1}) = f_i(q_{i+1} - q_i). \tag{5.13}$$

Differentiating equation (5.13) with respect to time $t$ gives us

$$f_{i+1}'(q_{i+2} - q_{i+1})(\dot{q}_{i+2} - \dot{q}_{i+1}) = f_i'(q_{i+1} - q_i)(\dot{q}_{i+1} - \dot{q}_i). \tag{5.14}$$

Again since $\dot{q}_i \equiv \dot{q}_{i+1} \equiv 0$, as before equation (5.14) yields $\dot{q}_{i+2} \equiv 0$ (and consequently $\ddot{q}_{i+2} \equiv 0$). Continuing this process of recursive substitution into the constrained equation of motion of the

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$(i + 2)$th mass, and then the $(i + 3)$th mass, and so on and so forth until the $n$th mass of the lattice, we obtain

$$\dot{q}_k \equiv \ddot{q}_k \equiv 0, \quad k = i - 1, i, i + 1, i + 2, \ldots, n. \tag{5.15}$$

On the other hand, since $\dot{q}_{i-1} \equiv \ddot{q}_{i-1} \equiv 0$, following the steps in equations (5.9)–(5.12) for the constrained equation of motion of the $(i - 1)$th mass of the lattice, we obtain $\dot{q}_{i-2} \equiv 0$ (and hence $\ddot{q}_{i-2} \equiv 0$). Continuing this process of recursive substitution into the constrained equation of motion of the $(i - 2)$th mass, and then the $(i - 3)$th mass, and so on and so forth until the first mass of the lattice, we obtain

$$\dot{q}_k \equiv \ddot{q}_k \equiv 0, \quad k = i, i - 1, i - 2, i - 3, \ldots, 1. \tag{5.16}$$

Thus, for an $n$-d.f. inhomogeneous nonlinear lattice:

   (i) If $i_x, i_y \in S_C$ and $|i_x - i_y| = 1$, then whenever $\dot{q}_C \equiv 0$, we have $\dot{q}_{i_x} \equiv \dot{q}_{i_y} \equiv 0$ with $|i_x - i_y| = 1$ and for $i = i_x$ in equations (5.9)–(5.16), we obtain $\dot{q} \equiv \ddot{q} \equiv 0$.
  (ii) For a fixed–fixed lattice, if we actuate the first mass of the lattice, when $\dot{q}_C \equiv 0$ we have $\dot{q}_1 \equiv 0$. Then $i = 1$ in equations (5.9)–(5.15) and with $\dot{q}_o \equiv 0$, one obtains $\dot{q} \equiv \ddot{q} \equiv 0$. On the other hand, if we actuate the last mass of the lattice, when $\dot{q}_C \equiv 0$, we have $\dot{q}_n \equiv 0$, then $i = n$ in equations ((5.9)–(5.12), (5.16)) along with $\dot{q}_{n+1} \equiv 0$ yields $\dot{q} \equiv \ddot{q} \equiv 0$.
 (iii) For a fixed–free lattice, actuating the first mass follows a derivation similar to the fixed–fixed case. On the other hand, if we actuate the last mass of the lattice, when $\dot{q}_C \equiv 0$, we have $\dot{q}_n \equiv 0$. Then $i = n$ in equations ((5.9)–(5.12), (5.16)) and with $f_n \equiv 0$, one obtains $\dot{q} \equiv \ddot{q} \equiv 0$.

From appendix B, we know that for an $n$-d.f. nonlinear lattice, if $\dot{q} \equiv \ddot{q} \equiv 0$, then $q \equiv 0$. Thus, for the three cases discussed above, it follows that the origin O ($q \equiv \dot{q} \equiv 0$) is the only invariant point satisfying $\dot{q}_C \equiv 0$. ∎

Now, since the origin O is excluded from the set $\Omega$, the largest invariant set in $E$ is

$$P = \{(q, \dot{q}) \in \Re^{2n} \mid H(q, \dot{q}) = H^*; \; H^* > 0\}. \tag{5.17}$$

Then, by the invariance principle, every solution $x(t)$ starting in $\Omega$ approaches $P$ as $t \to \infty$. Thus, global asymptotic convergence to the set $H(q, \dot{q}) = H^*$ has been established in $\Omega$. Now, since $c$ can be chosen arbitrarily large and $\epsilon$ can be chosen arbitrarily small, all trajectories in $\Re^{2n} - \{O\}$ asymptotically tend to $P$.

**Remark.** Since the open region around the origin can be made arbitrarily small through a proper choice of $\epsilon$, the origin ($q \equiv \dot{q} \equiv 0$) is an unstable fixed point.

Hence, this proves that the control force $F^C$ derived in equation (4.8) for an $n$-d.f. nonlinear lattice with fixed–fixed (or fixed–free) ends gives us global asymptotic convergence to any given desired energy state $H^*$ in $\Re^{2n} - \{O\}$ provided that the first mass, or the last mass, or alternatively, any two consecutive masses of the lattice are included in the subset of masses that are controlled. Since actuation at only one mass or any two consecutive masses could guarantee global asymptotic energy control for an $n$-mass lattice, the control could be highly underactuated. This is illustrated in the next section where the energy of a 101-mass lattice is controlled using just two actuators placed on consecutive masses.

## 6. Results and simulations

In this section, numerical simulations involving a nonlinear lattice with fixed–fixed boundary conditions are presented to illustrate the ease and efficacy with which the control methodology can be applied. Since $n$ can be any finitely large number, a 101-mass lattice is chosen. Given any non-zero initial energy state of the lattice, our aim is to control the energy of the nonlinear lattice and bring it to a desired energy level. To achieve this desired energy level, control can be applied

to one or more of these 101 masses (provided of course that the first mass, or the last mass, or any two consecutive masses, are included in the set of masses that are controlled). In both the examples that we consider in this section, the spring elements in the lattice are all taken to be nonlinear and control is applied to two consecutive masses, $m_{75}$ and $m_{76}$, located at about a quarter of the lattice's length from the right end (figure 1). In the first example, for the sake of later comparison, a homogeneous FPU $\beta$-lattice [1,3] consisting of 101 unit masses is considered. The nonlinear potential of the FPU $\beta$-lattice is given by $u_i(x) = (a_i/2)x^2 + (b_i/4)x^4$, where the spring constants are chosen to be $a_i = b_i = 1 \,\forall\, i$.

