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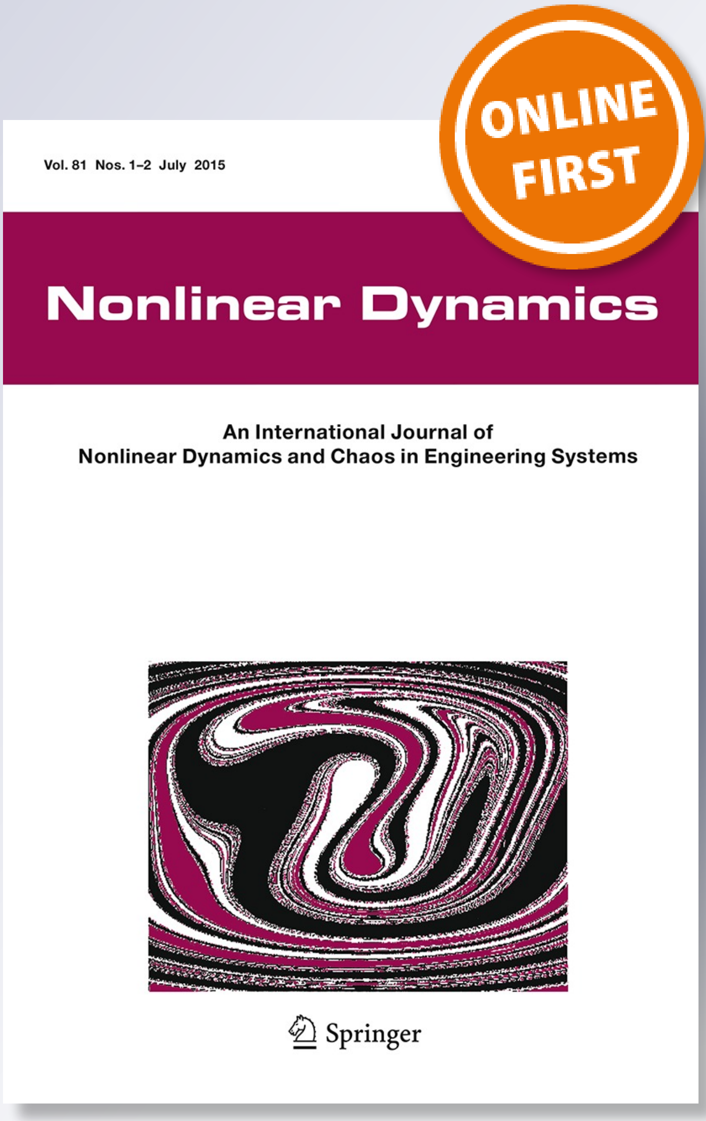

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ORIGINAL PAPER

Optimal stable control for nonlinear dynamical systems: an analytical dynamics based approach

Firdaus E. Udwadia · **Prasanth B. Koganti**

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Abstract This paper presents a method for obtaining optimal stable control for general nonlinear nonautonomous dynamical systems. The approach is inspired by recent developments in analytical dynamics and the observation that the Lyapunov criterion for stability of dynamical systems can be recast as a constraint to be imposed on the system. A closed-form expression for control is obtained that minimizes a user-defined control cost at each instant of time and enforces the Lyapunov constraint simultaneously. The derivation of this expression closely mirrors the development of the fundamental equation of motion used in the study of constrained motion. For this control method to work, the positive definite functions used in the Lyapunov constraint should satisfy a consistency condition. A class of positive definite functions has been provided for mechanical systems that meet this criterion. To illustrate the broad scope of the method, for linear systems it is shown that a proper choice of these positive definite functions results in conventional LQR control. Control of the Lorenz system and a multi-degree of freedom nonlinear mechanical system are considered. Numerical examples demonstrating the efficacy and simplicity of the method are provided.

Keywords Nonlinear dynamical systems · Minimization of control cost · Lyapunov constraint · Analytical dynamics approach · Global asymptotic stability · Consistent constraint · Control of Lorenz and mechanical systems

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1 Introduction

The general approach in control design of large, multi-scale nonlinear dynamical systems is to postulate a controller first, often based on experience or heuristic considerations, and then to check its stability. Lyapunov's second method is the most popular method for checking the stability of the control design. In this approach, the analyst searches for a suitable Lyapunov function such that the dynamics of the system ensure that its time derivative is nonpositive [1–6]. If such a function can be found, then stability is ensured. Otherwise, the fact that such a function cannot be found does not

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necessarily mean that the proposed control is unstable, and therefore, the stability of the proposed controller remains uncertain.

However, there are methods available that first start with a suitable candidate Lyapunov function and obtain control by using this function in such a manner that the resulting controlled system is stable in the Lyapunov sense. Two such methods that are more prominent and are often used are Sontag's formula and the back-stepping approach [1,7–10]. Although these methods have several differences between them, a common theme is to use a positive definite function to obtain stable control by ensuring that the rate of change of the chosen positive definite function is always negative along the controlled system's trajectories.

In Sontag's formula method, which was first proposed in 1989, a closed-form control is derived for a dynamical system by prescribing a positive definite function (such a function is called a control Lyapunov function, or CLF for short) that satisfies a certain criterion [7]. The development of Sontag's formula is inspired by the linear quadratic control problem, and the problem is framed in terms of a family of linear stabilizable systems parametrized by the state. In [8], Sontag's method is extended and a control is obtained that minimizes the L_2 norm of the control variable at each instant of time while ensuring that the time derivative of the prescribed positive definite function is negative. In [9], it has been observed that the solution of the LQ control problem can be recovered using this method for a carefully chosen CLF.

The back-stepping method is developed for dynamical systems that can be viewed as consisting of several cascaded sub-systems. The output of one sub-system is viewed as an input to the next. In this method, a Lyapunov function is obtained in a recursive manner by successively modifying it at each level of the cascaded system in such a way that the cascaded system is stable. Due to the gradual morphing of the Lyapunov function as it progresses through the cascade, it is difficult to enforce a user-specified Lyapunov function for the entire system. For further details, the reader may refer to the text written by some of the pioneers of this method [10].

The state-dependent Riccati equation (SDRE) method, which is a control method for nonlinear autonomous systems, gets its inspiration from LQR theory. Here, the system is described through factorization of the nonlinear dynamics into a state-dependent matrix and the state vector, thereby yielding for the nonlinear system a nonunique linear structure. A performance index with a quadratic-like structure is minimized by solving an algebraic Riccati equation to give the suboptimal control law at each point in state space. Thus, the SDRE approach is considerably more complex than the one presented herein both from analytical and a computational standpoint. Since solving the Riccati equations online is computationally very intensive, especially for systems with a large number of degrees of freedom, the method has substantial limitations, besides being applicable to only autonomous systems. For an extensive list of references on SDRE method, see Ref. [11].

Recently, a control approach has been proposed in Ref. [12] for nonlinear, nonautonomous mechanical systems described by second-order differential equations that typically arise in studying the dynamics of mechanical system through the use of Lagrange's and/or Newton's equations of motion. The method proposed utilizes recent developments in analytical dynamics. Lyapunov's stability condition is cast as a constraint (referred to as Lyapunov constraint from here on) to be imposed on the mechanical system. Two user-specified positive definite functions are utilized to synthesize the Lyapunov constraint, which is imposed on the system in the form of an *equality*, and an explicit expression for the control force that enforces it is obtained. The explicit closed-form control is based on the fundamental equation of mechanics [13–16]. The full nonlinear dynamical system is treated without any linearizations/approximations. Of equal importance is the fact that the control so obtained minimizes a user-specified control cost at each instant of time. Recently, Udwadia and Koganti [17] have shown the use of this approach for the stable control needed to swing-up a 10-body planar pendulum from its static equilibrium position so that it stands in various 'inverted' configurations.

The current paper extends the approach developed in Ref. [12] to general nonlinear nonautonomous dynamical systems described by first-order differential equations. Its compass of applicability is thus substantially expanded to include a much wider class of general nonlinear systems, including mechanical systems. Of special importance is its applicability to general systems that can be more conveniently described using Hamiltonian formulations. Unlike Ref. [12] that dealt exclusively with full-state control, the formulation herein is expanded to include systems where such full-state control is either (i) not possible, for example, due to economic limitations (e.g., on the deployment of actuators) for systems with a large number of degrees of freedom, or (ii) not feasible, for example, due to large data processing burdens. The current formulation is thus applicable to underactuated control of nonlinear nonautonomous systems—a topic of considerable interest in several specialized areas of mechanical engineering, such as robotics and tele-operator design.

