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ON THE OPTIMAL DESIGN OF STRUCTURES SUBJECTED TO PERIODIC BASE EXCITATIONS*

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Abstract. This paper considers the optimal apportionment of the stiffness of a building structure modeled as an undamped shear beam when subjected to a periodic base excitation of frequency ω. A suitable "cost" associated with the stiffness distribution is minimized subject to constraints on the lowest natural vibration frequency of the system, the base shear generated by the periodic excitations, and the given lower-bound stiffness distribution. Closed form solutions for such constrained optimization problems are generally very difficult to obtain; numerical techniques suffer from a host of problems. The paper uses Farkas's Theorem to investigate the underlying structure of the optimization problem and obtains closed-form solutions in several cases of engineering importance.

Key words. optimal design, continuous system, stiffness distribution, frequency and base shear constraints, closed-form solutions, Farkas's theorem

AMS(MOS) subject classification. 49

Introduction. In the field of earthquake engineering we often model tall building structures by one-dimensional shear of bending beams. Such beams are assumed to be fixed at one end (ground level) and free at the other. The effect of strong earthquake ground shaking is then modeled by a suitable base excitation of the system. In most structural systems, the mass distribution of the structure with height is reasonably well prescribed by the building codes and depends on the use to which the structure will be put. The structural designer is then left with the problem of apportioning the stiffness of the structure as a function of height, so that the system is safe, not only under the static gravity loading created by the assumed mass distribution throughout the structure, but also under the dynamic loads induced by the base excitations from earthquake ground shaking that the structure is likely to experience during its useful life. In the design for earthquake safety, a critical parameter that the analyst must consider is the shear force induced at the base of the structure by the ground shaking. Most building codes around the world require this base shear to be calculated and within safe limits for any particular structural configuration on which the designer settles.

In this paper we broach this problem of optimally distributing the stiffness (with height) of a structure so that the base shear generated is less than some fixed value F_0. We assume that the base motion is harmonic with frequency ω and amplitude A_0, and that the system is required to have its lowest fundamental frequency ω_0, to be greater than the base excitation frequency ω. This prevents resonance. In addition, we take the stiffness distribution to be constrained from below, by the continuous function $k^0(x)$. This function is presumably obtained by considering the safety requirements pertinent to the structure under the static loading condition.

The optimal design of structures subjected to harmonic excitations was first studied by Icerman [1] when he considered a "response constraint" in the form of the virtual work of the load amplitude on the displacement amplitude at its point of application.

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Plaut [2] and Huang [3] extended the minimum-weight design to structures subjected to periodic excitations by using constraints on the prescribed deflections at a specified point of the structure. Johnson and Rizzi [4] studied a cantilever bar excited by a harmonic tip force and subject to a maximum allowable stress constraint. They discretized the bar into finite elements, and used variational methods to obtain numerical solutions to their problem. They show that the global minimum of the objective function may be difficult to obtain. In a previous study, Johnson [5] used a displacement constraint to obtain the optimal weight design, again numerically. Recently, Ivanova [6] has used the variational method with constraints on the stiffness to obtain the gradient of the objective function with respect to the weight distribution. This method, which is most commonly used in such problems, leads to a nonlinear programming problem that needs to be worked out numerically. Here, we study a different variety of optimium design problems where the mass distribution is given a priori, and a suitably defined "cost" associated with the stiffness distribution must be minimized. Constraints are imposed on the values of the base shear, the natural vibration frequency of the system, and on the stiffness distribution of the system. Such optimization problems, as indicated above, have been attacked in the past by the use of variational calculus, which in turn transforms the problem to one involving nonlinear programming. Numerical results are then obtained by one of several standard computational procedures. Often the nature of the constraints make the problem notoriously difficult to solve numerically and lead to numerical results which may, at best, be uncertain. Comparing the characteristics of the problem addressed in this paper and our method of approach with some of the work done in the past, we note the following.

(1) The number of constraints that we deal with here are more than those considered by, for example, Ivanova. We obtain closed-form solutions rather than expressions for the gradients which then require numerical computations, in general.

(2) The variational calculus method is difficult to implement when there are frequency constraints. Even if it were to be implemented in an approximate manner, the resulting equations for the adjoint variables, in our problem, would lead to a nonconstant coefficient, nonlinear differential boundary value problem. At each step in the optimization process, these adjoint variables would have to be numerically solved for, to obtain the gradient of the objective function. This gradient would then be used in the numerical minimization scheme. The stiffness distribution would need to be discretized. The handling of the nonlinear constraints would lead to a difficult numerical optimization problem, whose results may leave us yet unsatisfied because of the vulnerability of the computational method to local minima. Besides, such numerical procedures seldom provide insight (unless we have large computing budgets) into the basic structure of the optimization problem.

(3) We expose the structure of the optimization problem in this paper through the use of the Farkas Theorem, which yields closed-form solutions for the global optimum under most situations of engineering interest. Consequently, no numerical schemes are involved, and we do not discretize the stiffness distribution to obtain the minimum cost. The search is carried out in function space. The method does not require us to solve either the nonconstant coefficient differential boundary value problem characterizing the dynamic response of the system, or the nonlinear adjoint equation.

No doubt the approximation of assuming the base excitation as harmonic may be unrealistic in the seismic environment. We use it here, following common engineering practice, realizing that the ground excitations are perhaps best modeled by a nonstationary stochastic process [7]. Though the problem statement in this paper has been

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motivated by an application from the field of aseismic design of structures, it is equally applicable to the design of cantilever beams subjected to harmonic base excitations. The results of this work will find wide application in the area of mechanical, nuclear, and aerospace engineering. An area of particular interest may be the design of machine foundations.

1. Problem definition. Consider a structure whose relative response $u(x, t)$ is modeled by the following differential problem:

$$\begin{aligned} &\rho(x)u_{tt} = (k(x)u_x)_x - \rho(x)\ddot{u}_g(t), \quad x \in (0,1), \quad -\infty < t < \infty, \\ &u(0,t) = 0, \\ &k(x)u_x(x,t)|_{x=0} = 0 \end{aligned} \tag{1.1}$$

with

$$u_g(t) = A_0 \cos \omega t, \quad -\infty < t < \infty, \quad A_0 > 0, \quad \omega > 0.$$

The end $x = 0$ (is assumed fixed while the end $x = 1$, is assumed stress-free. The positive function $k(x)$ represents the distribution of the shear stiffness along the height of the structure, and $\ddot{u}_g(t)$ represents the periodic base excitation (see Fig. 1).