The second example deals with a 101-mass inhomogeneous nonlinear lattice where the nonlinear potential of each spring in the lattice is chosen at random from the following set of potentials, $S_{sp}$.

$$S_{sp} = \left\{ \frac{a}{4}x^4, \frac{a}{2}x^2 + \frac{b}{4}x^4, \frac{a}{6}x^6, \frac{a}{4}x^4 + \frac{b}{6}x^6, \frac{a}{8}x^8, \frac{a}{6}x^6 + \frac{b}{8}x^8, \frac{a}{b}e^{bx} - ax - \frac{a}{b} \right\}. \qquad (6.1)$$

We note that each potential function in the set $S_{sp}$ is qualitatively different, and is characterized here, for illustration purposes, by at most two parameters $a$ and $b$. The potentials further satisfy all the requisite conditions listed in §2. For each spring in the 101-mass lattice, first a potential function is chosen at random from the set $S_{sp}$, and then its parameter values are selected at random from a uniformly distributed set of numbers between the limits $0.5 < a_i < 1.5$, and $0.5 < b_i < 1.5$. Likewise, each mass in the lattice is also chosen at random from a uniformly distributed set of numbers between the limits $0.5 < m_i < 1.5$. One realization of the inhomogeneous lattice from the ensemble of random lattices so produced is used in the example below.

In both examples, the lattice is initially excited with all the masses having zero initial displacement and zero initial velocity except for the mass at the centre of the lattice, $m_{51}$, which is initially displaced by 2 units. This causes the initial energy level, $H_o$, of the homogeneous and the inhomogeneous lattice to be 12 units and 28.4 units, respectively. The aim is to control the energy in these respective lattices and raise them to a desired energy level of $H^* = 150$ units in each case. The equations of motion are integrated using $ode113$ in the Matlab environment with a relative integration error tolerance of $10^{-10}$ and an absolute error tolerance of $10^{-13}$. All quantities are assumed to be in consistent units.

**Example 6.1.** Figure 2*a* shows a plot of the velocity field for the uncontrolled homogeneous FPU $\beta$-lattice where time is plotted on the *x*-axis, the location of the masses is plotted on the *y*-axis and the velocity of each mass in the lattice is shown through a colour variation (see colour scale on the right). The initial excitation of the mass at the centre of the lattice gives rise to, what appears to be, a breather structure (see figure 2*a*) located at the centre of the lattice amidst many small waves propagating through the field. This breather structure oscillates undisturbed throughout the duration of the simulation. Control is now applied to this homogeneous lattice (as described earlier) to raise its energy level to 150 units. Time histories of the control forces acting on each of the masses that are controlled, namely $m_{75}$ (solid line) and $m_{76}$ (dash line), are shown in figure 4*a* (top) for $\lambda_o = 0.1$ (see equation (4.8)). A finite amount of time is seen to elapse before the control sets in. This is because it takes a finite time for the initial excitation (at the centre of the lattice) to traverse through the lattice and reach the actuator locations (at $m_{75}$ and $m_{76}$). Since the control forces are proportional to the velocity of the actuated masses (see equation (4.8)), once the actuator masses are in motion at around 8.6 s, the control begins and the desired energy state of $H^* = 150$ is almost immediately achieved (see $\lambda_o = 0.1$ case of figure 4*b*, top). Further, to illustrate the effect that $\lambda_o$ has on the rate at which the controlled homogeneous lattice converges to the desired energy state, we show the time histories of energy convergence for $\lambda_o = 0.1$ and $\lambda_o = 0.005$ in figure 4*b* (top). Figure 2*b* shows a plot of the velocity field for the controlled homogeneous FPU $\beta$-lattice for $\lambda_o = 0.1$. From the figure, we observe that in addition to the breather structure generated by the initial excitation of the central mass, multiple soliton structures are generated (see black circle in the figure) coinciding with the application of the control forces. These structures have been further investigated, though for brevity their analysis is not shown
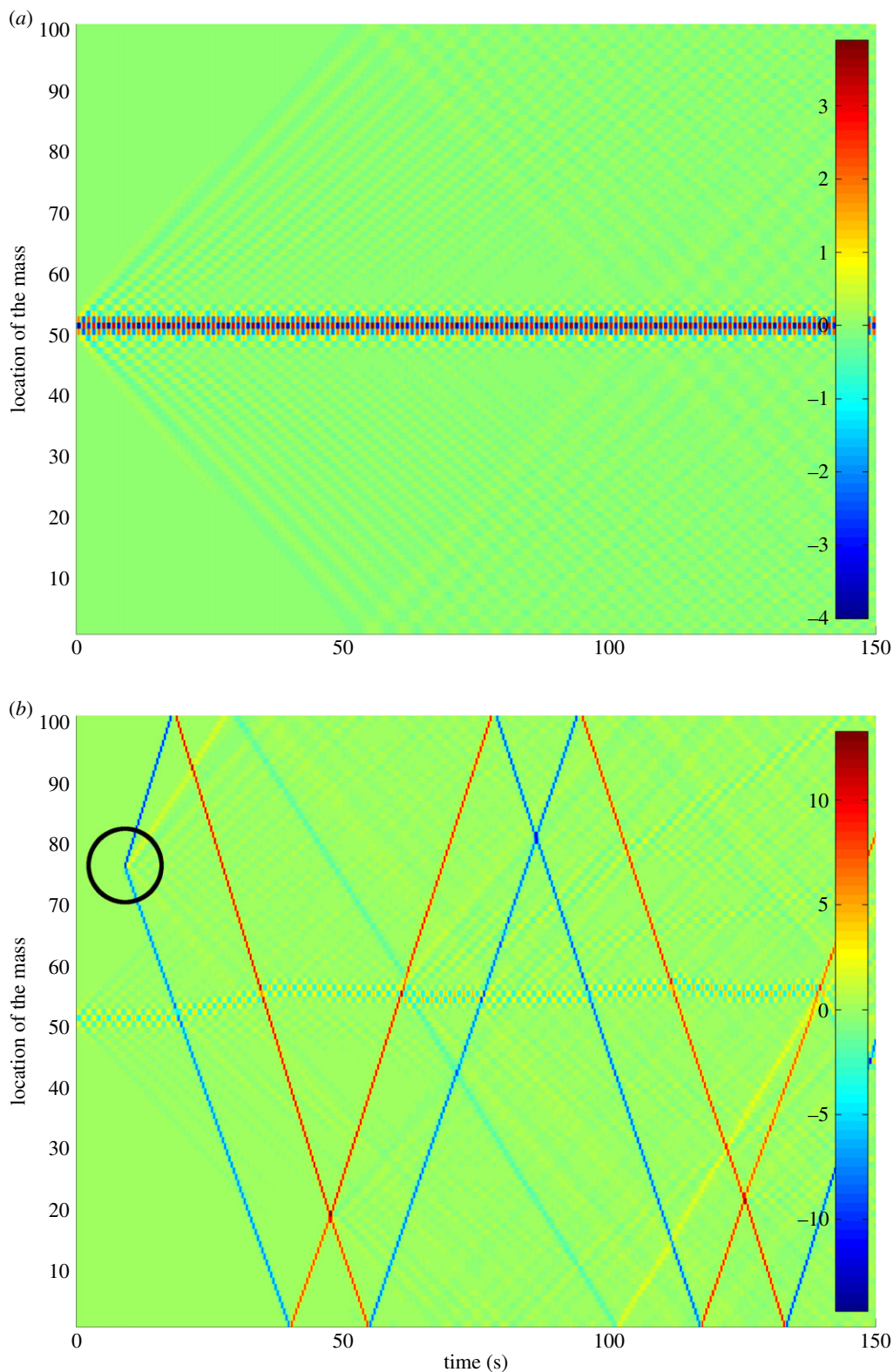
**Figure 2.** Velocity field of the homogeneous lattice. The lattice has parameters $a_i = 1$, $b_i = 1$, $m_i = 1 \, \forall \, i$, $H_o = 12$, $H^* = 150$, $\lambda_o = 0.1$ and the initial displacement of the centre mass, $m_{51}$, is 2 units. Actuators are located at $m_{75}$ and $m_{76}$. (*a*) Uncontrolled lattice and (*b*) controlled lattice. (Online version in colour.)