Furthermore, though Ref. [12] emphasizes that the methodology proposed therein requires that the imposed Lyapunov constraint must be consistent at all times, it does not provide a detailed discussion of this aspect, nor does it show how one can ensure this consistency. A not-so-obvious observation is that unlike the naturally arising physical constraints dealt with in the theory of constrained motion of mechanical systems (which is the inspiration for this control approach), the Lyapunov constraint that is required to be enforced to ensure stability is not guaranteed to be consistent at all times. In fact, a necessary (and sufficient) condition for the control approach in Ref. [12] to work is that the Lyapunov constraint be consistent at all times. This then places a burden on finding consistent Lyapunov constraints—that is, combinations of candidate Lyapunov functions, V, and positive definite functions, w (that describe the time rate of change of V)—that yield a consistent Lyapunov constraint at all times for the controlled nonlinear system. This task is taken up here.

As in Ref. [12], the current method also uses a user-defined quadratic control cost, J, which is required to be minimized at every instant of time, along with two user-defined positive definite functions (V, w) to describe the Lyapunov constraint. Sets of closed-form nonlinear controllers for a given nonlinear nonautonomous dynamical system are obtained in closed form that simultaneously (a) minimize a user-desired control cost and (b) guarantee the stability of the controlled system. They depend on: (i) the user-provided cost function, J; (ii) the desired candidate Lyapunov function chosen, V; and (iii) the desired stability requirement (as dictated by Lyapunov's second method) that is encapsulated in the function w, as described below. The issue of consistency of the Lyapunov constraint is discussed in detail. Sets of (V, w) pairs that ensure consistency for mechanical systems are obtained. A proper choice of these pairs can provide optimal, global, and asymptotically stable control of the nonlinear nonautonomous dynamical system. As an illustration of the wide applicability of the method, when applied to a linear system it is shown that by using a suitable (V, w) pair one can easily recover standard LQR controllers.

It is important to point out that the current method is quite different from methods that employ the calculus of variations, a subject with a rich and long scientific history [18–20] and one that underpins the conventional theory of optimal control that has been developed over the last 60 years or so. Such optimal control methods frame the control problem in terms of the minimization of an objective function that is expressed as an integral over time and then utilize techniques developed in the calculus of variations. In contrast, the current method does not use any notions from the calculus of variations whatsoever. In this sense, conventional optimal control theory is not used, and the method herein is instead inspired by results from a different field, namely analytical dynamics. It is interesting that though the method developed deals with the control of nonlinear nonautonomous dynamical systems, it minimizes a user-desired control cost at each instant of time (instead of an integral over time), and it relies on a few simple results from linear algebra (instead of the calculus of variations).

In summary, the differences between the current approach and those extant in the literature to date are: (i) the underlying philosophy of the present approach is totally different in that it is inspired by recent developments in analytical dynamics; (ii) a user-specified control cost is minimized at *each instant* of time; (iii) the approach is much simpler and uses only elementary linear algebra; and (iv) no notions from the calculus of variations are used.

This paper is organized as follows. In Sect. 2, a general nonlinear, nonautonomous dynamical system is considered and optimal stabilizing control is obtained in closed form. The consistency condition, which is necessary for the control method to work, is also discussed. Section 3 contains an application of the current method to linear systems and it is shown that the current control approach, with proper choices of V, w, and J recovers standard results of LQR theory. In Sect. 4, mechanical systems are considered and sets of (V, w) pairs that satisfy the consistency condition are obtained. Examples are provided along the way to show the performance of the approach. Section 5 gives the conclusions.

2 General dynamical systems

In this section, control is derived for general nonlinear nonautonomous dynamical systems described by a set of first-order differential equations. In Sect. 2.1 an explicit expression is obtained for the control input that minimizes a user-desired control cost at each instant of time and ensures that the dynamical system has an asymptotic equilibrium point at the origin. First, a candidate Lyapunov function is chosen, and then, a set of nonlinear controllers are obtained in closed form such that (a) the control effort is minimized and (b) the dynamics ensure that the candidate Lyapunov function *is* indeed a Lyapunov function for the system. No linearizations and/or approximations of the dynamical system are made. In Sect. 2.2, the consistency condition, which can be used to check if a given pair of positive definite functions (V, w) can be used with the current method to produce stable control, has been formalized. Numerical examples have been provided in Sect. 2.3 that demonstrate the efficacy of the current method and the ease and simplicity of its application. They also demonstrate the use of the consistency condition to check the positive definite function pairs before using them with the current method.

2.1 Main result

Consider a general dynamic system described by the first-order ordinary differential equation

$$\dot{x} = f(x, t) + B(x, t)u(x, t), \tag{1}$$

where $f : D \times [0, \infty) \to R^n$, $u : D \times [0, \infty) \to R^p$, and, $B : D \times [0, \infty) \to R^{n \times p}$ are all piecewise continuous in t and locally Lipschitz on $D \times [0, \infty)$, $D \subset R^n$ is a domain that contains the origin. In addition, $B(x, t)$ and $u(x, t)$ are assumed to be bounded on $D \times [0, \infty)$. In the above, $u(x, t)$ is the control input vector, and f and B are known. If the dynamics ensure that there exists a continuously differentiable positive definite Lyapunov function $V(x, t)$ such that $V_L(x) \le V(x, t) \le V_U(x)$, where $V_L(x)$ and $V_U(x)$ are continuous positive definite functions on D, and a positive definite function $w(x)$ on D, such that the derivative of $V(x, t)$ along the trajectories of Eq. (1) satisfies the relation

$$\frac{\mathrm{d}V(x, t)}{\mathrm{d}t} := \dot{V}(x, t) = -w(x), \tag{2}$$

then the system has a uniformly asymptotically stable equilibrium point at $x = 0$ [1–6].

Equation (2) can be expanded as,

$$\dot{V}(x, t) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial t} = -w(x). \tag{3}$$

Denoting

$$A := \frac{\partial V}{\partial x} \quad \text{and} \quad b := -w(x) - \frac{\partial V}{\partial t}, \tag{4}$$

Equation (3) can be expressed concisely as,

$$A(x, t)\,\dot{x} = b(x, t). \tag{5}$$

We refer to any continuously differentiable function $V(x, t)$ such that $V_L(x) \le V(x, t) \le V_U(x)$, where $V_L(x)$ and $V_U(x)$ are continuous positive definite functions on D, as a candidate Lyapunov function. Thus, a candidate Lyapunov function $V(x, t)$ that satisfies Eq. (3) or equivalently Eq. (5) ensures that the equilibrium point $x = 0$ of the controlled system described by Eq. (1) is uniformly asymptotically stable [1–6].

The problem at hand is the following. Given a candidate Lyapunov function $V(x, t)$ and a positive definite function $w(x)$ we want to devise a control input $u(x, t)$, such that for the controlled system described by Eq. (1), the control cost given as,

$$J(t) = u(x, t)^{\mathrm{T}} N(x, t)\, u(x, t) \tag{6}$$

is minimized at each instant of time *and* relationship (5) is simultaneously satisfied. The weighting matrix $N(x, t)$ above is a symmetric positive definite matrix. The former ensures that for the given pair (V, w) the control is optimal, and the latter ensures that the controlled system has a uniformly asymptotically stable equilibrium point at $x = 0$. We will assume that such a control u exists. In what follows, the arguments of the various quantities will be suppressed unless required for clarity.

Result 1 The control input that minimizes the control cost given in Eq. (6) at *each* instant of time t and ensures the asymptotic stability of the controlled dynamical system by satisfying relation (5), is given by

$$u = N^{-1/2} G^{+} (b - Af) = \frac{N^{-1} B^{\mathrm{T}} A^{\mathrm{T}}}{A B N^{-1} B^{\mathrm{T}} A^{\mathrm{T}}} (b - Af), \quad (7)$$

In the above equation, A and b are as defined in Eq. (4) and G is defined as,

$$G := A B N^{-1/2}. \quad (8)$$

The matrix G^{+} in Eq. (7) denotes the Moore–Penrose inverse of the matrix G.

Proof Using Eq. (1) in the Lyapunov constraint Eq. (5), the relation

$$A\dot{x} = Af + ABu = b \quad (9)$$

is obtained. The last equality can be rewritten as,

$$ABu = b - Af. \quad (10)$$

It is necessary (and sufficient) that Eq. (10) should have at least one solution at every instant of time for a control vector u to exist that makes the controlled system satisfy relation (5). This is called the consistency condition which is discussed further in Sect. 2.2.

