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Research paper

On generalized inverses of dual matrices

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ABSTRACT

Generalized inverses of real dual matrices are classified according to the set of Moore–Penrose conditions satisfied.

The present paper offers theoretical and numerical insights regarding the types of dual generalized inverses that can be computed from a recently proposed formula. Moreover, the usefulness of the formula is demonstrated solving different kinematic problems. In particular, thanks to the dual matrix generalized inverse formula availability, the computation of infinite and infinitesimal screw parameters motion from redundant point and line features is obtained within a unified theoretical treatment. Numerical examples and comparison with the results from previous investigations are provided.

1. Introduction

The solution of systems of linear dual equations (SLDU) is a task often required in many spatial mechanisms kinematic analysis [3,4,6–8] and synthesis problems [5] or sensor calibration [6]. A common approach is based on the splitting of the system of dual linear equations into two systems of real linear equations. The first system contains the real part and the second the dual part of the system of the SLDU. The solutions of the two systems are then computed separately. The availability of a dual generalized inverse of the coefficient matrix allows the simultaneous computation of the dual solution in a single step.

In this paper some new results about the properties of a dual generalized inverse formula provided in [2] are demonstrated and discussed. The paper is organized in two main parts. The first part is dedicated to analytical proofs whereas the second part presents some numerical tests that confirm the findings. An application of the results to rigid body kinematics is discussed in Section 4.

2. Analytical results

Consider two m by n matrices A and B, and a real scalar ε. Denote its dual by the m by n matrix $\hat{A} = A + \varepsilon B$, ε, $B \neq 0$, and define the matrix [2,3]

$$G = A^{+} - \varepsilon A^{+} B A^{+}, \tag{1}$$

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where A^+ is the n by m Moore–Penrose (MP) inverse of the matrix A [2]. The matrix A^+ is often also called the {1,2,3,4}-*generalized inverse of* A, since it satisfies the four standard Moore–Penrose conditions. The four MP conditions that the n by m generalized inverse X of an m by n matrix A is required to satisfy are ordered as [1]:

$$\begin{aligned}&(1)\quad AXA = A\\ &(2)\quad XAX = X\\ &(3)\quad AX = (AX)^T, \quad \textit{and}\\ &(4)\quad XA = (XA)^T\end{aligned} \tag{2}$$

The real unit ε is subjected to the rules:

$$\varepsilon = 0, \quad 0\varepsilon = \varepsilon 0 = 0, \quad 1\varepsilon = \varepsilon 1 = \varepsilon, \quad \varepsilon^2 = 0$$

In what follows, when we say, for example, that the matrix G is a {*i*,*k*,*j*}-dual generalized inverse of the matrix $\hat{A}$, we mean that the dual generalized inverse G satisfies the *i*th, *j*th, and *k*th, MP conditions *under the* proviso *that* $\varepsilon^2 = 0$.

Result 1. The necessary and sufficient condition for the matrix G given in Eq. (1) to be a {1}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$ for all m by n matrices A and B is

$$(I_m - AA^+)B(I_n - A^+A) = 0 \tag{3}$$

Proof. The condition for G to be a {1}-dual generalized inverse of $\hat{A}$ is

$$\hat{A}G\hat{A}\Big|_{\varepsilon^2=0} = \hat{A}. \tag{4}$$

Using the expressions for $\hat{A}$ and G in the left hand side of Eq. (4), we get

$$\begin{aligned}\hat{A}G\hat{A}\Big|_{\varepsilon^2=0} &= (A + \varepsilon B)(A^+ - \varepsilon A^+BA^+)(A + \varepsilon B)\\ &= (A + \varepsilon B)(A^+A - \varepsilon A^+BA^+A + \varepsilon A^+B)\\ &= AA^+A + \varepsilon(BA^+A - AA^+BA^+A + AA^+B)\\ &= A + \varepsilon(BA^+A - AA^+BA^+A + AA^+B).\end{aligned} \tag{5}$$

The second and third equality result by noting that $\varepsilon^2 = 0$. In the fourth equality we have used the fact that since A^+ is the MP-inverse of the matrix A, it satisfies the first MP condition $AA^+A = A$ (see the first equation of the set in Eq. (2)).

For the expression on the right hand side of Eq. (5) to equal $\hat{A} = A + \varepsilon B$, the necessary and sufficient condition is that

$$B = BA^+A - AA^+BA^+A + AA^+B, \tag{6}$$

which can be rewritten as

$$(I_m - AA^+)B(I_n - A^+A) = 0, \tag{7}$$

as can be simply verified by multiplication of the three terms in Eq. (7). □

Remark 1. Note that Eq. (3) is satisfied if:

i $(I_n - A^+A) = 0$, and or,
ii $(I_m - AA^+) = 0$, and or,
iii $B(I_n - A^+A) = 0$, and or,
iv $(I_m - AA^+)B$, and or $(I_m - AA^+)B(I_n - A^+A) = 0$.