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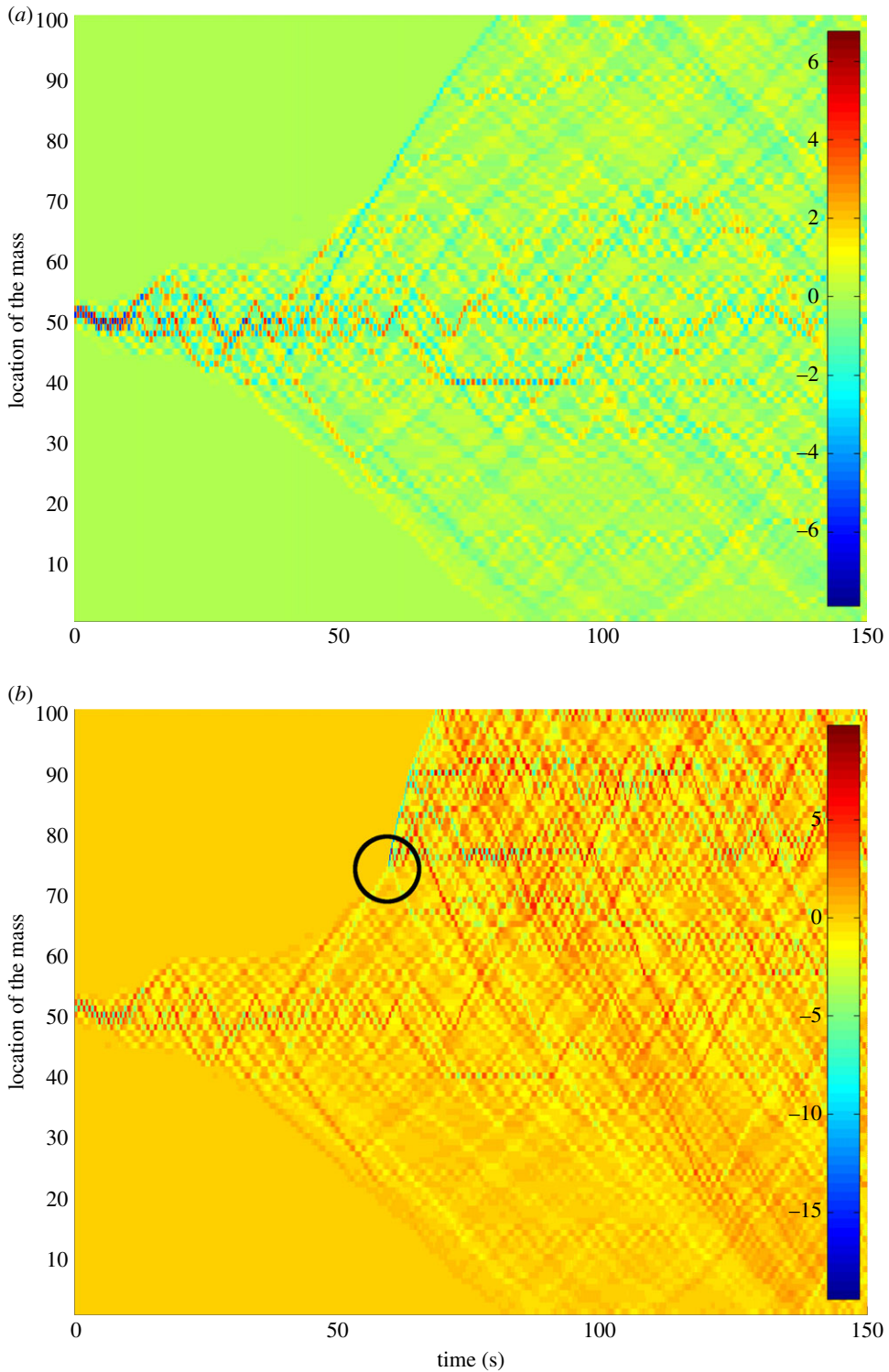rspa.royalsocietypublishing.org   *Proc. R. Soc. A* **471**: 20140694