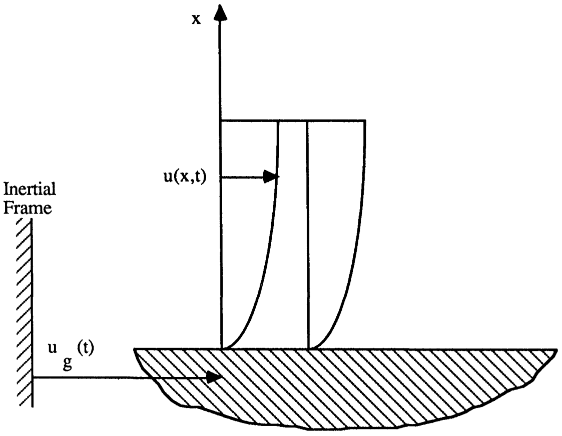

FIG. 1

Given the function $\rho(x) \in C[0,1]$ with $\rho(x) \geqq \rho_0 > 0$, our aim is to find a function $k(x) \in C[0,1]$ such that the cost of penalty defined by

$$J(k) = \int_0^1 g^2(x)k(x)\,dx \tag{1.2}$$

is minimized, subject to the following three constraints:

(1.3a) The lowest natural frequency of the system, ω_0, which depends on the choice of $k(x)$, is greater than ω, the given forcing frequency. This condition prevents the system from undergoing resonance.

(1.3b) The base shear induced by the excitation, $u_g(t)$, is less than or equal to a fixed value, F_0. This constraint is relevant to design procedures followed in assessing the safety of the soil-structural system.

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(1.3c) $k(x) \geqq k^0(x) \geqq \varepsilon > 0$, where $k^0(x)$ is a prescribed function belonging to $C^1[0,1]$ and ε is a positive number. This lower bound on the stiffness distribution comes about from static design conditions.

The function $g^2(x)$ is a weighting function and $J(k)$ provides the cost of choosing a given stiffness distribution $k(x)$. The function $g^2(x)$ is often a monotone increasing function; it is increasingly more expensive, from a construction standpoint, to add stiffness at higher and higher levels in a structure. The condition that the end, $x=1$, is stressfree would imply that there is no need to provide stiffness at that location, and therefore it appears appropriate not to penalize the stiffness at $x=1$. We therefore define the positive function $g(x) \in C[0,1]$ such that $g^2(x) > 0$, $x \in [0,1)$ and $g(1) = 0$.

The continuity requirements on $\rho(x)$ and $k(x)$ are imposed on physical grounds and could be relaxed.

It would be worthwhile at this point to express the constraints (1.3) in a more formal fashion. Let $\hat{H}$ be a Sobolev space of the form $\hat{H} := \{v \in H^1(0,1)\colon v(0) = 0\}$. Then the lowest natural frequency can be expressed as

$$\omega_0^2(k) = \operatorname*{Min}_{z \in H} \left[\frac{(kz_x, z_x)}{(\rho z, z)} \right], \tag{1.4}$$

and constraint (1.3a) can be written as

$$\omega_0^2(k) > \omega^2. \tag{1.3$'$a}$$

We are thus restricted to considering those designs $k(x)$ which satisfy (1.3′a).

The shear force induced at the base by the base excitation, for a given design $k(x)$ is given by Newton's Law as

$$L(t) = -\int_0^1 \rho(u_{tt} + \ddot{u}_g)\, dx$$

where $u(x,t)$ is the solution of (1.1) corresponding to the chosen function $k(x)$. For the steady-state response of the system, defined by

$$u(x,t) = U(x) \cos \omega t, \tag{1.5a}$$

and the steady-state base shear force, defined by

$$L(t) = F \cos \omega t, \tag{1.5b}$$

we get

$$F = \omega^2(\rho, U) + M_t \omega^2 A_0 \tag{1.6}$$

where $M_t \triangleq \int_0^1 \rho(x)\, dx$, is the total mass of the structure and $(\rho, U) \triangleq \int_0^1 \rho U\, dx$.

If the constraint (1.3b) is to be satisfied, we are further restricted to those functions $k(x)$ for which

$$|F(U(k))| \leqq F_0 \tag{1.3$'$b}$$

for some given fixed value of F_0. We note that the shear force F depends on the solution $U(x)$ of (1.1), which in turn depends on the choice of $k(x)$. From here on we shall denote the dependence of $U(x)$ on the choice of $k(x)$ by $U(k)$. Let us introduce the set $\mathcal{K}$ of functions $k(x) \geqq k^0(x) \in C^1(0,1)$ such that the constraints (1.3′a) and (1.3′b) are satisfied by each element of the set. The optimization problem may now be stated as follows:

Find $\hat{k}(x) \in \mathcal{K}$ such that

$$J(\hat{k}) = \min_{k \in \mathcal{K}} J(k). \tag{1.7}$$

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We next provide three useful results pertinent to the steady-state response of the shear beam system. In the interest of brevity the results have been provided without proof. They can be derived using [8]. Physical interpretation of the results will be provided as we go along.

LEMMA 1. *Consider the shear beam problem represented by the differential problem* (1.1). *Let* $k(x)$ *be such that the lowest fundamental natural vibration frequency of the system* ω_0 *is greater than the given frequency of the base excitation* ω. *Then the steady-state response* $U(x)$ *defined by relation* (1.5a) *is such that* $U_x(x) \neq 0$ *for* $x \in [0, 1)$.

Lemma 1 implies the following: Consider a shear beam which is subjected to a base excitation whose frequency, ω, is less than the lowest natural frequency of vibration of the beam. Then there is no point in the interior of the beam at which the stresses, induced by the steady-state vibratory response of the beam to this base excitation, are zero.

LEMMA 2. *Consider the steady-state response of the system described by* (1.1). *Then as long as the forcing frequency* ω *is less than* ω_0, *where* ω_0 *is the lowest natural frequency of the system, the inner product* (ρ, U) *is always positive.*

This result implies that the base shear force F defined in relation (1.6) is always positive. We note that the shear force induced at the base is a consequence of (a) the rigid body motion induced in the entire beam, given by the term $M_t A_0 \omega^2$; and (b) the vibratory response of the beam, given by $\omega^2(\rho, U)$. We have thus shown that as long as the base excitation frequency ω is less than the lowest fundamental vibration frequency of the system, these two contributions are always in phase and augment each other. Further, the base shear is always in phase with the base displacement.