Note that the first four conditions constitute a set of *sufficient* (though not necessary) conditions for the matrix G to be a {1}-dual generalized inverse of $\hat{A}$.

Result 2. The matrix G given in Eq. (1) is a {2}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$ for all m by n matrices A and B.

Proof. We need to show that under the proviso $\varepsilon^2 = 0$, G satisfies the 2nd MP condition (see the second equation in Eq. set (2)), namely,

$$G\hat{A}G\Big|_{\varepsilon^2=0} = G. \tag{8}$$

Using the expressions for $\hat{A}$ and G in the left hand side of Eq. (8) we get

$$\begin{aligned}G\hat{A}G\Big|_{\varepsilon^2=0} &= (A^+ - \varepsilon A^+BA^+)(A + \varepsilon B)(A^+ - \varepsilon A^+BA^+)\\ &= (A^+A - \varepsilon A^+BA^+A + \varepsilon A^+B)(A^+ - \varepsilon A^+BA^+)\\ &= A^+AA^+ - \varepsilon A^+BA^+AA^+ + \varepsilon A^+BA^+ - \varepsilon A^+AA^+BA^+\\ &= A^+ - \varepsilon(A^+BA^+ + A^+BA^+ - A^+BA^+)\\ &= A^+ - \varepsilon A^+BA^+ = G.\end{aligned} \tag{9}$$

In the second and third equality we have used the fact that $\varepsilon^2 = 0$, and in the fourth equality the fact that since A^+ is the MP-inverse of the matrix A, it satisfies the second MP condition $A^+AA^+ = A^+$ (see the second equation of the set in Eq. (2)). □

Remark 2. Result 2 is valid for *all* dual matrices $\hat{A} = A + \varepsilon B$.

Result 3. The necessary and sufficient condition for the matrix G given in Eq. (1) to be a {3}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$ for all m by n matrices A and B is

$$(I_m - AA^+)BA^+ = (A^T)^+B^T(I_m - AA^+) \tag{10}$$

Proof. The condition for G to be a {3}-dual generalized inverse of $\hat{A}$ is

$$\hat{A}G\big|_{\varepsilon^2=0} = (\hat{A}G)^T\Big|_{\varepsilon^2=0} \tag{11}$$

Using the expressions for $\hat{A}$ and G in the left hand side of Eq. (11) we get

$$\begin{aligned}\hat{A}G\big|_{\varepsilon^2=0} &= (A + \varepsilon B)(A^+ - \varepsilon A^+BA^+) = AA^+ + \varepsilon(BA^+ - AA^+BA^+)\\ &= AA^+ + \varepsilon(I_m - AA^+)BA^+\end{aligned} \tag{12}$$

where the second equality is obtained by setting $\varepsilon^2 = 0$. The necessary and sufficient condition for Eq. (11) to be satisfied is that the right hand side of Eq. (12) be a symmetric matrix. Since A^+ is the generalized inverse of A, the matrix AA^+ is symmetric (see the third equation in Eq. (2)). Thus, the necessary and sufficient condition that G be a {3}-dual generalized inverse is that the matrix $(I_m - AA^+)BA^+$ be symmetric, which is stated in Eq. (10) since $(I_m - AA^+)$ is a symmetric matrix. □

Result 4. The necessary and sufficient condition for the matrix G given in Eq. (1) to be a {4}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$ for all m by n matrices A and B is

$$A^+B(I_n - A^+A) = (I_n - A^+A)B^T(A^T)^+ \tag{13}$$

Proof. The condition for G to be a {4}-dual generalized inverse of $\hat{A}$ is

$$G\hat{A}\big|_{\varepsilon^2=0} = (G\hat{A})^T\Big|_{\varepsilon^2=0}. \tag{14}$$

The left hand side of Eq. (14) is

$$\begin{aligned}G\hat{A}\big|_{\varepsilon^2=0} &= (A^+ - \varepsilon A^+BA^+)(A + \varepsilon B)\\ &= A^+A - \varepsilon(A^+BA^+A - A^+B)\\ &= A^+A + \varepsilon A^+B(I_n - A^+A).\end{aligned} \tag{15}$$

Since A^+A is a symmetric matrix (see the fourth equation in Eq. (2)), for the right hand side of Eq. (15) to be a symmetric matrix, the necessary and sufficient condition given in Eq. (13) follows. □

Since full rank matrices often arise in engineering systems, we begin by restricting the m by n matrix A to have either (i) rank n, or (ii) rank m, for which we provide the following two results.