Observing the form of the control cost $J(t)$ in Eq. (6), let a transformation of u be defined as,

$$z := N^{1/2} u. \quad (11)$$

Thus, the control cost is now simply $J(t) = z^{\mathrm{T}} z$, and the control input can be recovered back from the transformed variable as,

$$u = N^{-1/2} z. \quad (12)$$

Using relation (12), Eq. (10) simplifies to

$$A B N^{-1/2} z = b - Af \quad (13)$$

or alternatively,

$$Gz = b - Af, \quad (14)$$

where the definition of matrix G in Eq. (8) has been used. Thus, the problem is now reduced to finding the solution z to Eq. (14) which simultaneously minimizes $J(t) = z^{\mathrm{T}} z$. The solution is found by using the Moore–Penrose pseudoinverse as simply [13],

$$z = G^{+} (b - Af). \quad (15)$$

The control input can then be obtained in closed form using Eq. (12) as,

$$u = N^{-1/2} G^{+} (b - Af). \quad (16)$$

Since, G is a row vector, this can be further simplified as,

$$u = N^{-1/2} \frac{G^{\mathrm{T}}}{G G^{\mathrm{T}}} (b - Af). \quad (17)$$

Substituting for G from Eq. (8), the simplified expression for the control input is

$$u = \frac{(b - Af)}{A B N^{-1} B^{\mathrm{T}} A^{\mathrm{T}}} N^{-1} B^{\mathrm{T}} A^{\mathrm{T}}. \quad (18)$$

□

2.2 Consistency of the Lyapunov constraint

Result 2 When the matrix $AB \neq 0$, the Lyapunov constraint is always consistent. When $AB = 0$, the Lyapunov constraint is consistent if and only if $(b - Af) = 0$.

Proof In the derivation of the previous result, it has been assumed that at all time, there exists a vector u that ensures that the system satisfies the Lyapunov constraint (5). In the derivation of the main result in Sect. 2.1, it is shown that this is equivalent to the statement that at least one solution u to Eq. (10) exists at all instants of time. The necessary and sufficient condition for the existence of a solution to Eq. (10) is [13]

$$(AB)(AB)^{+} (b - Af) = (b - Af). \quad (19)$$

Since AB is a row vector, this equation can be expanded as,

$$\frac{(AB)(AB)^{\mathrm{T}}}{(AB)(AB)^{\mathrm{T}}} (b - Af) = (b - Af), \quad (20)$$

which is always true when $AB \neq 0$. When AB is identically zero, the left-hand side of Eq. (19) is zero, and hence, for the Lyapunov constraint to be consistent, we require that the right-hand side of Eq. (19) is also zero, so that

$$b - Af = 0. \tag{21}$$

Thus, if $b - Af = 0$ whenever $AB = 0$, the pair (V, w) used in obtaining the corresponding A and b [see relations (4) and (5)] can be used to obtain a suitable control, u. □

From here on, we then say, that the positive definite function pair (V, w) provides a consistent Lyapunov (stability) constraint, or that the (V, w) *pair is consistent*, for short.

The next example illustrates the importance of this consistency requirement.

2.3 Numerical examples

Example 1(a) Consider the Lorenz system with control applied to only the first state, described by the equations

$$\begin{aligned} \dot{x}_1 &= \sigma (x_2 - x_1) + u, \\ \dot{x}_2 &= x_1 (\rho - x_3) - x_2, \\ \dot{x}_3 &= x_1 x_2 - \beta x_3. \end{aligned} \tag{22}$$

where $\sigma, \rho, \beta > 0$ are a given set of parameters. Since the control is only applied to the first state, the control input $u \in R$ is a scalar and the matrix B in Eq. (1) is a 3-vector,

$$B = [1, 0, 0]^{\mathrm{T}}. \tag{23}$$

The vector f is simply,

$$f = [\sigma (x_2 - x_1), x_1 (\rho - x_3) - x_2, x_1 x_2 - \beta x_3]^{\mathrm{T}}. \tag{24}$$

Consider the candidate positive definite function pair (V, w) given by

$$V(x) = \frac{1}{2}\left(\alpha x_1^2 + \upsilon_2 x_2^2 + x_3^2\right), \alpha, \upsilon_2 > 0, \tag{25}$$

and

$$w(x) = \upsilon_1 x_1^2 + \upsilon_2 x_2^2 + \beta x_3^2 \tag{26}$$

where $\upsilon_1 > 0$ and $\upsilon_2 > 0$ are positive scalars, and β is the parameter of the Lorenz oscillator. Using this pair, the Lyapunov constraint is

$$\dot{V}(x, t) = \frac{\partial V}{\partial x}\dot{x} = -w. \tag{27}$$

For the pair (V, w) given by Eqs. (25) and (26) to be usable, it must be consistent.

Here, A and b are computed as,

$$\begin{aligned} A &= \frac{\partial V}{\partial x} = [\alpha x_1, \upsilon_2 x_2, x_3], \\ b &= -w = -\left(\upsilon_1 x_1^2 + \upsilon_2 x_2^2 + \beta x_3^2\right). \end{aligned} \tag{28}$$

Since $AB = \alpha x_1$, $AB = 0$ implies $x_1 = 0$. Using Eqs. (28) and (24), it can be verified that whenever $x_1 = 0$,

$$\begin{aligned} b - Af &= -\left(\upsilon_1 x_1^2 + \upsilon_2 x_2^2 + \beta x_3^2\right) - x_1 \alpha \sigma (x_2 - x_1) \\ &\quad - \upsilon_2 x_2 x_1 (\rho - x_3) + \upsilon_2 x_2^2 - x_1 x_2 x_3 + \beta x_3^2 \\ &= -\left(\upsilon_2 x_2^2 + \beta x_3^2\right) + \upsilon_2 x_2^2 + \beta x_3^2 = 0. \end{aligned} \tag{29}$$

Thus, the positive definite function pair (V, w) provides a consistent Lyapunov constraint and can be used to obtain a stabilizing control for the system.

The control cost to be minimized is specified as $J(t) = u(x, t)^{\mathrm{T}} N(x, t) u(x, t)$. Since u is a scalar in this example, N is just a positive scalar and without loss of generality, it can be set to unity. Since $AB = x_1$, the expression for the control force can be simplified as,

$$u = \frac{(b - Af)}{ABN^{-1}B^{\mathrm{T}}A^{\mathrm{T}}} N^{-1} B^{\mathrm{T}} A^{\mathrm{T}} = \frac{1}{\alpha x_1}(b - Af). \tag{30}$$

Since

$$\begin{aligned} b - Af &= -x_1 x_2 (\alpha\sigma + \upsilon_2 \rho) - x_1^2 (\upsilon_1 - \alpha\sigma) \\ &\quad - x_1 x_2 x_3 (1 - \upsilon_2), \end{aligned} \tag{31}$$

the explicit control is given by

$$u = -\frac{x_2(\alpha\sigma + \upsilon_2 \rho) + x_1(\upsilon_1 - \alpha\sigma) + x_2 x_3 (1 - \upsilon_2)}{\alpha}. \tag{32}$$

The Lyapunov function V in Eq. (25) is radially unbounded, and hence, the controlled system is globally asymptotically stable. It is important to realize that the control given in Eq. (32) not only ensures global asymptotic stability but it is also simultaneously optimal in the sense that for the chosen (V, w) pair it provides a control that minimizes $J(t) = u^2(t)$ at each instant of time.

For the simulation, the system parameter values are chosen as, $\sigma = 10$, $\rho = 28$, and $\beta = 2/3$. The (uncontrolled) system's behavior for this set of parameters is chaotic. The parameters defining the (V, w) pair are chosen as $\alpha = 1$, $\upsilon_1 = 0.5$, and $\upsilon_2 = 0.5$ [see Eqs. (25, 26)]. The initial conditions are as follows: $x_1(0) = -6$, $x_2(0) = 4$, and $x_3(0) = 1$.

The controlled system given in Eq. (22) is integrated numerically using the ODE15s package in the MATLAB environment. The error tolerances for numerical integration are chosen as 10^{-8} for the relative error, and 10^{-12} for the absolute error. The results of the simulation are shown in Figs. 1, 2 and 3. Figure 1 shows the response of the controlled system, showing asymptotic convergence to the equilibrium point $x_1 = x_2 = x_3 = 0$. Since V is radially unbounded, the origin is globally attractive, and thus, the system can be controlled starting from any set of initial conditions.