Thus if condition (1.3′a) is satisfied, then condition (1.3′b) can be restated as

$$|F(U(k))| = F(U(k)) = \omega^2 M_t A_0 + \omega^2(\rho, U) \leqq F_0. \tag{1.8}$$

LEMMA 3. *Consider the system represented by* (1.1). *If the forcing frequency of the base excitation* $\omega < \omega_0$, *then the steady-state response function* $U(x)$ *is a strictly positive monotone increasing function in* (0, 1).

2. Inequality constraints.

LEMMA 4. *Let* $k(x)$ *be a candidate design which satisfies condition* (1.3′a). *Let* $u(x, t) = U(x) \cos \omega t$ *be the solution of* (1.1) *for this candidate design. Then for any function* $W(x) \in \hat{H}(0, 1)$, *we have*

$$(kW_x, W_x) - \omega^2(\rho W, W) - 2A_0\omega^2(\rho, W) + A_0\omega^2(\rho, U) \geqq 0. \tag{2.1}$$

Moreover, the equality holds for $W(x) = U(x)$.

Proof. Since $u = U(x) \cos \omega t$ is a solution of (1.1) we have

$$(k(x)U_x(x))_x + (A_0 + U)\rho\omega^2 = 0, \quad U(0) = 0, \quad U_x(1) = 0. \tag{2.2}$$

Then $U(x) \in \hat{H}(0, 1)$. Let $W = U + V$. Inserting $U = W - V$ in (2.2) we get

$$(k(x)W_x(x))_x + (A_0 + W)\rho\omega^2 = (k(x)V_x(x))_x + \omega^2 \rho V. \tag{2.3}$$

Multiplying by $W(x)$ in the $L_2(0, 1)$ sense and integrating by parts, we get

$$(k(x)W_x, W_x) - \omega^2(\rho W, W) - A_0\omega^2(\rho, W) = (kV_x, W_x) - \omega^2(\rho V, W). \tag{2.4}$$

Denoting the right-hand side of (2.4) by P we can restate it, using $W = U + V$, as

$$P = P_1 + P_2$$

where

$$P_1 = (kV_x, V_x) - \omega^2(\rho V, V),$$

$$P_2 = (kV_x, U_x) - \omega^2(\rho V, U).$$

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Noting that $V(x)\in \hat{H}(0,1)$, $P_1\geqq 0$ because of (1.3′a) and (1.4). Also taking the inner product of (2.2) with $V(x)$ we get

$$(kU_x, V_x)-\omega^2(\rho U, V)=A_0\omega^2(\rho, V). \tag{2.5}$$

Thus, $P_2=A_0\omega^2(\rho, V)=A_0\omega^2(\rho, W)-A_0\omega^2(\rho, U)$. Relation (2.4) now yields the result. The equality, which holds for $W(x)=U(x)$, is obvious when (2.2) is multiplied by $U(x)$ in $L_2(0,1)$.

LEMMA 5. *Let $k(x)$ be such that it satisfies conditions* (1.3′a) *and* (1.8), *i.e.,*

$$\omega_0^2(k)>\omega^2$$

and

$$F(U(k))\leqq F_0.$$

Then for any $W\in \hat{H}(0,1)$,

$$(kW_x, W_x)-\omega^2(\rho W, W)-2A_0\omega^2(\rho, W)+A_0F_0-M_t\omega^2A_0^2\geqq 0. \tag{2.6}$$

Proof. When we use (1.6) and (1.8) in (2.1), the result follows.

LEMMA 6. *If* (2.6) *is satisfied for all* $W\in \hat{H}(0,1)$ *with* $F(k)\leqq F_0$, *then* $k(x)$ *is such that*

$$\omega_0^2\geqq \omega^2-\Delta \tag{2.7}$$

where

$$\Delta=\frac{A_0[F_0-F(k)]}{(\rho V^0, V^0)} \tag{2.8}$$

where V^0 is the eigenfunction corresponding to the lowest eigenvalue ω_0^2 of

$$\begin{aligned}&(k(x)V_x(x))_x+\lambda^2\rho(x)V(x)=0, \qquad x\in(0,1),\\ &V(0)=V_x(1)=0.\end{aligned} \tag{2.9}$$

Proof. Let $W=U+V$ where $U(x)$ is the solution of

$$\begin{aligned}&(k(x)U_x(x))_x+(A_0+U(x))\rho\omega^2=0, \qquad x\in(0,1),\\ &U(0)=0, \qquad U_x(1)=0\end{aligned}$$

for the chosen $k(x)$. Then by (2.6), and using (1.6), we get

$$\begin{aligned}&(k(U_x+V_x), U_x+V_x)-\omega^2(\rho U+\rho V, U+V)-2A_0\omega^2(\rho, U+V)\\ &\qquad +A_0\omega^2(\rho, U)+A_0[F_0-F(k)]\geqq 0.\end{aligned} \tag{2.10}$$

Since U is the solution of (1.12) we have

$$(kU_x, U_x+V_x)-A_0(\rho, U+V)\omega^2-\omega^2(\rho U, U+V)=0.$$

Using this in (2.10) and again noting that U is a solution of (2.2) we get

$$(kV_x, V_x)-\omega^2(\rho V, V)\geqq -A_0[F_0-F(k)].$$

Thus

$$\frac{(kV_xV_x)}{(\rho V, V)}\geqq \omega^2-\frac{A_0[F_0-F(k)]}{(\rho V, V)} \quad \text{for all } V\in \hat{H}(0,1).$$

Choosing V^0 to be the eigenfunction corresponding to $\lambda=\omega_0$ for (2.9) we get

$$\omega_0^2\geqq \omega^2-\Delta.$$

COROLLARY 1. *If* (2.6) *is satisfied for all* $W \in \hat{H}(0,1)$ *with* $F(k)=F_0$, *then* $k(x)$ *must be such that* $\omega_0^2 \geqq \omega^2$.

Proof. Since $k(x)$ is such that $F_0 = F(k)$, $\Delta = 0$.