Result 5. Let the m by n matrix A have rank m. The n by m matrix G given in Eq. (1) is then a {1,2,3}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$ all m by n matrices B.

Proof. (i) We note that if the m by n matrix A has rank m, then $I_m - AA^+ = 0$ [1], hence Eq. (3) is satisfied and G is a {1}-dual generalized inverse of $\hat{A}$. Similarly, Eq. (10) is satisfied, and hence, G is a {3}-dual generalized inverse also. From this the result follows. □

Result 6. Let the m by n matrix A have rank n. The n by m matrix G given in Eq. (1) is then a {1,2,4}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$ all m by n matrices B.

Proof. (i) We note that if the m by n matrix A has rank n, then $I_n - A^+A = 0$ [1], hence Eq. (3) is satisfied and G is a {1}-dual generalized inverse of $\hat{A}$. Similarly, Eq. (13) is satisfied, and hence, G is a {4}-dual generalized inverse also. From this the result follows. □

Remark 3. We observe from Eq. (12) that when the m by n matrix A has rank m, $\hat{A}G = I_m$, since $AA^+ = I_m$. Eq. (15) shows that when the m by n matrix A has rank n, $G\hat{A} = I_n$, since $A^+A = I_n$.

Result 7. If the m by m matrix A has full rank, the matrix G given in Eq. (1) simplifies to $G = A^{-1} - \varepsilon A^{-1}BA^{-1}$ and it is a {1,2,3,4}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$ for all m by n matrices B.

Proof. Since $m = n$, Results 5 and 6 are both true, and therefore G is a {1,2,3,4}-dual generalized inverse of $\hat{A}$. Since A is square and has full rank, $A^+ = A^{-1}$ so that G simplifies to

$$G = A^{-1} - \varepsilon A^{-1}BA^{-1} \tag{16}$$

This result is first obtained in Refs. [2,4–7]. □

Remark 4. The requirement that the matrices be full rank in Results 5 and 6 is a sufficient condition. Thus, for example, Result 5 says that when A has rank m the matrix G is at least a {1,2,3}-dual generalized inverse of $\hat{A}$. It may, or may not, be a {4}-dual generalized inverse of $\hat{A}$.

Results 5–7 deal with matrices A that have full rank, and they provide the appropriate dual generalized inverse G with *no* restrictions on the nature of the matrix B. Having considered full rank matrices, we are next lead to question what happens when the matrix A does not have full rank. That is, when the m by n matrix A has $n < m$ but its rank r is less that n, or, when the matrix A has $m < n$ but its rank is r is less that m. As we shall shortly see, the matrix G now serves again as an appropriate dual generalized inverse (depending on whether r is less than m or n), but no longer for all m by n matrices B contained in the dual matrix $\hat{A}$ That is, when $r < \min(m, n)$, G remains an appropriate dual generalized inverse when the m by n matrix B satisfies addition conditions in relation to the matrix A. We next investigate this situation.

For the matrix G to be a {1}-dual generalized inverse of the matrix $\hat{A}$, we need Eq. (3) to be true. As stated in Remark 1, Eq. (3) can be satisfied is many ways. The first two ways listed in Remark 1 are: $I_m - AA^+ = 0$ and/or $I_n - A^+A = 0$. These two ways are used in developing Results 5 and 6. We will consider more general ways (the third and fourth ways listed in Remark 1) of satisfying Eq. (3). This is because when the rank of A is $r < \min(m, n)$, then neither $(I_m - AA^+)$ nor $(I_n - A^+A)$ are zero [1]. To make further progress we therefore need to place restrictions on the nature of the matrix B so that the matrix G given in Eq. (1) continues to have useful dual generalized inverse properties. We have the following results which are valid irrespective of the rank r of the matrix A.

Result 8. If $null(A) \subseteq null(B)$ and/or $null(A^T) \subseteq null(B^T)$ then Eq. (3) is satisfied, and hence G given in Eq. (1) is a {1}-inverse of $\hat{A} = A + \varepsilon B$.

Proof. Eq. (3) is satisfied when

$$B(I_n - A^+A) = 0 \; and/or \; (I_m - AA^+)B = 0 \tag{17}$$

But the column space of the matrix $(I_n - A^+A)$ is the null space of A [1], and hence when $null(A) \subseteq null(B)$, then the column space of $(I_n - A^+A)$ belongs to the $null(B)$, and therefore $B(I_n - A^+A) = 0$, so that Eq. (3) is satisfied.