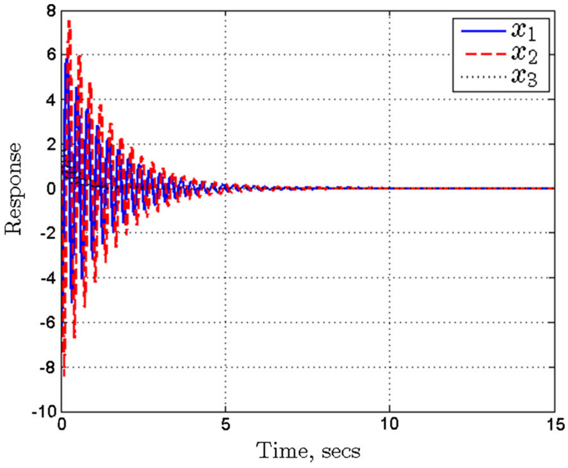

Fig. 1 Time history of response of the controlled Lorenz system with control only applied to first state

Figure 2a, b show the projections of the phase portrait of the controlled system on the $x_1 - x_2$ and $x_1 - x_3$ planes, respectively. The initial position is marked by a square and a circle indicates the position at the end of 15 seconds of integration. The final position can be seen to have converged to the origin. Figure 3a shows the time history of the control input $u(t)$. Figure 3b shows the error

$$e(t) := \frac{\partial V}{\partial x}\dot{x} + w = A\dot{x} - b \tag{33}$$

in satisfying the Lyapunov constraint. It is found to be of the same order of magnitude as the error tolerance (10^{-12}) used for numerically integrating the equations.

Example 1(b) To demonstrate the consequences of choosing a positive definite function pair (V, w) that leaves the Lyapunov constraint inconsistent at some point on the trajectory, let us again consider the Lorenz system, this time with control applied only to the third state. The equation of the controlled system is now

$$\begin{aligned}
\dot{x}_1 &= \sigma\,(x_2 - x_1)\,,\\
\dot{x}_2 &= x_1\,(\rho - x_3) - x_2,\\
\dot{x}_3 &= x_1 x_2 - \beta x_3 + u.
\end{aligned} \tag{34}$$

The various quantities defining this dynamical system are identified as,

$$\begin{aligned}
B &= [0, 0, 1]^{\mathrm{T}}\,,\\
f &= [\sigma\,(x_2 - x_1)\,, x_1\,(\rho - x_3) - x_2, x_1 x_2 - \beta x_3]^{\mathrm{T}}.
\end{aligned} \tag{35}$$

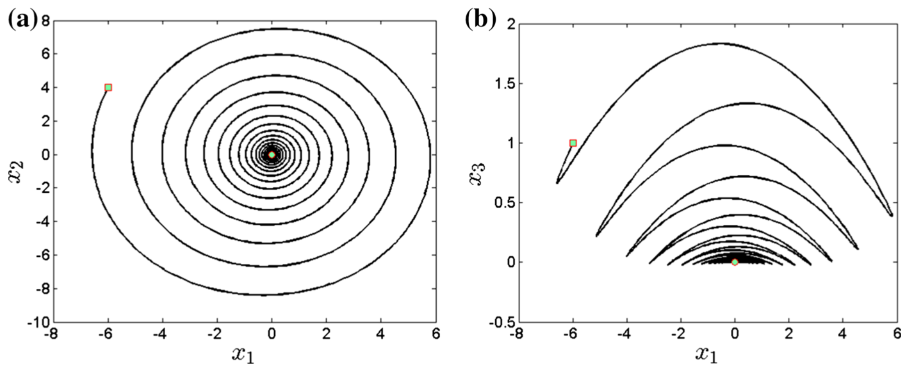

Fig. 2 **a** Projection of phase portrait of the controlled system on the x_1–x_2 plane. **b** Projection of phase portrait of the controlled system on the x_1–x_3 plane

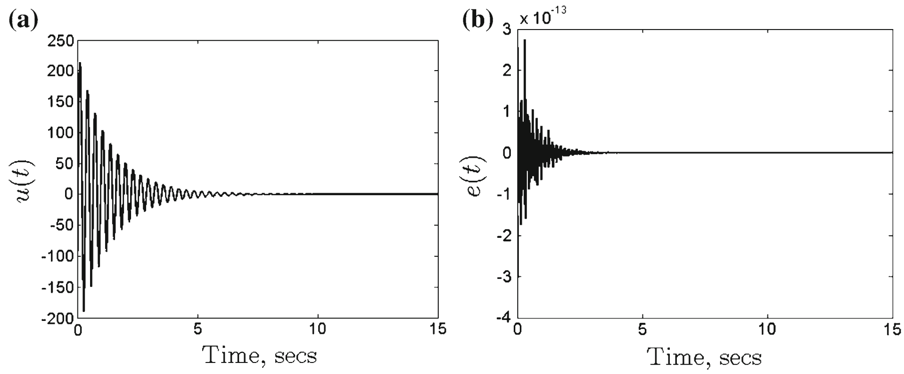

Fig. 3 **a** Time history of the control input $u(t)$. **b** Time history of the error $e(t)$ in enforcing Lyapunov stability constraint [see Eq. (33)]

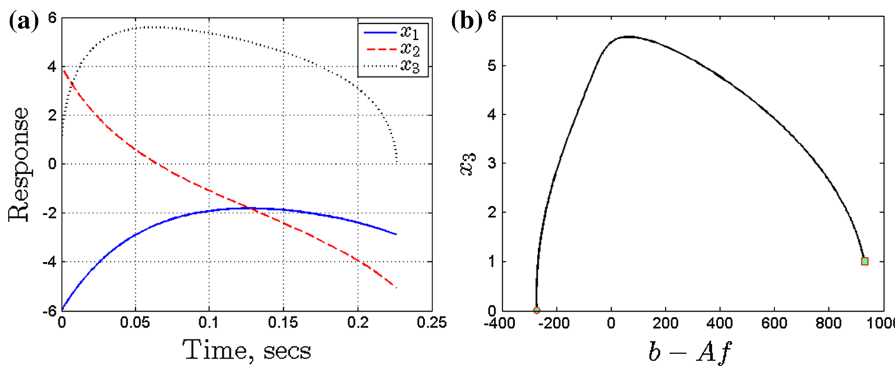

Fig. 4 **a** Time history of response of the controlled Lorenz system (control only applied to third state). **b** Variation of x_3 and $b - Af$ along the trajectory of the system

The positive definite function pair (V, w) is chosen to be identical to that in Example 1(a) [see Eqs. (25) and (26)]. Thus, the Lyapunov constraint given in Eq. (27) is still valid with the quantities A, b defined in Eq. (28). But the matrix B is different in this case and AB is now x_3. When $x_3 = 0$, $b - Af$ is found to be

$$\begin{aligned} b - Af &= -x_1x_2(\alpha\sigma + \upsilon_2\rho) - x_1^2(\upsilon_1 - \alpha\sigma) \\ &\quad - x_1x_2x_3(1 - \upsilon_2) \\ &= -x_1x_2(\alpha\sigma + \upsilon_2\rho) - x_1^2(\upsilon_1 - \alpha\sigma) \end{aligned} \qquad (36)$$

The parameter υ_2 cannot be chosen such that the above quantity is zero whenever x_3 is zero (υ_2 cannot be chosen to be $-\alpha\sigma/\rho$ as then, it will be negative). Thus, the Lyapunov constraint is not guaranteed to be consistent for this (V, w) pair. The explicit expression for the control is obtained by substituting $AB = x_3$ in the first equality of Eq. (30) and is given in simplified form by the expression

$$\begin{aligned} u = \frac{-x_1}{x_3} &(x_2(\sigma + \upsilon_2\rho) + x_1(\upsilon_1 - \alpha\sigma) \\ &+ x_2x_3(1 - \upsilon_2)) . \end{aligned} \qquad (37)$$

All the parameter values are kept the same as in Example 1(a). Using the control given in Eq. (37), the controlled system shown in Eq. (34) is integrated starting from identical initial conditions as before. The integration fails to meet the prescribed error tolerances at around 0.23 seconds and stops when the Lyapunov constraint becomes inconsistent.

Figure 4a shows the evolution of the state of the controlled system with time. Figure 4b shows the variation of x_3 and $b - Af$ along the trajectory of the system. The square marker indicates the initial state and the circular marker shows the final position. It can be seen that when the integration fails, x_3 has reached zero but not $b - Af$ (recall, $AB = x_3 = 0$ when the Lyapunov constraint becomes inconsistent).