**Figure 3.** Velocity field of the inhomogeneous lattice. For each spring in the lattice, first a nonlinear potential is chosen at random from the set $S_{sp}$ and then its parameters $a$, $b$ and $m$ are chosen randomly from a uniform distributed set of numbers between the limits $0.5 < a_i < 1.5$, $0.5 < b_i < 1.5$ and $0.5 < m_i < 1.5$, respectively. The initial displacement of the centre mass, $m_{51}$, is 2 units and $H_o = 28.4$, $H^* = 150$ and $\lambda_o = 0.1$. Actuators are located at $m_{75}$ and $m_{76}$. (a) Uncontrolled lattice and (b) controlled lattice. (Online version in colour.)
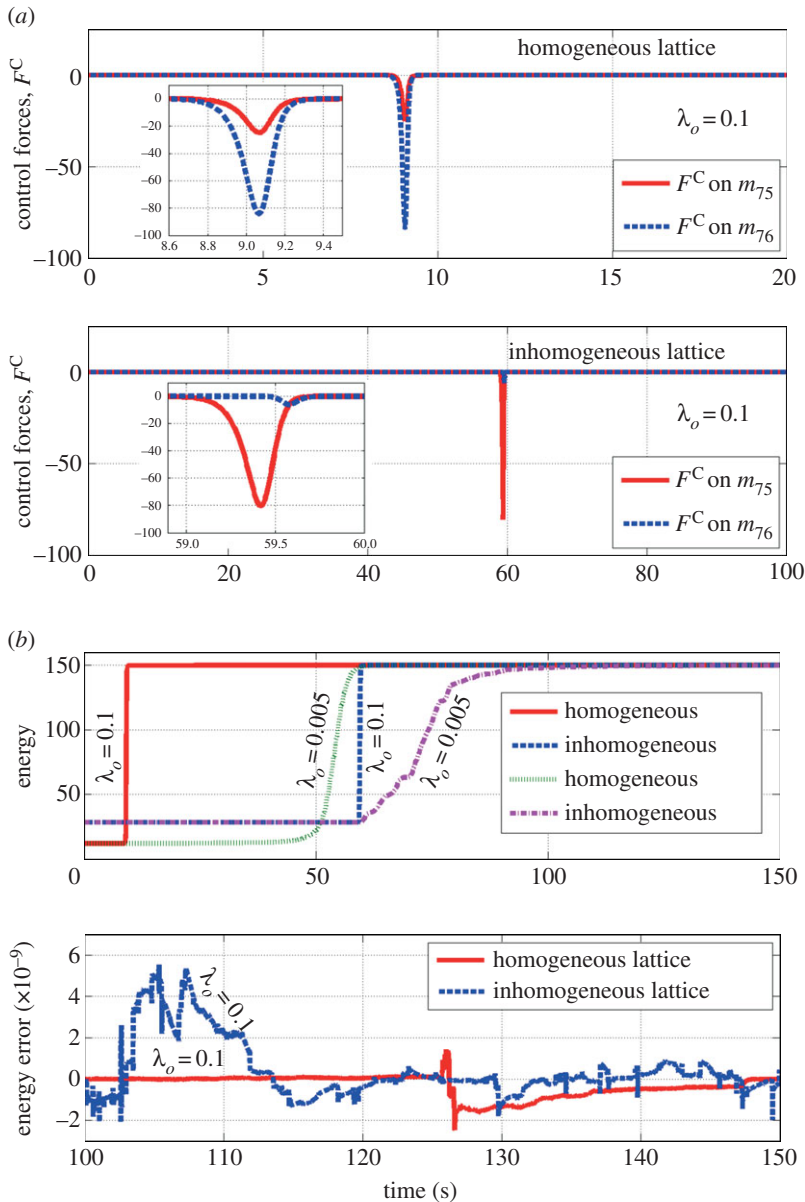
**Figure 4.** Time history of control forces, energy convergence and energy errors. (*a*) Control forces acting on the two actuator masses $m_{75}$ (solid line) and $m_{76}$ (dash line) of the homogeneous lattice (top) and the inhomogeneous lattice (bottom) for $\lambda_o = 0.1$. (*b*) Energy of the system from $t = 0$ to $t = 150$ s (top) and energy error ($e(t) = H(t) - H^*$) from $t = 100$ to $t = 150$ s (bottom). Solid ($\lambda_o = 0.1$) and dotted ($\lambda_o = 0.005$) lines denote the homogeneous lattice whereas dash ($\lambda_o = 0.1$) and dash-dot ($\lambda_o = 0.005$) lines denote the inhomogeneous lattice. (Online version in colour.)

here. Figure 2*b* shows the interaction of these structures and their propagation through the velocity field.