Similarly, taking the transpose on both sides, the second equality in Eq. (17) can be rewritten as

$$B^T(I_m - AA^+) = 0 \tag{18}$$

since AA^+ is a symmetric matrix. Also, the column space of $(I_m - AA^+)$ is the null space of A^T, hence when $null(A^T) \subseteq null(B^T)$, then the column space of $(I_m - AA^+)$ belongs to $null(B^T)$ and hence $B^T(I_m - AA^+) = 0$. Thus, Eq. (3) is again satisfied. □

Remark 5. Result 8 can be thought of as generalization that is applicable to both full rank and to rank deficient (rank $r < \min(m, n)$) matrices. This is because when the rank of the matrix A is n, then $(I_n - A^+A) = 0$, which is in the null space of *any* matrix B. On the other hand when the rank of the matrix A is m, then $(I_m - AA^+) = 0$, and therefore again $(I_m - AA^+)$ is in the null space of *any* matrix B^T. Result 8 is more general as it does not require either $(I_n - A^+A)$ or $(I_m - AA^+)$ to be zero, i.e., it does not require A to be full rank. See also Remark 1.

Result 9. If $null(A^T) \subseteq null(AB^T)$, then the n by m matrix G given in Eq. (1) is a {3}-dual generalized inverse of the matrix $\hat{A}$. The matrix $\hat{A}G = AA^+$, which is symmetric.

Proof. For G to be a {3}-dual generalized inverse, Eq. (10) must be satisfied. One way in which it can be satisfied is by having

$$(I_m - AA^+)BA^+ = (A^T)^+B^T(I_m - AA^+) = 0. \tag{19}$$

Considering the last equality in Eq. (19), we note that the column space of $(A^T)^+$ is the column space of A [1], and the column space of the matrix $(I_m - AA^+)$ is the null space of the matrix A^T [1]. Hence, the relation

$$(A^T)^+B^T(I_m - AA^+) = 0 \tag{20}$$

can be restated as $null(A^T) \subseteq null(AB^T)$. From Eq. (12) we see that now$\hat{A}G = AA^+$. □

Remark 6. We note that when

$$B^T(I_m - AA^+) = 0, \tag{21}$$

Eq. (20) is automatically satisfied. But Eq. (21) states that $null(A^T) \subseteq null(B^T)$, and when this is true, then $null(A^T) \subseteq null(AB^T)$. Hence when $null(A^T) \subseteq null(B^T)$ the n by m matrix G is a {3}-dual inverse of $\hat{A}$.

Result 10. If $null(A) \subseteq null(A^T B)$, then the n by m matrix G given in Eq. (1) is a {4}-dual generalized inverse of the matrix $\hat{A}$. The matrix $G\hat{A} = A^+ A$, which is symmetric.

Proof. For G to be a {4}-dual generalized inverse, Eq. (15) must be satisfied. One way in which Eq. (15) can be satisfied is by having

$$(I_n - A^+ A) B^T (A^T)^+ = A^+ B (I_n - A^+ A) = 0. \tag{22}$$

Consider the second equality in Eq. (22). The column space of A^+ is the same as the column space of A^T and the column space of the matrix $(I_n - A^+ A)$ is the null space of A [1]. Hence, the relation

$$A^+ B (I_n - A^+ A) = 0 \tag{23}$$

can be rewritten as $null(A) \subseteq null(A^T B)$.$G\hat{A} = A^+ A$, follows from Eq. (15). □

Remark 7. We note that when

$$B(I_m - A^+ A) = 0, \tag{24}$$

Eq. (23) is automatically satisfied. But Eq. (24) states that $null(A) \subseteq null(B)$, and hence when this is true, then $null(A) \subseteq null(A^T B)$ is also true. Hence when $null(A) \subseteq null(B)$ the n by m matrix G is a {4}-dual inverse of $\hat{A}$.

Result 11. When $null(A^T) \subseteq null(B^T)$ then the matrix G given in Eq. (1) is a {1,2,3}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$.

Proof. We know that G is a {2}-dual inverse for all matrices A and B. Result 8 states that G is a {1}-dual generalized inverse; Remark 6 shows that it is a {3}-dual generalized inverse. Hence, the result. □

Result 12. When $null(A) \subseteq null(B)$ and then the matrix G given in Eq. (1) is a {1,2,4}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$.

Proof. We know that G is a {2}-dual inverse for all matrices A and B. Result 8 states that G is a {1}-dual generalized inverse; Remark 7 shows that it is a {4}-dual generalized inverse. Hence, the result. □

Result 13. If $null(A^T) \subseteq null(B^T)$, and (iii) $null(A) \subseteq null(B)$ then the n by m matrix G given in Eq. (1) is a {1,2,3,4}-dual generalized inverse of the matrix $\hat{A}$.