THEOREM 1. *Let* $k(x)$ *belong to the set* $\mathcal{K}$. *If an optimal design* $\hat{k}(x)$ *exists such that it minimizes* $J(k)$ *as defined in* (1.7), *then* $\hat{k}(x)$ *must satisfy the following relations*:

$$(\hat{U}_x, k_1(x)\hat{U}_x) + A_0\alpha(\hat{k}) \geqq 0, \tag{2.11a}$$

$$k_1 + \hat{k}(x) - k^0(x) \geqq 0, \tag{2.11b}$$

$$(g^2(x), k_1(x)) \geqq 0 \tag{2.11c}$$

where $k_1(x) = k(x) - \hat{k}(x)$, $\hat{U}(x)$ *is the solution of* (2.2) *corresponding to* $\hat{k}(x)$, *and* $\alpha(k) \triangleq F_0 - F(\hat{k}) \geqq 0$.

Proof. The second and third inequalities follow directly from relation (1.2) and condition (1.3c). We prove the first inequality as follows: Since $\hat{U}(x) \in \hat{H}(0,1)$, let $W(x) = \hat{U}(x)$ in (2.6). Thus

$$(k\hat{U}_x, \hat{U}_x) - \omega^2(\rho\hat{U}, \hat{U}) - 2A_0\omega^2(\rho, \hat{U}) + A_0F_0 - M_t\omega^2A_0^2 \geqq 0. \tag{2.12}$$

Since $\hat{U}(x)$ is the solution of (2.2) for $k(x) = \hat{k}(x)$, by taking the inner product of (2.2) with $\hat{U}(x)$, we get

$$(\hat{k}\hat{U}_x, \hat{U}_x) - \omega^2(\rho\hat{U}, \hat{U}) - 2A_0\omega^2(\rho, \hat{U}) + A_0F(\hat{k}) - M_t\omega^2A_0^2 = 0. \tag{2.13}$$

When we subtract (2.13) from (2.12) the result follows.

In order to find the optimal stiffness distribution $\hat{k}(x)$, we shall use the Farkas Theorem (see [9]). We begin by establishing the following nomenclature. In what follows we shall assume that an element $\hat{k}(x) \in \mathcal{K}$ exists such that $J(\hat{k})$ is a minimum.

3. Nomenclature and the Farkas Theorem.

Let V_1 and V_2 be two Hilbert spaces and let V_1^* and V_2^* be their corresponding duals. For $x \in V_i$, $f \in V_i^*$, let $f(x) = \langle f|x\rangle_i$ denote the duality pairing bilinear form on $V_i^* \times V_i$, $i = 1, 2$. Let A be a linear bounded operator from V_1 to V_2, and denote by A^* its transpose. Then $\langle A^*f_2|x_1\rangle_1 = \langle f_2|Ax_1\rangle_2$ for any $f_2 \in V_2^*$ and any $x_1 \in V_1$.

Let M be a cone in V_1. We define the positive polar cone of M as

$$M^+ = \{f \in V^*: \langle f|x_1\rangle_1 \geqq 0 \text{ for all } x \in M\}. \tag{3.1}$$

Note that in [9] the set M^+ is denoted $-M^-$. Using these definitions, the Farkas Theorem states the following [9].

Let V_1 and V_2 be Hilbert spaces and let A be a bounded linear operator from V_1 to V_2. Let P be a closed convex cone in V_1. Then, the following two statements are equivalent:

$$\text{For all } f_2 \in V_2^*, \quad A^*f_2 \in P^+ \text{ implies } \langle f_2|x_2\rangle_2 \geqq 0, \tag{3.2a}$$

$$\exists x_1 \in P \quad \text{such that } Ax_1 = x_2 \text{ for given } x_2 \in V_2. \tag{3.2b}$$

Figure 2 presents the "geometry of the Farkas Theorem." We note that the closed convex cone P is in the space V_1 and thus its positive polar cone P^+ belongs to the space V_1^*. $A(P)$ is a cone in the space V_2. If we can show that $A(P)$ is a closed set in V_2, then the Farkas Theorem states that (3.2a) and (3.2b) are equivalent assertions.

This means that an element x_2 in V_2 is in $A(P)$, i.e., x_2 is in the range of the restriction of A to P, if and only if, for arbitrary bounded linear functionals f_2 acting on V_2, we have $f_2(x_2) = \langle f_2|x_2\rangle_2 \geqq 0$ whenever the functional A^*f_2 is in P^+ of V_1^*. In turn, A^*f_2 is in P^+ when the image of P under A^*f_2 is a subset of $[0, \infty)$.

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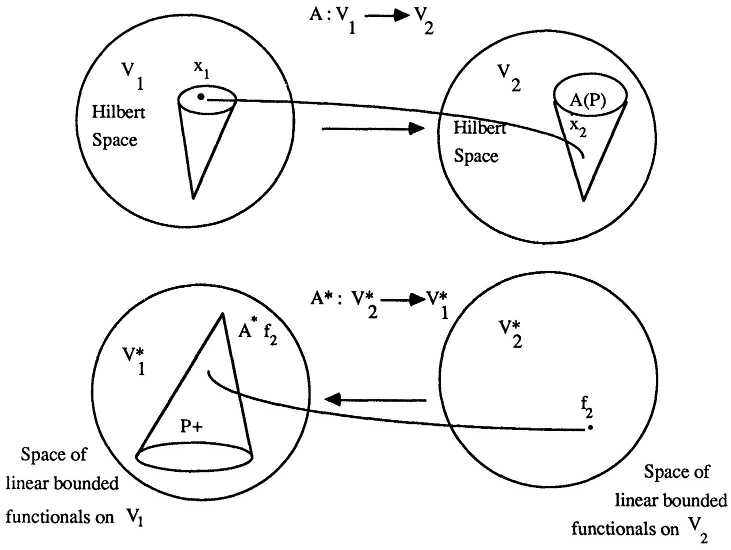

P is a closed Convex Cone in V_1; A(P) is a closed cone in V_2

$(\forall f_2 \varepsilon V_2^*)$, $(A^* f_2 \varepsilon P+ \Rightarrow f_2(x_2) \geq 0) \Longleftrightarrow (\exists x_1 \varepsilon P \ni Ax_1 = x_2$ for given $x_2 \varepsilon V_2)$

FIG. 2. *Geometry of Farkas' theorem.*

In order to use the theorem we shall need to define the spaces V_1 and V_2 appropriately, and also the operator A. We specify a set P in V_1 and an element $x_2 \in V_2$. We then prove several assertions, stating the following: P is a closed convex cone in V_1 (Lemma 7); A is a bounded linear operator (Lemma 8); $A(P)$ is a closed cone in V_2 (Lemma 9); $A^* f_2 \in P^+ \subseteq V_1^*$ (Lemma 10); $f_2(x_2) \geqq 0$ (Lemma 11); and finally we apply Farkas's result (in Theorem 2).