**Example 6.2.** Figure 3*a* shows a plot of the velocity field for the uncontrolled inhomogeneous nonlinear lattice. The initial excitation of the mass at the centre of the lattice generates waves, which traverse through the length of the lattice as can be inferred from the figure. Until about 40 s, the dynamics of the lattice seems confined to only a few masses, following which it spreads

out more rapidly to the other masses in the lattice. The criss-cross pattern shown in the figure is generated by the propagation of the waves and their reflection at the boundaries. Once again, we apply control as before to this inhomogeneous lattice to raise its energy level to 150 units. A time history of the control forces acting on the inhomogeneous lattice is shown in figure 4a (bottom) for $\lambda_o = 0.1$. The control begins at around 59 s and stabilizes the lattice at the desired energy level of 150 units (figure 4b, top). Like in example 6.1, the time histories of energy convergence are plotted for two different values of $\lambda_o$ in figure 4b(top) for the controlled lattice. Figure 3b shows a plot of the velocity field for the controlled system for $\lambda_o = 0.1$. The control generates its own velocity field causing waves to emanate (see black circle in figure 3b) in addition to those generated by the initial displacement of the centre mass. Some of these newly generated waves appear to have larger amplitudes and higher propagation speeds when compared to those extant.

For both examples in this section, the energy error ($e(t) = H(t) - H^*$) in achieving the desired energy state is plotted as a function of time over the duration of the last 50 s of the simulation for $\lambda_o = 0.1$ (figure 4b, bottom). The figure shows that this error is small and is close to the error tolerance levels specified in our integration algorithm, thus showing the efficacy of the control methodology in achieving the desired energy state. From figure 4a, we observe that once the desired energy state is achieved, the control forces automatically become zero, and the conservative nature of the lattice is thereafter used to maintain its energy at the desired level for all future time. Similar examples can also be generated for a fixed–free nonlinear lattice, but we do not present them here for the sake of brevity.

# 7. Conclusion

This paper deals with the problem of energy control of an $n$-d.f. general nonlinear lattice with fixed–fixed and fixed–free boundary conditions. The nonlinear lattice is composed of a chain of masses wherein each mass is connected to its nearest neighbour by a nonlinear or linear memoryless spring element. The masses in the lattice are assumed to be different from one another. The *qualitative* nature of the nonlinear spring elements along the lattice is also assumed to be different as are the parameters of the functions of the potentials describing each of the spring elements. To the best of the authors' knowledge, neither the dynamics nor the control of such general nonlinear lattice systems has been hitherto addressed in the literature.

The control approach adopted in this paper is inspired by recent results in analytical dynamics that deal with the theory of constrained motion. Despite the general nature of the nonlinear lattice considered in this study, closed form expressions for the explicit nonlinear control forces are obtained with relative ease without the need for any approximations and/or linearizations of the nonlinear dynamical system. The equations of motion of the controlled nonlinear lattice resemble that of a self-excited system. The control forces, $F^C$, are continuous in time and are optimal; they minimize the control cost given by $J(t) = [F^C]^T M^{-1}[F^C]$ at each instant of time while causing the energy constraint (4.3) to be exactly satisfied. The control forces act on the $n$-d.f. nonlinear lattice to bring it to the desired energy level. Once this desired value is reached, the control forces terminate and the conservative nature of the lattice is used to maintain it at the desired energy level for all future time.

The nonlinear lattice is underactuated. Global asymptotic convergence to the desired energy state is guaranteed provided that (i) the first mass, or (ii) the last mass, or (iii) any two consecutive masses of the lattice are included in the set of actuated masses. The manifold $H(q, \dot{q}) = H^*$ forms a globally attracting limit hypersurface in $2n$-dimensional phase space and the trajectories of the controlled system asymptotically tend to it.

Numerical simulations contrasting the behaviour of homogeneous and inhomogeneous nonlinear lattices containing 101 masses is shown. The value of each mass in the inhomogeneous lattice is chosen at random from a uniformly distributed set of numbers. Each spring potential is randomly chosen from a set of seven qualitatively different potential functions (that satisfy the requisite conditions on the potentials described in §2), and the parameter values defining each

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potential function are also chosen at random. These simulations demonstrate the ease, simplicity and accuracy with which the control methodology works.

# Appendix A. Closed form expression for the control force

In this appendix, a closed form expression for the explicit nonlinear control force $F^C$ is derived using equation (3.7). The constraint matrices $A$ and $b$ are expressed in terms of matrices $C$ and $N$ (see §4). And therefore before we compute the control force, let us list some properties of the matrices $C$ and $N$.

(a) $C^T C + N^T N = I_n$, where $I_n$ denotes the identity matrix of size $n$.
(b) $CC^T = I_k$, where $I_k$ denotes the identity matrix of size $k$.
(c) $NN^T = I_{n-k}$, where $I_{n-k}$ denotes the identity matrix of size $(n-k)$.
(d) $C^T C \Lambda = \Lambda C^T C$ for all diagonal matrices $\Lambda$.
(e) $N^T N \Lambda = \Lambda N^T N$ for all diagonal matrices $\Lambda$.
(f) $NC^T = NMC^T = [O]_{(n-k) \times k}$, where $[O]$ denotes the zero matrix.
(g) $CN^T = CMN^T = [O]_{k \times (n-k)}$, where $[O]$ denotes the zero matrix.

The computation of the control force, $F^C$, involves the evaluation of the Moore–Penrose (MP) inverse of the $(n - k + 1)$-by-$n$ matrix $B$ [13] given by

$$B = AM^{-1/2} = \begin{bmatrix} \dot{q}^T M \\ NM \end{bmatrix} M^{-1/2} = \begin{bmatrix} \dot{q}^T M^{1/2} \\ NM^{1/2} \end{bmatrix}. \tag{A 1}$$

Given any $(n - k + 1)$-by-$n$ matrix $B$, there exists a unique $n$-by-$(n - k + 1)$ matrix $B^+$, called the MP inverse of the matrix $B$, which satisfies the following four conditions [13].

$$(BB^+)^T = BB^+; \quad (B^+B)^T = B^+B; \quad BB^+B = B \quad \text{and} \quad B^+BB^+ = B^+.$$

For a matrix $B$ given by (A 1), we claim that $B^+$ is given by

$$B^+ = \left[ \frac{M^{1/2}C^T C\dot{q}}{\dot{q}^T C^T CM\dot{q}} \,\middle|\, M^{-1/2}N^T - \frac{M^{1/2}C^T C\dot{q}\dot{q}^T N^T}{\dot{q}^T C^T CM\dot{q}} \right]. \tag{A 2}$$

Assuming that $B^+$ given by equation (A 2) is indeed the correct expression for the MP inverse of $B$, we show that it satisfies all four conditions of the MP inverse.