Example 1(c) We next consider the system when only the second state is controlled. The controlled nonlinear system is described by the equations

$$\begin{aligned} \dot{x}_1 &= \sigma\,(x_2 - x_1)\,, \\ \dot{x}_2 &= x_1\,(\rho - x_3) - x_2 + u, \\ \dot{x}_3 &= x_1x_2 - \beta x_3. \end{aligned} \qquad (38)$$

The control input u for this case is again a scalar and the matrix B is simply $B = [0, 1, 0]^{\mathrm{T}}$. Using the same pair (V, w) given in Eqs. (25) and (26), the scalar

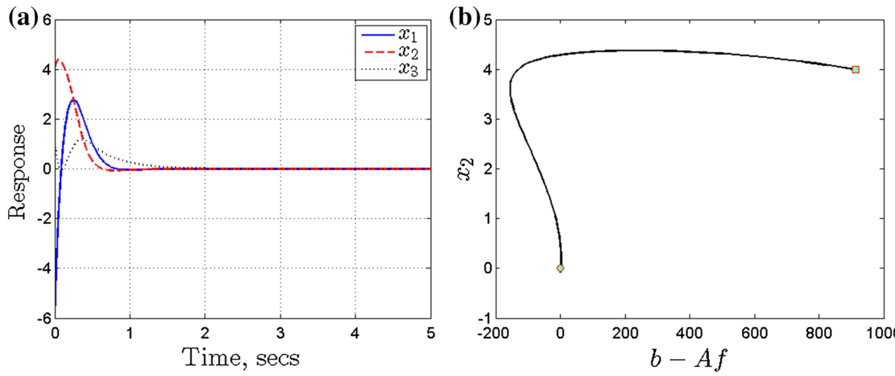

Fig. 5 **a** Time history of response of the controlled Lorenz system (control only applied to second state). **b** Variation of x_2 and $b - Af$ along the trajectory of the system

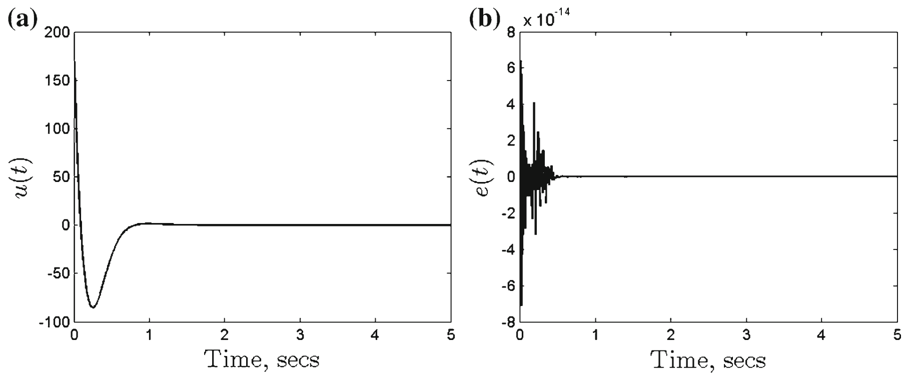

Fig. 6 **a** Time history of the control input $u(t)$. **b** Time history of the error $e(t)$ in enforcing Lyapunov stability [see Eq. (33)]

AB is found to be equal to $\upsilon_2 x_2$. Therefore, $AB = 0$ implies that $x_2 = 0$. We again obtain, as before,

$$b - Af = -x_1x_2(\alpha\sigma + \upsilon_2\rho) - x_1^2(\upsilon_1 - \alpha\sigma) - x_1x_2x_3(1 - \upsilon_2). \quad (39)$$

The choice $\upsilon_1 = \alpha\sigma$ makes the quantity $b - Af$ zero when $x_2 = 0$. The Lyapunov constraint is then consistent, and this (V, w) pair can be used to yield a control that is globally asymptotically stable.

Using Eq. (18) (with $\upsilon_1 = \alpha\sigma$), the globally asymptotically stable control is explicitly given by

$$u = -\frac{x_1(\alpha\sigma + \upsilon_2\rho) + x_1x_3(1 - \upsilon_2)}{\upsilon_2}. \quad (40)$$

This control minimizes at each instant of time the control cost $J(t) = u^2$ for the pair (V, w) chosen in Eqs. (25) and (26) with $\upsilon_1 = \alpha\sigma$.

All the parameter values are kept the same as in Example 1(a) except α, υ_1 which are, respectively, set to be $\alpha = 0.1$, $\upsilon_1 = \alpha\sigma = 1$.

Figure 5a shows the evolution of the state of the controlled system with time. Figure 5b shows the variation of x_2 and $b - Af$ along the trajectory of the system. The square marker again indicates the initial state and the circular marker shows the final position. Figure 6a shows the time history of the control input that enforces the Lyapunov constraint and simultaneously minimizes the control cost u^2 at each instant of time. Similarly, Fig. 6b shows the error in the satisfaction of the Lyapunov constraint. The error, $e(t)$, in satisfying the Lyapunov constraint [see Eq. (33)] is found to be of the same order of magnitude as the error tolerances used in numerically integrating the equations describing the controlled system.

Example 1(d) We next consider the system when both the second and the third state are simultaneously controlled and the control input is a vector. The controlled nonlinear system is now described by the equations

$$\dot{x}_1 = \sigma\,(x_2 - x_1)\,,$$
$$\dot{x}_2 = x_1\,(\rho - x_3) - x_2 + u_2,$$
$$\dot{x}_3 = x_1x_2 - \beta x_3 + u_3, \quad (41)$$

and the matrix B is the 3 by 2 matrix given by

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (42)$$

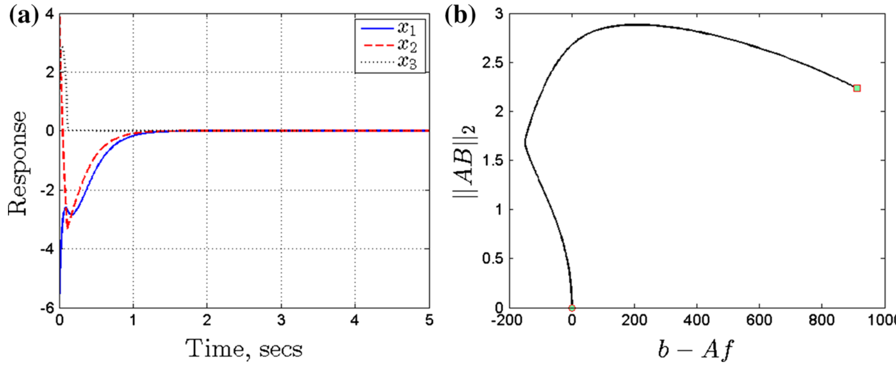

Fig. 7 **a** Time history of response of the controlled Lorenz system (control applied to second and third states only). **b** Variation of $\|AB\|_2$ and $(b - Af)$ along the trajectory of the system

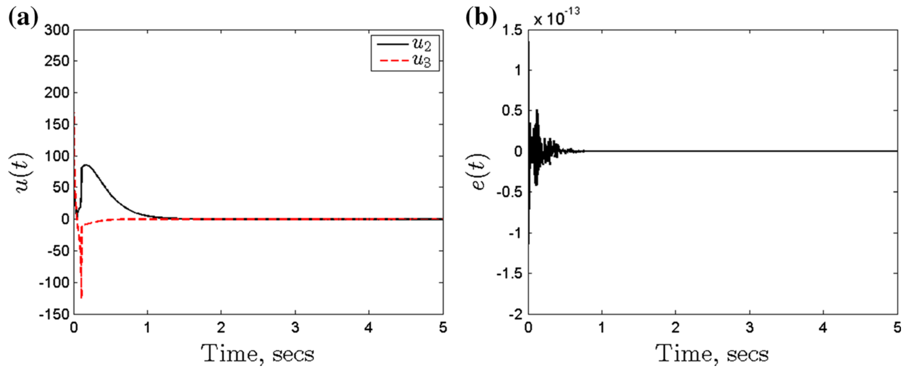

Fig. 8 **a** Time history of the control input $u(t)$. **b** Time history of the error $e(t)$ in enforcing Lyapunov stability [see Eq. (33)]

The control input is the 2-vector, $u = [u_2, u_3]^{\mathrm{T}}$. Using the same (V, w) pair given in Eqs. (25) and (26) as before, the row vector AB is given by $AB = [\upsilon_2 x_2, x_3]$, so that $AB = 0$ implies that $x_2 = x_3 = 0$. We again obtain, as before,

$$b - Af = -x_1 x_2(\alpha\sigma + \upsilon_2\rho) - x_1^2(\upsilon_1 - \alpha\sigma) - x_1 x_2 x_3(1 - \upsilon_2). \tag{43}$$

As in example 1(c), the choice $\upsilon_1 = \alpha\sigma$ makes the quantity $b - Af$ zero when $x_2 = x_3 = 0$. This (V, w) pair is thus made consistent. Choosing the weighting matrix to be

$$N = \mathrm{Diag}(1 + x_2^2, 1 + x_3^2), \tag{44}$$

the requisite globally asymptotically stable control is found using Eq. (18) (with $\upsilon_1 = \alpha\sigma$) and is given in closed form by

$$u = -\frac{x_1 x_2(\alpha\sigma + \upsilon_2\rho) + x_1 x_2 x_3(1 - \upsilon_2)}{\upsilon_2^2 x_2^2(1 + x_3^2) + x_3^2(1 + x_2^2)} \times \begin{bmatrix} \upsilon_2 x_2(1 + x_3^2) \\ x_3(1 + x_2^2) \end{bmatrix}. \tag{45}$$

This control minimizes at each instant of time the control cost $J(t) = u^{\mathrm{T}} N u$ for the pair (V, w) chosen in Eqs. (25) and (26), and the aforementioned weighting matrix N.