DEFINITIONS OF V_1 AND V_2. Let $V_1 = R \times H^1(0, 1)$ and define the inner product and linear space operations for $x_1 = (\nu_0, \nu(x))$, $y_1 = (\mu_0, \mu(x))$ in V_1 by

$$\begin{aligned} &x_1 + y_1 = (\nu_0 + \mu_0, \nu(x) + \mu(x)), \\ (3.3) \qquad &\alpha x_1 = (\alpha\nu_0, \alpha\nu(x)), \\ &\langle x_1, y_1 \rangle_1 = \nu_0\mu_0 + \int_0^1 \nu(x)\mu(x)\, dx + \int_0^1 \nu'(x)\mu'(x)\, dx. \end{aligned}$$

It is clear that V_1 is a Hilbert space. Let V_2 be the Hilbert space $L_2(0, 2)$.

DEFINITION OF P. Let

$$(3.4) \qquad P = \{x_1 = (\nu_0, \nu(x)) \in V_1 \colon \nu_0 \geqq 0 \text{ and } \forall x \in (0, 1), \nu(x) \geqq 0\}.$$

LEMMA 7. *P is a closed convex cone in V_1.*

Proof. The proof is straightforward and is therefore omitted.

FUNCTION AND OPERATOR DEFINITION. Consider the functions $k_1(x)$, $\hat{U}_x^2(x)$, and $g^2(x) \in C[0, 1)$. We extend these functions as follows:

$$(3.5) \qquad K_1(x) = \begin{cases} k_1(x), & 0 \leqq x < 1, \\ 1, & 1 \leqq x \leqq 2, \end{cases}$$

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$$\mathcal{U}_x^2(x)=\begin{cases}\hat{U}_x^2(x), & 0\leqq x<1,\\ A_0\alpha(\hat{k}), & 1\leqq x\leqq 2,\end{cases}\tag{3.6}$$

$$G(x)=\begin{cases}g^2(x), & 0\leqq x<1,\\ 0, & 1\leqq x\leqq 2.\end{cases}\tag{3.7}$$

Define the function $D(x,\xi)$ on the rectangle $[0,2]\times[0,1]$ as

$$D(x,\xi)=\delta(\xi-x)+[\hat{k}(x-1)-k^0(x-1)]H(x-1)\delta(\xi-(x-1))\tag{3.8}$$

where δ is the Dirac delta distribution, and H is the unit-step function defined by

$$H(x-1)=\begin{cases}0, & 0\leqq x<1,\\ 1, & 1\leqq x\leqq 2.\end{cases}\tag{3.9}$$

Let $A: V_1\rightarrow V_2$ be defined by the relation

$$Ax_1=A(\nu_0,\nu(x))=\mathcal{U}_x^2(x)\cdot\nu_0+\int_0^1 D(x,\xi)\nu(\xi)\,d\xi=x_2.\tag{3.10}$$

LEMMA 8. *The operator A defined by* (3.10) *is a bounded linear operator from V_1 to V_2.*

Proof. Using (3.8) we can expand (3.10) to read

$$Ax_1=\begin{cases}\hat{U}_x^2(x)\nu_0+\nu(x), & x\in[0,1),\\ A_0\alpha(\hat{k})\nu_0+\nu(1)+[\hat{k}(0)-k^0(0)]\nu(0), & x=1,\\ A_0\alpha(\hat{k})\nu_0+[\hat{k}(x-1)-k^0(x-1)]\nu(x-1), & x\in(1,2].\end{cases}\tag{3.11}$$

It is clear that $Ax_1\in L_2[0,2]$ and the linearity of A can be easily shown. We show that it is bounded.

Let $x_1=(\nu_0,\nu(x))\in V_1$ such that

$$\|x_1\|_{V_1}^2=\nu_0^2+\int_0^1\nu^2(x)\,dx+\int_0^1\nu'^2(x)\,dx\leqq 1.\tag{3.12}$$

Using (3.11), $\|x_2\|_{V_2}^2=\|Ax_1\|_{L_2[0,2]}^2$ can be written as follows:

$$\begin{aligned}\|x_2\|_{V_2}^2={}&\nu_0^2\int_0^1\{\hat{U}_x^4(x)+[\alpha(\hat{k})A_0]^2\}\,dx\\ &+\int_0^1\{1+[\hat{k}(x)-k^0(x)]^2\}\nu^2(x)\,dx\\ &+2\nu_0\int_0^1[\hat{U}_x^2(x)+\alpha(\hat{k})A_0(\hat{k}(x)-k^0(x))]\nu(x)\,dx.\end{aligned}\tag{3.13}$$

Since $\hat{U}_x^2(x)$, $\hat{k}(x)$, $k^0(x)\in C[0,1)$, $\alpha(\hat{k})A\in\mathbf{R}$, inequality (3.12) implies that $\sup_{\|x_1\|<1}\|Ax_1\|_{V_2}<\infty$ and therefore a is bounded.

LEMMA 9. *If P is the closed convex cone defined by* (3.4) *in V_1, then $A(P)$ is a closed cone in V_2.*

Proof. If $x_2\in A(P)\subseteq V_2$, then there is an $x_1=(\nu_0,\nu(x))\in P\subseteq V_1$ such that $Ax_1=x_2$. If we note the definition of the operator A, it is obvious that $A(P)$ is a cone. Since A is a continuous operator, $A(P)$ is closed.

Let f_2 be a bounded linear functional on $V_2 = L_2[0, 2]$ whose action on the elements of V_2 can be expressed by the Riesz Representation Theorem as

$$f_2(*) = \langle *, K_1(x)\rangle_{V_2} \tag{3.14}$$

where $K_1(x)$ is defined in (3.5).