(i)

$$BB^+ = \begin{bmatrix} \dot{q}^T M^{1/2} \\ NM^{1/2} \end{bmatrix} \left[ \frac{M^{1/2}C^T C\dot{q}}{\dot{q}^T C^T CM\dot{q}} \,\middle|\, M^{-1/2}N^T - \frac{M^{1/2}C^T C\dot{q}\dot{q}^T N^T}{\dot{q}^T C^T CM\dot{q}} \right]$$

$$= \begin{bmatrix} \dfrac{\dot{q}^T MC^T C\dot{q}}{\dot{q}^T C^T CM\dot{q}} & \dot{q}^T N^T - \dfrac{(\dot{q}^T MC^T C\dot{q})\dot{q}^T N^T}{\dot{q}^T C^T CM\dot{q}} \\ \dfrac{(NMC^T)C\dot{q}}{\dot{q}^T C^T CM\dot{q}} & NN^T - \dfrac{(NMC^T)C\dot{q}\dot{q}^T N^T}{\dot{q}^T C^T CM\dot{q}} \end{bmatrix} = I_{n-k+1}. \tag{A 3}$$

By applying property (d), the $(1, 1)$ block of $BB^+$ is unity and the $(1, 2)$ block simplifies to a $(n - k)$ sized zero row vector. The $(2, 1)$ block is a $(n - k)$ sized column vector which is zero by virtue of property (f). Similarly, the $(2, 2)$ block is an $(n - k)$ sized square matrix which reduces to $NN^T$ by applying property (f), which further simplifies to $I_{n-k}$ by applying property (c). This reduces the matrix $BB^+$ to an $(n - k + 1)$ sized identity matrix. Hence, the first MP condition is satisfied.

(ii)

$$
\begin{aligned}
B^+B &= \left[\;\frac{M^{1/2}C^TC\dot q}{\dot q^TC^TCM\dot q}\;\middle|\; M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot q\dot q^TN^T}{\dot q^TC^TCM\dot q}\;\right]\left[\begin{array}{c}\dot q^TM^{1/2}\\ NM^{1/2}\end{array}\right]\\[2mm]
&= \left[\;\frac{M^{1/2}C^TC\dot q\dot q^TM^{1/2}}{\dot q^TC^TCM\dot q} + M^{-1/2}N^TNM^{1/2} - \frac{M^{1/2}C^TC\dot q\dot q^TN^TNM^{1/2}}{\dot q^TC^TCM\dot q}\;\right]\\[2mm]
&= \left[\;\frac{M^{1/2}C^TC\dot q\dot q^T(I-N^TN)M^{1/2}}{\dot q^TC^TCM\dot q} + M^{-1/2}N^TNM^{1/2}\;\right]\\[2mm]
&= \left[\;\frac{M^{1/2}C^TC\dot q\dot q^TC^TCM^{1/2}}{\dot q^TC^TCM\dot q} + N^TN\;\right].
\end{aligned}
\tag{A 4}
$$

To arrive at the last equality of (A 4), properties (a) and (e) have been used. Clearly, the matrix $B^+B$ is symmetric and thus the second MP condition is satisfied.

(iii) $BB^+B = I_{n-k+1}B = B$, which directly follows from equation (A 3).

(iv)

$$
B^+BB^+ = \left[\;\frac{M^{1/2}C^TC\dot q\dot q^TC^TCM^{1/2}}{\dot q^TC^TCM\dot q} + N^TN\;\right] \times \left[\;\frac{M^{1/2}C^TC\dot q}{\dot q^TC^TCM\dot q}\;\middle|\; M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot q\dot q^TN^T}{\dot q^TC^TCM\dot q}\;\right]
$$

The $B^+BB^+$ matrix is a 1-by-2 block matrix, where the $(1,1)$ block is given by

$$
\begin{aligned}
(1,1) &= \left[\;\frac{M^{1/2}C^TC\dot q\dot q^TC^TCMC^TC\dot q}{(\dot q^TC^TCM\dot q)^2} + \frac{N^T(NM^{1/2}C^T)C\dot q}{\dot q^TC^TCM\dot q}\;\right]\\[2mm]
&= \left[\;\frac{M^{1/2}C^TC\dot q\dot q^TC^T(CC^T)CM\dot q}{(\dot q^TC^TCM\dot q)^2}\;\right] = \left[\;\frac{M^{1/2}C^TC\dot q(\dot q^TC^TCM\dot q)}{(\dot q^TC^TCM\dot q)^2}\;\right] = \left[\;\frac{M^{1/2}C^TC\dot q}{\dot q^TC^TCM\dot q}\;\right].
\end{aligned}
\tag{A 5}
$$

In the derivation (A 5) above, the second term of the first equality drops out by virtue of property (f) and the first term is simplified by using properties (b) and (d). Next, the $(1,2)$ block of the matrix $B^+BB^+$ is given by

$$
\begin{aligned}
(1,2) = &\left[\;\frac{M^{1/2}C^TC\dot q\dot q^TC^T(CM^{1/2}M^{-1/2}N^T)}{\dot q^TC^TCM\dot q} + N^TNM^{-1/2}N^T\right.\\[2mm]
&\left.- \frac{M^{1/2}C^TC\dot q\dot q^TC^TCM^{1/2}M^{1/2}C^TC\dot q\dot q^TN^T}{(\dot q^TC^TCM\dot q)^2} - \frac{N^T(NM^{1/2}C^T)C\dot q\dot q^TN^T}{\dot q^TC^TCM\dot q}\;\right].
\end{aligned}
\tag{A 6}
$$