Proof. Combining the assertions from Results 11 and 12, assertion follows. □

Remark 8. Results 8 to 13 are applicable to general matrices A that may or may not be rank deficient.

Remark 9. When $r = m$ and $m < n$, which is the full rank case handled in Result 5, then the condition $null(A^T) \subseteq null(B^T)$ is always satisfied. This is because $null(A^T) = col(I - AA^+)$. But $(I - AA^+) = 0$, hence the $null(A^T)$ is spanned by the zero column vector which obviously is included in the null space of B^T. Since $AA^+ = I_m$, from Result 9 $\hat{A}G = AA^+ = I_m$ as observed in Remark 3. Thus Result 11 may therefore be thought of as a generalization of Result 5 when the matrix A does not have full rank.

Remark 10. When $r = n$ and $m > n$, which is the full rank case handled in Result 6, then the condition $null(A) \subseteq null(B)$ is always satisfied since $null(A) = col(I - A^+ A) = 0$, and $(I - A^+ A) = 0$. Since $A^+ A = I_n$, by Result 10, $G\hat{A} = A^+ A = I_n$, as observed in Remark 3.. Thus Result 12 may therefore be thought of as a generalization of Result 6.

Corollary 1. If (i) the m by n matrix A has rank m and (ii) $null(A) \subseteq null(A^T B)$, then the matrix G given in Eq. (1) is a {1,2,3,4}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$.

Proof. From Result 5 we know that condition (i) is sufficient to make the n by m matrix G a {1,2,3}-dual generalized inverse of $\hat{A}$ for all m by n matrices B. Condition (ii) further restricts the matrix B to be such that $null(A^T) \subseteq null(AB^T)$, which, by Result 10, ensures that G is a {4}-dual generalized inverse of $\hat{A}$.

Corollary 2. If (i) the m by n matrix A has rank n and (ii) $null(A^T) \subseteq null(AB^T)$, then the matrix G given in Eq. (1) is a {1,2,3,4}-dual generalized inverse of the matrix $\hat{A} = A + \varepsilon B$.

Proof. From Result 6 we know that condition (i) is sufficient to make the n by m matrix G a {1,2,4}-dual generalized inverse of $\hat{A}$ for all m by n matrices B. Condition (ii) further restricts the matrix B to be such that $null(A^T) \subseteq null(AB^T)$, which, by Result 9, ensures that G is also a {3}-dual inverse of $\hat{A}$. □

Remark 11. For any m by n matrix A, consider the dual matrix $\hat{A} = A + \varepsilon(\alpha A) = (1 + \varepsilon\alpha)A$, $\alpha \neq 0$, so that $B = \alpha A$. Then (i) $null(A^T) \subseteq null(B^T)$, since $B^T(I_m - AA^+) = \alpha A^T(I_m - AA^+) = 0$, and (ii) $null(A) \subseteq null(B)$, since $B(I_n - A^+ A) = \alpha A(I_n - A^+ A) = 0$. Hence the n by m matrix $G = A^+ - \varepsilon A^+ B A^+ = A^+ - \varepsilon\alpha A^+ A A^+ = (1 - \varepsilon\alpha)A^+$ is a {1,2,3,4}-dual generalized inverse of the matrix $(1 + \varepsilon\alpha)A$. Setting $\alpha = 1$, we find that the matrix $(1 - \varepsilon)A^+$ is the {1,2,3,4}-dual generalized inverse of the m by n matrix $(1 + \varepsilon)A$. This result could have been obviously obtained in a much easier fashion by simply using the expression for G given in Eq. (1).

Remark 12. Consider the dual matrix $\hat{A} = A + \varepsilon B$ where A and B are two m by n diagonal matrices, i.e., each matrix can only have its (i,i) elements non-zero. If the rank of A is n, then the matrix G given in Eq. (1) is a {1,2,3,4}-dual generalized inverse of $\hat{A}$.

We know that G is a {1,2,4}-dual generalized inverse from Result 6. Note that since A is diagonal and has rank n, none of its diagonal elements can be zero. The null space of A^T is therefore spanned by the $(m-n)$ column vectors each of which has m elements. The pth column vector in this set has all its elements zero except for its $(n+p)$th element, which is unity. The rank r of the matrix B equals the number of nonzero elements along its diagonal so that $r \le n$; its null space therefore has a dimension d with $d \ge (m-n)$. Since A and B are diagonal, it is easy to see that the null space of B^T includes the $(m-n)$ column vectors that span the null space of A^T. In addition, for each zero element on the diagonal of B located at (i,i), there is, in the null space of B^T, a column vector all of whose elements are zero except its ith element, which is unity. Clearly, then $null(A^T) \subseteq null(B^T)$, and by Corollary 2 and Remark 6 G is a {3}-dual generalized inverse of $\hat{A}$.