LEMMA 10. *Let relations* (2.11a) *and* (2.11b) *be satisfied. Let* $A^*: V_2^* \to V_1^*$ *be the transpose operator of* A, *and let* P^+ *be the positive polar cone of* P. *Then* $A^*f_2 \in P^+ \subseteq V_1^*$, *where* f_2 *is given by* (3.14).

Proof. Let $x_1 = (\nu_0, \nu(x)) \in P$. We have $\nu(\xi) \geqq 0$ for any $\xi \in (0, 1)$. By (3.5) and (3.6), (2.11a) and (2.11b) become, respectively,

$$\int_0^2 K_1(x)\mathcal{U}_x^2(x)\, dx \geqq 0 \tag{3.15}$$

and

$$\int_0^2 D(x, \xi)K_1(x)\, dx \geqq 0 \quad \text{for any } \xi \in [0, 1]. \tag{3.16}$$

Multiplying both sides of (3.16) by $\nu(\xi)$ and integrating from $\xi = 0$ to $\xi = 1$, we get

$$\int_0^2 K_1(x)\left[\int_0^1 D(x, \xi)\nu(\xi)\, d\xi\right] dx \geqq 0.$$

Noting that $\langle A^*f_2|x_1\rangle_1 = \langle f_2|Ax_1\rangle_2 = f_2(Ax_1) = \langle Ax_1, K_1(x)\rangle_{V_2}$ and using the linearity of the inner product, we get

$$\langle A^*f_2|x_1\rangle_1 = \nu_0 \int_0^2 K_1(x)\mathcal{U}_x^2(x)\, dx + \int_0^2 K_1(x)\left[\int_0^1 D(x, \xi)\nu(\xi)\, d\xi\right] dx. \tag{3.17}$$

Using (3.15) and (3.16) and remembering that $x_1 \in P$, so $\nu_0 \geqq 0$, we have $A^*f_2 \in P^+ \subseteq V_1^*$ because $\langle A^*f_2|x_1\rangle_1 \geqq 0$.

LEMMA 11. *The relation* (2.11c) *can be expressed as* $\langle f_2|x_2\rangle_2 \geqq 0$, *where* $x_2 = G(x)$ *is defined in* (3.7).

Proof. Because $f_2 \in V_2^*$,

$$\langle f_2|x_2\rangle_2 = \langle x_2, K_1(x)\rangle_{V_2} = \langle G(x), K_1(x)\rangle_{V_2} = \int_0^1 k_1(x)g^2(x)\, dx. \tag{3.18}$$

Hence we have the result.

4. The optimal design $\hat{k}(x)$.

THEOREM 2. *If relations* (2.11a) *and* (2.11b) *imply relation* (2.11c), *then*

$$\nu_0\hat{U}_x^2(x) + \nu(x) = g^2(x), \qquad x \in [0, 1) \tag{4.1}$$

and

$$\nu_0A_0\alpha(\hat{k}) + \nu(x)[\hat{k}(x) - k^0(x)] = 0, \qquad x \in (0, 1]. \tag{4.2}$$

Proof. By Lemmas 7–11, we see that the Farkas Theorem is applicable. Thus there exists an $x_1 \in P$ such that

$$Ax_1 = x_2$$

where $x_2 = G(x)$, the operator A is defined in relation (3.11), and the equality holds in the L_2 sense. Noting that $\nu(x)$, $\hat{k}(x)$, $k^0(x)$, and $\hat{U}_x(x)$ are continuous in [0, 1], the result follows pointwise in the respective regions.

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THEOREM 3. *When the optimal design is such that the base shear constraint is binding, i.e.,* $F_0 = F(\hat{k})$, *then we have the following*:

$$\begin{aligned} &(1)\ \textit{If } \nu_0 = 0,\ \hat{k}(x) = k^0(x),\ x \in [0,1]; \\ &(2)\ \textit{If } \nu_0 > 0, \textit{ and } \hat{k}(x) > k^0(x),\ x \in [0,1], \textit{ then} \end{aligned} \tag{4.3}$$

$$\hat{U}_x^2 = \frac{g^2(x)}{\nu_0}, \qquad x \in [0, 1). \tag{4.4}$$

Proof. (1) If $\nu_0 = 0$, (4.1) gives

$$\nu(x) = g^2(x), \qquad x \in (0, 1).$$

Equation (4.2), for $\alpha(\hat{k}) = 0$, gives

$$\hat{k}(x) = k^0(x), \qquad x \in (0, 1).$$

the result follows from the assumed continuity of $\hat{k}(x)$ and $k^0(x)$ in $[0, 1]$.

(2) When we use (4.2), $\nu(x) = 0$, $x \in (0, 1)$. Then relation (4.1) gives

$$\hat{U}_x^2 = \mathrm{g}^2\frac{(\mathrm{x})}{\nu_0}, \qquad \mathrm{x} \in [0, 1).$$

COROLLARY 2. *If the set* $B = \{x\colon \hat{k}(x) - k^0(x) > 0\}$ *is dense in* $[0, 1]$, *then relation* (4.4) *is valid.*

Proof. For each $x \in B$, by (4.2), $\nu(x) = 0$. But $\nu(x)$ is continuous in $[0, 1]$, and since B is dense in $[0, 1]$, $\nu(x) = 0$ on $[0, 1]$. Hence we have the result.

In particular, if the optimal stiffness $\hat{k}$ coincides with the lower bound k^0 at only a finite number of points, then B is dense in $[0, 1]$ and relation (4.4) holds.

THEOREM 4. *When the shear force constraint is binding and relation* (4.4) *is valid, the optimal stiffness distribution is given by*

$$k(x) = \frac{\int_x^1 \omega^2 \rho(x) \int_0^x g(\alpha)\, d\alpha}{g(x)} + \frac{\sqrt{\nu_0} A_0 \omega^2 \int_x^1 \rho(x)\, dx}{g(x)}, \qquad x \in [0, 1) \tag{4.5}$$

where

$$\sqrt{\nu_0} = \omega^2 \frac{\int_0^1 \rho(y)[\int_0^y g(x)\, dx]\, dy}{F_0 - \omega^2 M_t A_0}.$$

Proof. Using (2.2), we have

$$k(x) = \frac{\int_x^1 \omega^2 \rho(x) U(x)\, dx}{U_x(x)} + \frac{A_0 \omega^2 \int_x^1 \rho(x)\, dx}{U_x(x)}.$$

Since $\alpha(\hat{k}) = 0$, $\hat{F} = F_0 = \omega^2(\rho, U) + \omega^2 M_t A_0$.