The first and the fourth terms of the $(1,2)$ block above drop out by virtue of properties (g) and (f), respectively. When property (e) is applied to the second term and property (d) is applied to the third term, the $(1,2)$ block reduces to

$$
\begin{aligned}
(1,2) &= \left[\;M^{-1/2}N^T(NN^T) - \frac{M^{1/2}C^TC\dot q\dot q^TC^T(CC^T)CM\dot q\dot q^TN^T}{(\dot q^TC^TCM\dot q)^2}\;\right]\\[2mm]
&= \left[\;M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot q(\dot q^TC^TCM\dot q)\dot q^TN^T}{(\dot q^TC^TCM\dot q)^2}\;\right]\\[2mm]
&= \left[\;M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot q\dot q^TN^T}{\dot q^TC^TCM\dot q}\;\right].
\end{aligned}
\tag{A 7}
$$

We note that properties (b) and (c) have been used to simplify the first equality of (A 7) above. Hence, we obtain $B^+BB^+$ as

$$
B^+BB^+ = \left[\;\frac{M^{1/2}C^TC\dot q}{\dot q^TC^TCM\dot q}\;\middle|\; M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot q\dot q^TN^T}{\dot q^TC^TCM\dot q}\;\right] = B^+,
\tag{A 8}
$$

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which satisfies the fourth MP condition. Since, all four MP conditions are satisfied, we ascertain that the $B^+$ given by (A 2) is indeed the correct expression for the Moore–Penrose inverse of the matrix $B$.

*Main result*: The control force can now be calculated as

$$F^C(q, \dot{q}, t) = M^{1/2}(AM^{-1/2})^+ (b - Aa) = M^{1/2}B^+ \left[ b - \begin{bmatrix} \dot{q}^T M \\ NM \end{bmatrix} M^{-1}F \right]$$

$$= M^{1/2}B^+ \left[ \begin{bmatrix} \dot{q}^T F - \beta(H - H^*) \\ NF \end{bmatrix} - \begin{bmatrix} \dot{q}^T F \\ NF \end{bmatrix} \right] = M^{1/2}B^+ \begin{bmatrix} -\beta(H - H^*) \\ [O]_{(n-k) \times 1} \end{bmatrix}$$

$$= M^{1/2} \left[ \frac{M^{1/2}C^T \dot{q}}{\dot{q}^T C^T CM\dot{q}} \middle| M^{-1/2}N^T - \frac{M^{1/2}C^T \dot{q}\dot{q}^T N^T}{\dot{q}^T C^T CM\dot{q}} \right] \begin{bmatrix} -\beta(H - H^*) \\ [O]_{(n-k) \times 1} \end{bmatrix}$$

$$= \frac{-\beta(H - H^*)}{\dot{q}^T C^T CM\dot{q}} MC^T C\dot{q} = \frac{-\beta(H(q, \dot{q}) - H^*)}{\dot{q}_C^T M_C \dot{q}_C} C^T CM\dot{q}, \tag{A 9}$$

where $\dot{q}_C^T M_C \dot{q}_C = \sum_{g=1}^k (m_{i_g} \dot{q}_{i_g}^2)$ is twice the kinetic energy of the set of controlled masses. From the third equality of (A 9), we note that whenever an energy stabilization constraint is applied to a mechanical system, the term $\dot{q}^T F$ always drops out as long as the system under consideration is conservative.

## Appendix B. Origin O is a unique and isolated equilibrium point

Consider a $n$-d.f. nonlinear lattice with fixed–fixed (or fixed–free) boundary conditions. The equilibrium points of uncontrolled (unconstrained) and the controlled (constrained) system can be calculated by substituting $\dot{q} \equiv \ddot{q} \equiv 0$ in equations (2.2) and (4.9), respectively. In both cases, we obtain $F = [O]_{n \times 1}$, where the $i$th row of this relation can be written as

$$f_{i-1}(q_i(t) - q_{i-1}(t)) = f_i(q_{i+1}(t) - q_i(t)), \quad i = 1, 2, 3, \dots n. \tag{B 1}$$

For the fixed–fixed lattice, equation (B 1) implies

$$f_i(q_{i+1}(t) - q_i(t)) = c(t), \quad i = 0, 1, 2, 3, \dots n, \tag{B 2}$$

so that $f_i^{-1}(c(t)) = q_{i+1}(t) - q_i(t), i = 0, 1, \dots n$. Summing over $i$ on both sides, we have

$$\sum_{i=0}^n f_i^{-1}(c(t)) = \sum_{i=0}^n (q_{i+1}(t) - q_i(t)) = q_{n+1}(t) - q_o(t). \tag{B 3}$$

Since $q_o(t) \equiv q_{n+1}(t) \equiv 0$, we have $\sum_{i=0}^n f_i^{-1}(c(t)) = 0$ whose only solution is $c(t) = 0$ as each $f_i$ is a strictly increasing bijective function with $f_i(0) = 0$, $xf_i(x) > 0 \, \forall \, x \neq 0$ (see §2). From (B 2) then with $c(t) = 0$, we have $q_{i+1}(t) - q_i(t) = 0, i = 0, 1, \dots, n$, which implies $q_i(t) = 0, i = 1, 2, \dots n$, because $q_o(t) \equiv 0$. For the fixed–free case, when $i = n, f_n \equiv 0$ and hence equation (B 1) yields $f_i(q_{i+1}(t) - q_i(t)) = 0, i = 0, 1, \dots (n - 1)$, as in the fixed–fixed case. And since $q_o(t) \equiv 0$, we again obtain $q_i(t) = 0$, $i = 1, 2, \dots n$. Therefore, in $2n$-dimensional phase space, the origin $O(q \equiv \dot{q} \equiv 0)$ is a unique and isolated equilibrium point of the unconstrained (and the constrained) $n$-d.f. nonlinear lattice.