As we shall see in the last example in Section 3, if the diagonal matrix A has rank $r < n$, then there is no guarantee that G given in Eq. (1) will give the {1, 2, 3, 4}-dual generalized inverse of $\hat{A}$. In fact it can be shown that, if the A has its diagonal elements $A(i_k, i_k) = 0,\ k = 1, 2, \dots (n-r)$, and the diagonal matrix B has any one (or more) of its corresponding diagonal elements $B(i_k,\ i_k) \neq 0,\ k = 1, 2, \dots (n-r)$ then the dual matrix $\hat{A}$ has *no* {1}-dual generalized inverse!

Remark 13. Consider the dual matrix $\hat{A} = A + \varepsilon B$ where A and B are two m by n diagonal matrices, i.e., each matrix can only have its (i,i) elements non-zero. If the rank of A is m, then the matrix G given in Eq. (1) is a {1,2,3,4}-dual generalized inverse of $\hat{A}$. This can be proved in a manner similar to that stated in Remark 12. On the other hand, if the diagonal matrix A has rank $r < m$, then there is no guarantee that G given in Eq. (1) will give the {1,2,3,4}-dual generalized inverse of $\hat{A}$. In fact it can be shown that, if the A has its diagonal elements $A(i_k, i_k) = 0,\ k = 1, 2, \dots (m-r)$, and the diagonal matrix B has any one (or more) of its corresponding diagonal elements $B(i_k,\ i_k) \neq 0,\ k = 1, 2, \dots (m-r)$ then the dual matrix $\hat{A} = A + \varepsilon B$ has *no* {1}-dual generalized inverse!

3. Some computational results

In this section we show a few representative computational results to corroborate the analytical results obtained in Section 2. All computations have been done in the Matlab environment. It is important to note that all the results obtained in this paper (except for Result 1) only provide sufficient conditions for the existence of the relevant dual generalized inverses.

1) Consider the 4 by 3 matrices

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 6 & 4 \\ 3 & 7 & 6 \\ 4 & 8 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 6 & 5 \\ 2 & 3 & 4 \\ 7 & 7 & 6 \\ 4 & 8 & 18 \end{bmatrix} \tag{25}$$

so that

$$\hat{A} = A + \varepsilon B. \tag{26}$$

Here $m = 4$ and $n = 3$, and the

$$G = G_r + \varepsilon G_d = \\ = \begin{bmatrix} -0.1100 & -0.0450 & 0.0200 & 0.0850 \\ 0.2500 & 0.1250 & 0.0000 & -0.1250 \\ -0.2200 & -0.0900 & 0.0400 & 0.1700 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.2269 & 0.0923 & -0.0424 & -0.1771 \\ -0.3287 & -0.1544 & 0.0200 & 0.1944 \\ 0.4539 & 0.1846 & -0.0848 & -0.3541 \end{bmatrix} \tag{27}$$

which is a {2}-dual generalized inverse of $\hat{A}$ by Result 1. Here $G_r = A^+$ and $G_d = -A^+AA^+$.

We observe that (i) the rank of A is 2 because its third column is twice its first column, and its second column is independent of its first, and (ii) the rank of B is 3 (full rank). This says that the null space of A has dimension 1, while the null space of B contains only the zero column vector. Hence, $null(A) \not\subset null(B)$. Similarly, since $rank(A) = rank(A^T)$, the null space of A^T has a dimension of 2, and the null space of B^T has dimension 1. Hence, $null(A^T) \not\subset null(B^T)$. We note that these conditions regarding the respective non-inclusivity of the appropriate null spaces in this example have been carried out without the need to actually find what these null spaces are, i.e., without the need to find the actual column vectors that span these null spaces.

Thus we are not guaranteed that

$$\Delta = BA^+A - AA^+BA^+A + AA^+B - B = -(I_m - AA^+)B(I_n - A^+A) = 0 \tag{28}$$

and hence the matrix G computed in Eq. (27) may not a {1}-dual generalized inverse of the matrix $\hat{A}$.

In fact, the error $e_1 = \hat{A}G\hat{A}|_{\varepsilon^2=0} - \hat{A}$ in satisfaction of the {1}-dual generalized inverse condition is obtained from Eq. (5), as

$$\begin{aligned} e_1 &= \hat{A}G\hat{A}\big|_{\varepsilon^2=0} - \hat{A} = \hat{A}G\hat{A}\big|_{\varepsilon^2=0} - (A+\varepsilon B) = \\ &= \varepsilon[(BA^+A - AA^+BA^+A + AA^+B) - B] := \varepsilon\Delta. \end{aligned} \tag{29}$$

Notice from the right hand side of Eq. (29) that, theoretically speaking, this error is only in the term that multiplies ε (the dual part of the matrix e_1), since $G_r = A^+$, and we used the first MP condition for A (see the set of equations in Eq. (2)) in going from the third to the fourth equality in Eq. (5). Furthermore, the error term is proportional to $\Delta := (BA^+A - AA^+BA^+A + AA^+B) - B$.