All the parameter values are kept identical as in Example 1(c).

Figure 7a shows the evolution of the state of the controlled system with time. Figure 7b shows the variation of $\|AB\|_2$ and $b - Af$ along the trajectory of the system. The plot confirms that the Lyapunov constraint is consistent since the scalar $(b - Af)$ is zero when AB is zero.

Figure 8a shows the time history of the control input that enforces the Lyapunov constraint and simultaneously minimizes for this (V, w) pair the control cost given by Eq. (6) (at each instant of time) with the weighting matrix given by Eq. (44). Figure 8b shows the error, $e(t)$, in enforcing the constraint. The error in satisfying the Lyapunov constraint is again found to be within the error tolerances used in the numerical integration.

3 Application to linear systems

In this section, the control method developed in Sect. 2 is applied to linear systems and it is shown that the control method can recover the solution of the LQR control approach, thereby connecting this method with results obtained from conventional control theory.

Consider a linear system, described by a linear differential equation,

$$\dot{x} = \hat{A}x + Bu. \tag{46}$$

The matrices $\hat{A} \in R^{n\times n}$ and $B \in R^{n\times p}$ are constant matrices. It is assumed that the pair $\left(\hat{A}, B\right)$ is controllable. The control input is $u \in R^p$. The control objective is to minimize the cost defined as

$$\hat{J} = \int_0^\infty \left(x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru\right) \mathrm{d}t, \tag{47}$$

where $Q, R \in R^{n\times n}$ are constant positive definite matrices. This problem has been well studied in the literature under the name 'LQR control' [e.g., [21]]. The optimal control is found to be linear and is given by,

$$u = -R^{-1}B^{\mathrm{T}}Px, \tag{48}$$

where P is a symmetric, positive definite matrix obtained by solving the algebraic Riccati equation.

$$\hat{A}^{\mathrm{T}}P + P\hat{A} + PBR^{-1}B^{\mathrm{T}}P + Q = 2PBR^{-1}B^{\mathrm{T}}P. \tag{49}$$

Result 3 If the positive definite pair (V, w) is chosen as (see Remarks 1 and 2 below),

$$V = x^{\mathrm{T}}Px,\ w = x^{\mathrm{T}}\left(Q + PBR^{-1}B^{\mathrm{T}}P\right)x, \quad P > 0,\ Q > 0 \tag{50}$$

where the matrix P satisfies relation (49), and the positive definite weighting matrix N is chosen as

$$N = R, \tag{51}$$

the control vector u obtained using Eq. (18) is the same as that given in Eq. (48).

Proof From Eq. (50), the various quantities required for obtaining the control u using Eq. (18) are obtained as,

$$A := \frac{\partial V}{\partial x} = 2x^{\mathrm{T}}P, \tag{52}$$

$$b := -w - \frac{\partial V}{\partial t} = -x^{\mathrm{T}}\left(Q + PBR^{-1}B^{\mathrm{T}}P\right)x.$$

Since

$$f = \hat{A}x, \tag{53}$$

on substituting these quantities in Eq. (18), u is found explicitly to be

$$u = \frac{2R^{-1}B^{\mathrm{T}}Px}{4x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px} \times \left(-x^{\mathrm{T}}\left(Q + PBR^{-1}B^{\mathrm{T}}P\right)x - A\hat{A}x\right). \tag{54}$$

On substituting for A from Eq. (52), we obtain

$$u = \frac{R^{-1}B^{\mathrm{T}}Px}{2x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px} \times \left(-x^{\mathrm{T}}\left(Q + PBR^{-1}B^{\mathrm{T}}P + 2P\hat{A}\right)x\right). \tag{55}$$

As $x^{\mathrm{T}}P\hat{A}x$ is a scalar, Eq. (55) further simplifies to

$$u = \frac{R^{-1}B^{\mathrm{T}}Px}{2x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px} \times \left(-x^{\mathrm{T}}\left(Q + PBR^{-1}B^{\mathrm{T}}P + P\hat{A} + \hat{A}^{\mathrm{T}}P\right)x\right). \tag{56}$$

Using the algebraic Riccati Eqs. (49), (56) reduces to

$$u = \frac{R^{-1}B^{\mathrm{T}}Px}{2x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px}\left(-2x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px\right) = -R^{-1}B^{\mathrm{T}}Px. \tag{57}$$

□

The significance of Result 3 is that the LQR controller is obtained by using a particular (V, w) pair and applying the general methodology proposed herein.

Remark 1 Consider the matrix $PBR^{-1}B^{\mathrm{T}}P$. If we denote $\hat{P} = PB$, then

$$PBR^{-1}B^{\mathrm{T}}P = \hat{P}R^{-1}\hat{P}^{\mathrm{T}} \tag{58}$$

is an n by n symmetric semi-positive definite matrix of the same rank as $\hat{P}$ because R is a positive definite matrix. Hence, w is a positive definite function since Q is a positive definite matrix.

Remark 2 The (V, w) pair given in Eq. (50) is consistent. To verify this, using Eq. (52) we compute AB as,

$$AB = 2x^{\mathrm{T}}PB. \tag{59}$$

Following the same steps shown in Eqs. (54)–(57), we obtain

$$b - Af = -2x^{\mathrm{T}} PBR^{-1} B^{\mathrm{T}} Px, \tag{60}$$

so that $(b - Af)$ is zero whenever AB is zero.

Remark 3 While the result obtained through the use of Eq. (18) is identical to that given by LQR theory, what may not be known as well conventionally is that this LQR control u also minimizes, at each instant of time, the cost $u^{\mathrm{T}} Ru$ when using the (V, w) pair given in Eq. (50).

4 Consistent *(V, w)* pairs for mechanical systems

In this section, the special case of mechanical systems which are described by second-order nonlinear nonautonomous differential equations is considered. These systems have a specific structure with respect to how the control inputs (generalized control forces) influence the dynamics of the system. This structure is exploited here to provide a general class of (V, w) pairs which satisfy the consistency condition and hence can be used to design optimal stable control, in the sense described before. Furthermore, since the functions V used in these pairs are positive definite functions that are also radially unbounded, the control realized through their use becomes globally asymptotically stable. We assume that each degree of freedom of the mechanical system is subjected to a generalized control force [see Eq. (62) below].

Consider the nonautonomous mechanical system whose dynamics are described by the equations

$$M(q, t)\ddot{q} = F(q, \dot{q}, t) \tag{61}$$

where $M \in R^{n \times n}$ is a symmetric positive definite mass matrix and F is a prescribed force vector of dimension n. The n-vector q contains the generalized positions and the n-vector $\dot{q}$ is the vector of generalized velocities. The equation of motion of the controlled system is,

$$M(q, t)\ddot{q} = F(q, \dot{q}, t) + Q^C(q, \dot{q}, t) \tag{62}$$

where Q^C is the control force. The (generalized) control force is applied to each (generalized) degree of freedom of the system. The objective of the control design is to minimize the control cost

$$J(t) = Q^{C^{\mathrm{T}}} N(q, \dot{q}, t) Q^C \tag{63}$$

at each instant of time and simultaneously coerce the controlled system to have an asymptotic equilibrium point at $q = 0, \dot{q} = 0$ by enforcing a suitable Lyapunov constraint.

If we denote the generalized velocity of the system by $v (v := \dot{q})$, the controlled system can be represented in state-space form as

$$\begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ M^{-1}F \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}Q^C \end{bmatrix} \tag{64}$$

This equation can be represented in our standard generalized dynamical system form (Eq. (1)) by denoting

$$x = \begin{bmatrix} q \\ v \end{bmatrix}, f = \begin{bmatrix} v \\ M^{-1}F \end{bmatrix}, B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \text{ and, } u = Q^C. \tag{65}$$

Consider the positive definite function pair (V, w), where

$$V = \frac{1}{2} q^{\mathrm{T}} P_1 q + \frac{1}{2} v^{\mathrm{T}} P_2 v + v^{\mathrm{T}} P_{12} q, \tag{66}$$

$$w = \frac{1}{2} q^{\mathrm{T}} Q_1 q + \frac{1}{2} v^{\mathrm{T}} Q_2 v + v^{\mathrm{T}} Q_3 q, \tag{67}$$

and

$$P_1 = \mathrm{diag}(a_1), P_2 = \mathrm{diag}(a_2), P_{12} = \mathrm{diag}(a_{12}), \tag{68}$$

$$Q_1 = DP_1, Q_2 = DP_2, Q_3 = DP_{12}. \tag{69}$$

Here, a_1, a_2, a_{12}, α are each n-vectors, and the diagonal matrix $D = \mathrm{diag}(\alpha)$.