## Appendix C. $\Omega$ is compact

In this appendix, our aim is to show that the set $\Omega$ (described by equation (5.2)) is compact. The set $\Omega$ can be alternatively conceived as $\Omega = H^{-1}([\epsilon, c])$, where $0 < \epsilon < H^* < c$ and $H^{-1}$ denotes the pre-image of the energy function $H$ (described by equation (2.1)). To prove that $\Omega$ is compact, we use the following result [24,25].

*Let $X \subset \Re^{2n}$ and $Y \subset \Re^+$ be Euclidean spaces. A function $H : X \to Y$ is radially unbounded if and only if the pre-image $H^{-1}(K)$ of every compact set $K \subseteq Y$ is compact in $X$.*

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To use this result, we need to first establish that the energy $H$ is radially unbounded. The energy function $H$ is said to be radially unbounded if given any $M \in \mathfrak{R}^+$, there exists an $R \in \mathfrak{R}^+$ such that $H(x) > M$ for all $\|x\| > R$ [26]. We use the infinity norm to prove our results:

$$H(x) = H(q, \dot{q}) = T(\dot{q}) + U(q) = \sum_{i=1}^{n+1} \frac{1}{2} m_i \dot{q}_i^2 + \sum_{i=0}^{n} u_i(q_{i+1} - q_i) = \sum_i h_i. \qquad \text{(C 1)}$$

The basic idea behind the approach is to equate each individual term $h_i$ of the energy function $H$ to $M$, and find the supremum among the largest absolute values $|x_i|_{\max}$ of the $2n$-coordinates, such that for each of these terms, $h_i(|x_i|_{\max})$ equals $M$. This supremum value gives us $R$, which is the side length of the hypercube in $2n$-dimensional phase space. To find $R$, we adopt the following algorithm.

*Step* 1. Find the supremum $R_v$ among the largest absolute values of the $n$ velocities. To this end, we consider the kinetic energy terms $T(\dot{q})$ of the energy function $H$ (equation (C 1)). When each kinetic energy term is equated to $M$, out of all the $n$ terms, the term with the infimum mass $m_{k,\inf} = \inf\{m_1, m_2, \ldots, m_n\}$ gives us the supremum velocity. Therefore, $\frac{1}{2} m_{k,\inf} \dot{q}_k^2 = M$ yields $R_v = |\dot{q}_k| + \kappa_o = +\sqrt{2M/m_{k,\inf}} + \kappa_o$, where $\kappa_o > 0$ is included so that $R_v$ is strictly greater than the supremum $|\dot{q}_k|$.

*Step* 2. Find the supremum $R_d$ among the largest absolute values of the $n$ displacements. Consider the potential energy terms $U(q)$ of the energy function $H$ (see equation (C 1)). When the first term $u_0(q_1)$ is equated to $M$, for any given $M > 0$, there exist precisely two real values $r_1, r_2 \in \mathfrak{R}$ such that $u_0(r_1) = M$ and $u_0(r_2) = M$. This is because $u_0(q_1)$ is positive definite and strictly radially increasing (see §2). Therefore, given an $M$, a bound on the maximum value of $|q_1|$ is given by $R_1 = \max\{|r_1|, |r_2|\} + \kappa_1$, where $\kappa_1 > 0$.

Next, let us equate the second term of the potential energy, $u_1(q_2 - q_1)$, to $M$. Again, since $u_1$ is positive definite and strictly radially increasing, for a given $M$, there exist precisely two real values $r_3, r_4$ such that $u_1(r_3) = M$ and $u_1(r_4) = M$. Therefore, given an $M$, a bound on the maximum value of $|q_2|$ is given by $R_2 = R_1 + \max\{|r_3|, |r_4|\} + \kappa_2$, where $\kappa_2 > 0$. Continuing this recursive process, for the $i$th term, given an $M > 0$, a bound on the maximum value of $|q_i|$ is given by $R_i = R_{i-1} + \max\{|r_{2i-1}|, |r_{2i}|\} + \kappa_i$, where $\kappa_i > 0 \,\forall\, i$, $R_o = 0$, and $u_{i-1}(r_{2i-1}) = u_{i-1}(r_{2i}) = M$ for $i = 1, 2, \ldots n$.

For a fixed–free lattice, there are only $n$ potential energy terms in the energy expression and therefore the supremum among the $n$ displacements is given by $R_d = R_n$. On the other hand, for a fixed–fixed lattice, there are $(n + 1)$ terms and equating the last term of the energy expression $u_n(-q_n)$ to $M$ yields yet another estimate for a bound on the maximum value of $|q_n|$ given by $R'_n = \max\{|r_{2n+1}|, |r_{2n+2}|\} + \kappa_{n+1}$, where $\kappa_{n+1} > 0$ and $u_n(r_{2n+1}) = u_n(r_{2n+2}) = M$. Thus, for a fixed–fixed lattice, the supremum among the largest absolute values of the $n$ displacements is given by $R_d = \max\{R_n, R'_n\}$.

*Step* 3. Therefore, given any $M$, a corresponding bound on the side length $R$ of the hypercube in $2n$-dimensional phase space is given by $R = \max\{R_v, R_d\} + \triangle$ where $\triangle > 0$. Clearly, for all $\|x\|_\infty > R$, we have $H(x) > M$. Hence, the energy function $H(x)$ is radially unbounded.

Thus, by virtue of the result stated at the beginning of this appendix, since $H(x)$ is a radially unbounded function, it follows that the pre-image $H^{-1}(K)$ of every compact set $K = [\epsilon, c] \in \mathfrak{R}^+$ is compact in $\mathfrak{R}^{2n}$. But, $H^{-1}([\epsilon, c])$ is our set $\Omega$ and therefore, $\Omega$ is compact in $\mathfrak{R}^{2n}$.

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