The computed error is obtained as

$$e_1 = \begin{bmatrix} -0.0222 & -0.1776 & -0.0444 \\ -0.0444 & -0.1776 & -0.0888 \\ -0.0888 & -0.2665 & -0.1776 \\ -0.1332 & -0.2665 & -0.2665 \end{bmatrix} \times 10^{-14} + \varepsilon \begin{bmatrix} 1.4800 & -0.0000 & -0.7400 \\ -0.2400 & 0.0000 & 0.1200 \\ -3.9600 & 0 & 1.9800 \\ 2.7200 & -0.0000 & -1.3600 \end{bmatrix} \tag{30}$$

and since we know that the first matrix on the right hand side should be a zero matrix (theoretically speaking), its order of magnitude serves as a measure of our computational accuracy. What is important is that the dual part of e_1 is not zero, i.e., $\Delta \neq 0$. Since $\Delta \neq 0$, and as seen from Result 1 that G cannot be {1}-dual generalized inverse of $\hat{A}$.

The error in the satisfaction of the {2}-dual generalized inverse condition $e_2 := G\hat{A}G|_{\varepsilon^2=0} - G$ is obtained as

$$\begin{aligned} e_2 &= \begin{bmatrix} 0.1388 & 0 & -0.1388 & -0.2776 \\ -0.2776 & 0 & 0.1388 & 0.2776 \\ 0.5551 & 0.1388 & -0.2776 & -0.5551 \end{bmatrix} \times 10^{-16} \\ &+ \varepsilon \begin{bmatrix} -0.1665 & -0.0416 & 0.0555 & 0.1388 \\ 0.1665 & 0.0555 & -0.0451 & -0.1665 \\ -0.3331 & -0.1110 & 0.0971 & 0.3331 \end{bmatrix} \times 10^{-15}. \end{aligned} \tag{31}$$

The smallness of the numbers in the matrix on the right hand side of Eq. (31) comes from the fact that $G_r = A^+$, and A^+ satisfies the second MP condition (i.e., the second equation in the Eq. (2)). The error in satisfaction of the second dual generalized inverse condition is therefore, again, only in the dual part matrix e_2. This indicates that the matrix G is a {2}-dual generalized inverse of $\hat{A}$.

A sufficient condition for G in Eq. (27) to be a {4}-dual generalized inverse, is that $null(A) \subseteq null(B)$. But as shown before $null(A) \not\subset null(B)$ and therefore G may not be a {4}-dual generalized inverse of the matrix $\hat{A}$ given in Eqs. (25) and (26). While there is obviously no need to compute the null space of A, for completeness we compute it to be the span of [-0.8944, 0.0000, 0.4472]^T; the null space of B is zero. Using the last equality in Eq. (15), we find that the error, e_4, in satisfaction of the fourth dual generalized inverse condition is

$$\begin{aligned} e_4 :&= G\hat{A}\big|_{\varepsilon^2=0} - (G\hat{A})^T\big|_{\varepsilon^2=0} \\ &= \varepsilon[A^+B(I_n - A^+A) - (I_n - A^+A)B^T(A^T)^+]. \end{aligned} \tag{32}$$

We again notice that error arises only in the dual part of the matrix e_4, since $G_r = A^+$. The computational results show that

$$e_4 = \begin{bmatrix} 0 & -0.3053 & -0.2220 \\ 0.3053 & 0 & 0.7216 \\ 0.2220 & -0.7216 & 0 \end{bmatrix} \times 10^{-15} + \varepsilon \begin{bmatrix} 0 & -0.2000 & 0.3600 \\ 0.2000 & 0 & -0.1000 \\ -0.3600 & 0.1000 & 0 \end{bmatrix} \tag{33}$$

where, again, the first matrix on the right hand side, which should be theoretically speaking zero, gives an indication of our computational accuracy. The fact that the second matrix on the right hand side is not zero points out that $G\hat{A}|_{\varepsilon^2=0}$ is not a symmetric matrix, as required by the fourth dual generalized inverse condition.