But by Lemma 3 and (4.4), $\hat{U}(x) = \int_0^x (g(x)/\sqrt{\nu_0})\, dx$. Using this in the expression for F, we get the result.

THEOREM 5. *If* $\alpha(\hat{k}) > 0$, *then if an optimal solution exists satisfying the constraints, it must be* $\hat{k}(x) = k^0(x)$ *for* $x \in [0, 1]$.

Proof. If $\alpha(\hat{k}) > 0$ then by (4.2), $\nu_0 = 0$. Then (4.1) yields

$$\nu(x) = g^2(x) > 0,$$

and so, by (4.2),

$$\hat{k}(x) = k^0(x), \qquad x \in [0, 1].$$

From the continuity of $\hat{k}(x)$ and $k^0(x)$ in $[0, 1]$, the result follows.

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It should be noted that when the shear force constraint is binding, conditions (1.2) and (1.3) are equivalent to conditions (2.11). However, when the shear force $F(\hat{k}) < F_0$, while conditions (1.2) and (1.3) imply (2.11), conditions (2.11) do not imply (1.3'a) as shown in Lemma 6. The set, $\mathcal{K}_1$, of elements $k(x)$ that satisfy (2.11) with $\alpha(k) > 0$ is such that $\mathcal{K}_1 \supseteq \mathcal{K}$. Thus we need to check that the solution $\hat{k}(x) = k^0(x)$ obtained in Theorem 5 satisfies condition (1.3'a). If it does not, no solution to our constrained optimization problem exists, since $\mathcal{K}_1 \supseteq \mathcal{K}$.

5. Physical interpretation of the results. The results obtained in Theorems 2–5 can best be described by the flowchart shown in Fig. 3. The novelty of this paper's suggested approach to solving our constrained optimization problem lies in obtaining, in closed form, the optimal stiffness distribution without ever having to solve the nonconstant coefficient partial differential equation in (1.1). The penalty paid for this is that we do not obtain the optimal solution for all possible situations. However, we shall show that, from a practical standpoint, those situations for which we do not obtain the stiffness distributions in closed form, hardly ever occur.

The optimization problem is related to two constraints: (a) the base shear constraint and (b) the stiffness distribution constraint. They are the following:

$$\alpha(\hat{k}) = F_0 - F(\hat{k}) \geqq 0, \tag{5.1}$$

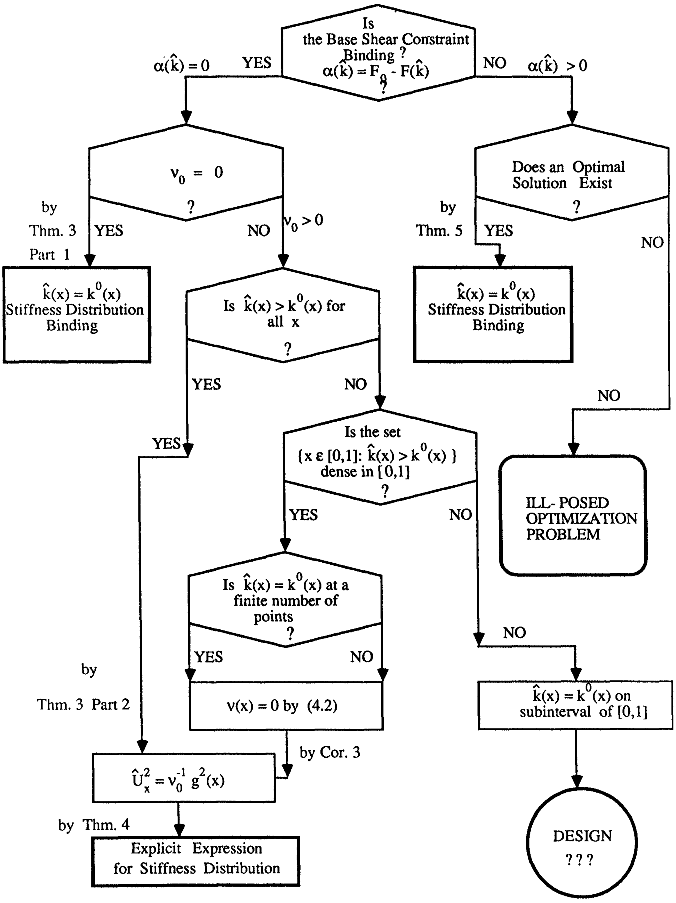

FIG. 3. *Design flowchart.*

$$h(x)=\hat{k}(x)-k^0(x)\geqq 0 \quad \text{for all } x\in[0,1]. \tag{5.2}$$

The solution x_1 of the operator equation $Ax_1=x_2$ is in the cone P of the space V_1. That means the following:

$$\nu_0\geqq 0, \tag{5.3}$$

$$\nu(x)\geqq 0 \quad \text{for all } x\in[0,1). \tag{5.4}$$

Inequalities (5.1) and (5.3) can be satisfied either by equality (base shear binding) or by strict inequality. Relation (5.2) can be satisfied in different ways. We have here a nonnegative continuous function defined on $[0,1]$. Its graph is a subset of the band $[0,1]\times[0,\infty)$ of R^2. We describe the different possibilities in Fig. 4. The different possible cases are as follows:

$$h(x)\equiv 0 \quad \text{on } [0,1], \tag{5.2a}$$

$$h(x)>0 \quad \text{on } [0,1], \tag{5.2b}$$

$$h(x)>0 \quad \text{on a set dense in } [0,1], \tag{5.2c}$$

$$h(x)=0 \quad \text{on a proper subinterval of } [0,1]. \tag{5.2d}$$

Case (5.2c) includes the possibility that the equation $h(x)=0$ has finitely many solutions in $[0,1]$.

The flowchart shows that the only case in which we cannot explicitly obtain $\hat{k}(x)$, although the optimization problem may have a solution, is the case (5.2d). However, from a practical point of view, the probability that the optimal stiffness distribution will coincide with its lower bound on a proper subinterval of $[0,1]$ is extremely small.