We shall denote, for convenience, the element-wise product of two n-vectors l and m by the n-vector $[lm]$, so that its i-th element, $[lm]_i$, is simply given by the product of l_i and m_i; also, for an n-vector l, we will mean by l^c (l raised to the power c) the n-vector whose i-th component is l_i^c.

Result 4 The positive definite function pair (V, w), given by Eqs. (66)–(69), satisfies the consistency condition if

$$a_1 \succ 0, a_2 \succ 0, \alpha \succ 0, 0 \prec a_{12} \prec [a_1 a_2]^{1/2}, \\ \alpha = 2[a_{12} a_2^{-1}], \tag{70}$$

where the signs '$\succ$ ($\prec$)' are to be interpreted as 'greater (less) than' element-wise.

Proof It can be easily verified that the functions V and w in Eqs. (66) and (67) are positive definite since

$$a_1 \succ 0, a_2 \succ 0, \alpha \succ 0, 0 \prec a_{12} \prec [a_1 a_2]^{1/2}. \quad (71)$$

It is important to note that the function V is radially unbounded. The matrix A and the scalar b are given by

$$A = \frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial q}, \frac{\partial V}{\partial v}\right], \quad \text{and} \quad b = -w. \quad (72)$$

Equation (66) can be differentiated to obtain the required partial derivatives,

$$\frac{\partial V}{\partial q} = q^{\mathrm{T}} P_1 + v^{\mathrm{T}} P_{12}, \text{ and, } \frac{\partial V}{\partial v} = v^{\mathrm{T}} P_2 + q^{\mathrm{T}} P_{12}. \quad (73)$$

The quantity AB can then be obtained using Eqs. (65) and (72) as,

$$AB = \frac{\partial V}{\partial v} M^{-1}. \quad (74)$$

Since M^{-1} is a positive definite matrix, AB is zero if and only if $\frac{\partial V}{\partial v}$ is zero. Using Eq. (73), it can be concluded that AB can be zero if and only if

$$v = -P_2^{-1} P_{12} q. \quad (75)$$

The next step in the proof is to compute the quantity $(b - Af)$ when AB is zero. When $\frac{\partial V}{\partial v}$ is zero, Af is computed as,

$$Af = \frac{\partial V}{\partial q} v + \frac{\partial V}{\partial v} M^{-1} F = \frac{\partial V}{\partial q} v. \quad (76)$$

In the last equality above, we have used the fact that $\frac{\partial V}{\partial v}$ is zero when AB is zero. On substituting for $\frac{\partial V}{\partial q}$ from Eq. (73), this simplifies to

$$Af|_{AB=0} = q^{\mathrm{T}} P_1 v + v^{\mathrm{T}} P_{12} v. \quad (77)$$

When AB is zero, Eq. (75) holds true, and therefore,

$$\begin{aligned} Af|_{AB=0} &= -q^{\mathrm{T}} P_1 P_2^{-1} P_{12} q + q^{\mathrm{T}} P_{12} P_{12}^2 P_2^{-2} q \\ &= -q^{\mathrm{T}} [P_{12} P_2^{-1} (P_1 - P_{12}^2 P_2^{-1})] q \end{aligned} \quad (78)$$

where we have made use of the commutative property of diagonal matrices. In order to compute $b = -w$ when AB is zero, Eq. (75) is used to obtain

$$\begin{aligned} w|_{AB=0} &= \frac{1}{2} q^{\mathrm{T}} D P_1 q + \frac{1}{2} q^{\mathrm{T}} D P_{12}^2 P_2^{-1} q \\ &\quad - q^{\mathrm{T}} D P_{12}^2 P_2^{-1} q \\ &= \frac{1}{2} q^{\mathrm{T}} [D(P_1 - P_{12}^2 P_2^{-1})] q. \end{aligned} \quad (79)$$

Since $\alpha = 2[a_{12} a_2^{-1}]$ [see Eq. (71)] implies that $D = 2 P_{12} P_2^{-1}$, hence

$$\begin{aligned} b|_{AB=0} &= -w|_{AB=0} \\ &= -q^{\mathrm{T}} \left[P_{12} P_2^{-1} \left(P_1 - P_{12}^2 P_2^{-1}\right)\right] q. \end{aligned} \quad (80)$$

From Eqs. (80) and (78), it is seen that $b - Af = 0$ whenever $AB = 0$. Hence, the positive definite function pairs (V, w), given in Eqs. (66)–(69), always satisfy the consistency condition. □

The significance of Result 4 is that the family of positive definite function pairs (V, w) provided here can now be used to obtain optimal globally stable control for *any* mechanical system.

Remark 4 When a_1, a_2, a_{12}, and α are each constant n-vectors, so that $a_1 = a_1[1, 1, 1, 1 \ldots, 1]^{\mathrm{T}}, a_2 = a_2[1, 1, 1, 1 \ldots, 1]^{\mathrm{T}}, a_{12} = a_{12}[1, 1, 1, 1 \ldots, 1]^{\mathrm{T}}$, $\alpha = \alpha[1, 1, 1, 1 \ldots, 1]^{\mathrm{T}}$, the (V, w) pairs in Result 3 simplify, and Eqs. (66) and (67) become

$$V = \frac{1}{2} a_1 q^{\mathrm{T}} q + \frac{1}{2} a_2 v^{\mathrm{T}} v + a_{12} q^{\mathrm{T}} v \text{ and } w = \alpha V. \quad (81)$$

Consistency of these $(V, \alpha V)$ pairs then requires that the constants a_1, a_2, a_{12}, and α satisfy the relations

$$a_1 > 0, a_2 > 0, 0 < a_{12} < \sqrt{a_1 a_2}, \alpha = \frac{2 a_{12}}{a_2}, \quad (82)$$

where the similarity between these relations and those in Eq. (70) is obvious.

Due to the particular (V, w) pair chosen in Eq. (81), the Lyapunov constraint can be written as,

$$\dot{V} = -\alpha V, \quad (83)$$

which when enforced ensures that the Lyapunov function V decays exponentially with time along the trajectory of the controlled system and hence that the controlled system is exponentially stable. One can alter the rate of convergence of the system to the origin by altering the decay rate α.

Remark 5 If we denote

$$\tilde{A} = \frac{\partial V}{\partial v} \text{ and } \tilde{b} = b - \frac{\partial V}{\partial q} v, \tag{84}$$

the quantities AB and Af can be expressed as

$$AB = \tilde{A} M^{-1},\ Af = \left[\frac{\partial V}{\partial q}, \tilde{A} \right] \begin{bmatrix} v \\ M^{-1} F \end{bmatrix} = \frac{\partial V}{\partial q} v + \tilde{A} M^{-1} F. \tag{85}$$

Substituting from the above equation in Eq. (18), the control force Q^C is obtained in the form,

$$u = Q^C = \frac{N^{-1} M^{-1} \tilde{A}^{\mathrm{T}}}{\tilde{A} M^{-1} N^{-1} M^{-1} \tilde{A}^{\mathrm{T}}} \left(\tilde{b} - \tilde{A} M^{-1} F \right), \tag{86}$$

which is in the same form as the fundamental equation of mechanics.