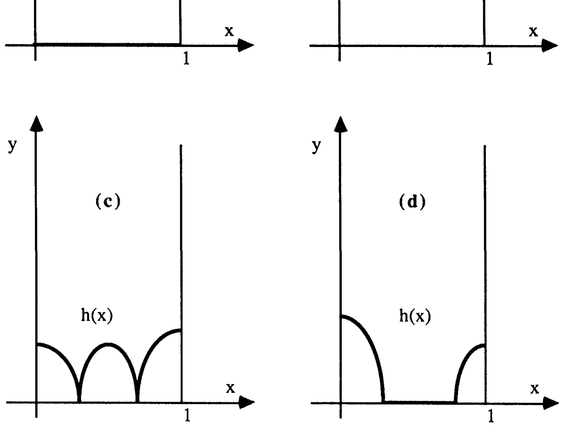

FIG. 4. *Graph of* $h(x)=\hat{k}(x)-k^0(x)$.

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Therefore this case is very unlikely to occur. This, in turn, means that whenever a solution to the optimization problem exists, in most cases we succeed in finding the solution in closed form.

Now we describe Fig. 3 in detail.

Case 1. *Base shear not binding.* In this case we have to check whether the solution to (2.11) satisfies (1.3). If not, then the optimization problem is ill-posed and has no solution. Otherwise, Theorem 5 proves that we are in situation (5.2a), which means that the optimal stiffness distribution is the same as the lower bound $k^0(x)$ for all x.

This describes the right branch of the flowchart. Next we describe the left branch of the flowchart.

Case 2. *Base shear binding.* For the next branching, we ask whether (5.3) is satisfied by equality. If so, then Theorem 3(1) shows that we are again in situation (5.2a). Otherwise, we ask if (5.2b) happens. If so, then by Theorem 3(2) we obtain explicitly $\hat{U}_x^2(x)$, and then Theorem 4 gives us the optimal stiffness distribution in closed form. If (5.2b) is not the case, we ask whether (5.2c) occurs. (Let us emphasize again that, from a practical point of view, the subcase of importance here is the situation where graph $h(x)$ hits the x-axis at a finite number of points). If so, then (5.4) is satisfied by identity; by Corollary 2, we again get $\hat{U}_x^2(x)$ explicitly; and Theorem 4 gives us the optimal stiffness distribution in closed form.

However, if (5.2c) is not the case, but $h(x)$ is equal to zero on a subinterval of $[0, 1]$, then we cannot find $\hat{k}(x)$ in closed form on $[0, 1]$. Thus we only know $\hat{k}(x)$ on the subinterval where it is equal to k^0. As mentioned, this is unlikely to happen in real-life situations.

As a final comment, we observe that whenever the optimization problem has a solution, in most cases, that solution is obtained, in closed form, via the Farkas Theorem, and this does not require the explicit solution of the differential problem (1.1).

6. Numerical example. The results of the previous section can be illustrated for the case when the shear force constraint is taken to be binding with $g(x)=1$, $\rho(x)=1$, $\omega=1$, $F_0=2$, $A_0=1$, and $k^0(x)=0$. We then obtain

$$\nu_0=\frac{1}{4} \quad \text{and} \quad k(x)=1-\frac{x}{2}-\frac{x^2}{2}. \tag{6.1}$$

Using the same parameters with $g(x)=1+x$, we get

$$\nu_0=\frac{4}{9} \quad \text{and} \quad k(x)=\left(\frac{4}{3}-\frac{2}{3}x-\frac{x^2}{2}-\frac{x^3}{6}\right)\Big/(1+x). \tag{6.2}$$

The results (6.1) and (6.2) are indicated in the Fig. 5, along with the result for $g(x)=(1-x/2)$.

7. Conclusions and discussion. This paper attempts to study the structure of the optimal design problem for a building structure subjected to harmonic base excitation. The aim is to find a stiffness distribution which minimizes a suitable cost function subject to constraints on the base shear, the lowest fundamental frequency, and a lower bound function for the stiffness distribution. The Farkas Theorem is used to study the underlying structure of the optimization problem. It is shown that when the base shear constraint is nonbinding, the optimal stiffness, if it exists, is given by its lower bound, $k^0(x)$. The Farkas Theorem is also used to find the optimal stiffness when the shear force constraint is binding and $\hat{k}(x)>k^0(x)$ for all x in $[0, 1]$ except perhaps at a discrete set of points. The closed-form solution of the optimal mode shape $\hat{U}(x)$ is first obtained and then $\hat{k}(x)$. The global optimum is thus analytically determined.

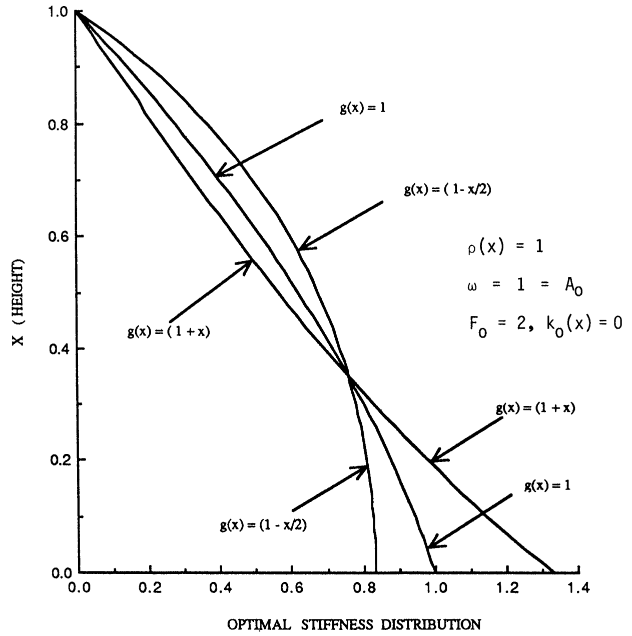

FIG. 5

These analytical solutions become all the more useful when it is realized that computational nonlinear programming methods for obtaining the optimal solution for such problems are fraught with numerical difficulties, even when the number of unknowns (the values of $\hat{k}(x)$ at a discrete set of points) is small.

This paper gives only simplistic modeling of a tall building structure subjected to ground shaking. However, such models are used commonly in the design and analysis of buildings in seismic areas, and therefore the results obtained here are of practical interest to the designer. It is anticipated that the results will find use in other application areas, such as the optimal design of space structures and the design of machine foundations.

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