Example 2 Let us consider a nonlinear, nonautonomous, mechanical system whose equation of motion is,

$$M\ddot{q} = F := -Kq - K_{nl}(q) + F_t(t). \tag{87}$$

In the above equation, $q = [q_1, q_2, q_3]^{\mathrm{T}} \in R^3$ is the displacement 3-vector, $K_{nl}(q)$ is a nonlinear stiffness term chosen as, $K_{nl}(q) = \left[(q_1 - q_2)^3, (q_2 - q_3)^3, (q_3 - q_1)^3 \right]^{\mathrm{T}}$, and $F_t(t)$ is a time-dependent force term, $F_t(t) = [10 \sin(2t), 25 \cos(8.5t), -15 \sin(5t + 2)]^{\mathrm{T}}$. The diagonal matrix M and the matrix K are given, respectively, as,

$$M = \mathrm{diag}\,(1, 2, 1.5)\,, \text{ and } K = \begin{bmatrix} 100 & -100 & 0 \\ -100 & 150 & -50 \\ 0 & -50 & 100 \end{bmatrix}. \tag{88}$$

The equation of motion of the controlled system is

$$M\ddot{q} = F + Q^c = -Kq - K_{nl}(q) + F_t(t) + Q^C(q, \dot{q}, t). \tag{89}$$

By denoting the generalized velocity vector as, $v = \dot{q}$, we can put the equation of motion of this controlled mechanical system in the generalized dynamical system form given in Eq. (1) using the definitions,

$$x = \begin{bmatrix} q \\ v \end{bmatrix},\ f = \begin{bmatrix} v \\ M^{-1} F \end{bmatrix},\ B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \text{ and, } u = Q^C. \tag{90}$$

Let us assume the control cost to be minimized at each instant of time is the Gaussian defined as,

$$J(t) = Q^{C^{\mathrm{T}}} M^{-1} Q^C. \tag{91}$$

Thus, the weighting matrix N in Eq. (6) is chosen to be $N = M^{-1}$. The explicit control force Q^C that minimizes the Gaussian at each instant of time and enforces the Lyapunov constraint $\dot{V} = -w$ is given as,

$$Q^C = u = \frac{\tilde{A}^{\mathrm{T}}}{\tilde{A} M^{-1} \tilde{A}^{\mathrm{T}}} \left(\tilde{b} - \tilde{A} M^{-1} F \right). \tag{92}$$

The above equation is obtained by substituting $N = M^{-1}$ in Eq. (86). For simplicity, the radially unbounded positive definite pair $(V, \alpha V)$ shown in Eq. (81) is used with $a_1 = 1, a_2 = 8, a_{12} = 1$, and $\alpha = 1/4$. These parameters satisfy the conditions given in relation (82), thus making the chosen pair consistent. We then have

$$\tilde{A} = \frac{\partial V}{\partial v} = a_{12} q^{\mathrm{T}} + a_2 v^{\mathrm{T}},$$
$$\tilde{b} = b - \frac{\partial V}{\partial q} v = -\alpha V - a_1 q^{\mathrm{T}} v - a_{12} v^{\mathrm{T}} v. \tag{93}$$

For the simulation, the initial conditions are, $q\,(0) = [1, -2, 1]^{\mathrm{T}}, v\,(0) = [-2, 3, 0]^{\mathrm{T}}$. The equation of motion of the controlled system given in Eq. (89) is integrated in the MATLAB environment, using the ODE15s package, with a relative error tolerance of 10^{-8} and an absolute error tolerance of 10^{-12}. The results of the numerical simulations are presented in Figs. 9, 10, and 11.

Figure 9a shows the time history of the displacement response of the controlled system for the first 40 seconds, showing asymptotic convergence to zero. Figure 9b shows the projection of the phase portrait of the controlled system on the $q_1 - v_1$ plane. For brevity, we do not show the other phase plots. The initial position is indicated by a square marker. The position at the end of 40 seconds of integration is shown by a circular marker. The plot shows the asymptotic convergence of the system to the origin.

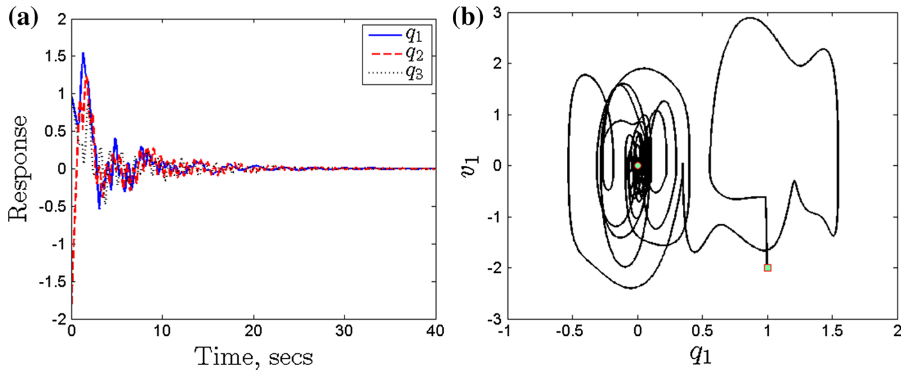

Fig. 9 **a** Displacement history of the controlled system. **b** Projection of phase portrait on the $q_1 - v_1$ plane

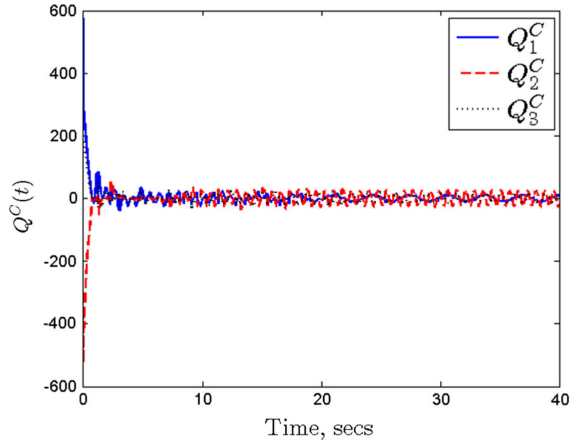

Fig. 10 Control force Q^C calculated using Eq. (92)

The control forces on the three masses given by Eq. (92) are shown in Fig. 10. These control forces minimize the control cost given by Eq. (91) at each instant of time while ensuring the stability of the system.

Figure 11a shows the variation of the Lyapunov function V with time. We see the expected exponential decay in the value of V, because of the constraint given in Eq. (83). Figure 11b shows the error

$$e(t) := \dot{V} + \alpha V \tag{94}$$

in the satisfaction of this constraint. We observe that this error is of the same order of magnitude as the error tolerance (10^{-12}) with which the numerical integration of the equations of motion of the controlled system is carried out.

Remark 6 Since Gauss's Principle requires that the generalized control force always minimize the control cost $J(t)$ given in Eq. (91), the control force obtained in Eq. (92) would be exactly the one nature would use with the pair $(V, \alpha V)$, with the function V given in Eq. (81) along with the chosen parameter values.

5 Conclusions

This paper provides a new method for obtaining sets of stable, optimal controllers for nonlinear, nonautonomous dynamical systems. A set of nonlinear controllers are obtained in closed form using a candidate Lyapunov function V and a positive definite function w, so that: (i) the controller minimizes at *each* instant

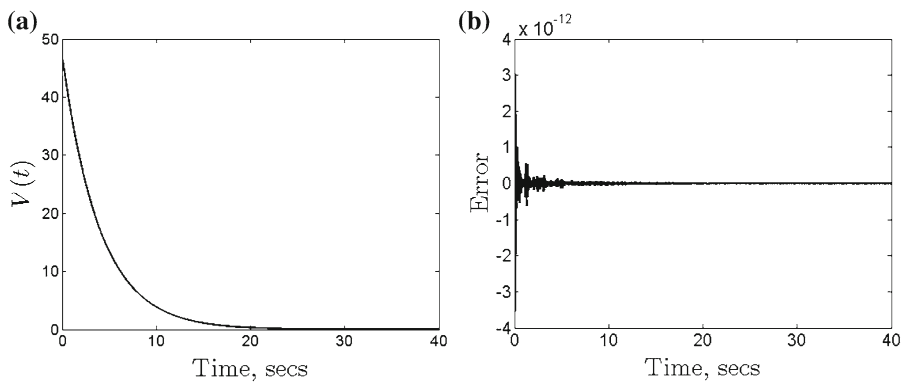

Fig. 11 **a** Variation of the Lyapunov function V with time. **b** Error in satisfying the constraint $e(t)$ given in Eq. (94)

of time a user-prescribed quadratic control cost, and (ii) the candidate Lyapunov function is indeed the Lyapunov function for the controlled system, thus ensuring stability. Depending on the Lyapunov function V chosen, the positive definite function w used—that is, the (V, w) pair chosen—the weighting matrix N desired by the user, and the specific parameter values involved in these three entities, one can obtain, in closed form, sets of stable controllers for a given dynamical system. These choices are governed by the fact that Lyapunov's stability criterion is interpreted as a constraint, and this constraint needs to be consistently satisfied at all instants of time. Conditions that guarantee consistency are obtained. A class of (V, w) pairs has been provided for nonlinear nonautonomous mechanical systems that guarantee consistency. For a linear system, a particular (V, w) pair is shown to yield the result obtained from conventional LQR control theory, thereby illustrating the broad scope and generality of the method. Numerical examples have been provided that illustrate the importance of the consistency condition and demonstrate the efficacy of the approach. The approach appears to be mathematically simple, yet effective. It does not invoke any concepts from linear control theory or variational calculus, and it does not involve any approximations/linearizations regarding the nonlinear, nonautonomous dynamical system.

Conflict of interest None.

Informed Consent Consent to submit has been received explicitly from all co-authors. The research did not involve any human participants.

Research involving Human Participants and/or Animals The research carried out in this article did not involve any human participants or animals.

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