Since a sufficient condition for G to be a {3}-dual generalized inverse of $\hat{A}$ is that $null(A^T) \subseteq null(B^T)$,which, as shown before, is not true, this indicates that the matrix G given in Eq. (27) may not be a {3}-dual generalized inverse. Using the last equality in Eq. (12), the error, e_3, in the satisfaction of the third dual generalized inverse condition is

$$\begin{aligned} e_3 :&= \hat{A}G\big|_{\varepsilon^2=0} - (\hat{A}G)^T\big|_{\varepsilon^2=0} \\ &= \varepsilon[(I_m - AA^+)BA^+ - (A^T)^+B^T(I_m - AA^+)]. \end{aligned} \tag{34}$$

As before, the error is only in the dual term because $G_r = A^+$. On computing e_3 we get

$$e_3 = \begin{bmatrix} 0 & 0.2220 & 0.2220 & 0 \\ -0.2220 & 0 & 0.1665 & 0.2220 \\ -0.2220 & -0.1665 & 0 & 0.0555 \\ 0 & -0.2220 & -0.0555 & 0 \end{bmatrix} \times 10^{-15} + \varepsilon \begin{bmatrix} 0 & -0.0025 & -0.6550 & 0.7625 \\ 0.0025 & 0 & -0.4275 & 0.0600 \\ 0.6550 & 0.4275 & 0 & -0.6675 \\ -0.7625 & -0.0600 & 0.6675 & 0 \end{bmatrix} \tag{35}$$

Thus, the matrix G given in Eq. (27) is only a {2}-generalized inverse of the matrix $\hat{A}$ given in Eqs. (25) and (26) .

One can also consider the transpose of the matrices A and B so that $\hat{A} = A^T + \varepsilon B^T$, where the matrices A and B are given in Eq. (25). Now one has an m by n matrix A^T with $m = 3$ and $n = 4$ whose rank is 2. Again, since $null(A) \not\subset null(B)$ and $null(A^T) \not\subset null(B^T)$, the matrix $G = (A^+)^T - \varepsilon (A^+)^T B^T (A^+)^T$ may not be a {1}-, or a {3}- or a {4}-dual generalized inverse of $\hat{A}$. Indeed, computations show that it remains only a {2}-dual generalized inverse of the matrix $\hat{A}$.

2) We next consider the 4 by 3 matrices

$$A = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 6 & 5 \\ 3 & 7 & 6 \\ 4 & 8 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 6 & 5 \\ 2 & 3 & 4 \\ 7 & 7 & 6 \\ 4 & 8 & 18 \end{bmatrix} \tag{36}$$

with

$$\hat{A} = A + \varepsilon B. \tag{37}$$

The matrix $G = G_r + \varepsilon G_d$ given in Eq. (1) computes to

$$G = \begin{bmatrix} 0.4643 & -2.0000 & 0.6071 & 0.6786 \\ 0.3214 & 0.0000 & 0.0357 & -0.1071 \\ -0.5714 & 1.0000 & -0.2857 & -0.1429 \end{bmatrix} + \varepsilon \begin{bmatrix} 2.1467 & -3.3214 & 0.5957 & -0.1798 \\ -0.5523 & 0.3929 & -0.1645 & 0.0293 \\ -0.1173 & 0.8571 & -0.0765 & -0.0561 \end{bmatrix}, \tag{38}$$

which is a {2}-dual generalized inverse of $\hat{A}$. Again, $m = 4$ and $n = 3$ for the matrix A, but now its rank is 3 because its columns are linearly independent. Since the rank of A equals to n, the matrix G given in Eq. (38) is, by Result 6, is at least a {1,2,4}-dual generalized inverse of the matrix $\hat{A}$ given in Eqs. (36) and (37).

The error $e_1 := \hat{A}G\hat{A}|_{\varepsilon^2=0} - \hat{A}$ in the satisfaction of the first dual generalized inverse condition, which is given in Eq. (29) is computed as

$$e_1 = \begin{bmatrix} 0 & 0.1776 & 0.2665 \\ 0 & 0.0888 & 0.1776 \\ -0.1776 & -0.2665 & -0.1776 \\ -0.1776 & -0.2665 & -0.0888 \end{bmatrix} \times 10^{-14} + \varepsilon \begin{bmatrix} 0.0178 & -0.0533 & 0.0178 \\ 0.0311 & 0.0622 & 0.0799 \\ -0.0178 & 0.0977 & 0.0888 \\ 0.0533 & -0.0533 & 0.1066 \end{bmatrix} \times 10^{-13}. \tag{39}$$

Recall that the error $\Delta := (BA^+A - AA^+BA^+A + AA^+B) - B$ is computed in the second matrix on the right hand side, while the first matrix, which ought to be zero, gives a measure of the accuracy of the computations. As expected, the matrix G is a {1}-dual generalized inverse of $\hat{A}$.

Using Eq. (32) to compute the error $e_4 := [G\hat{A} - (G\hat{A})^T]|_{\varepsilon^2=0}$, one gets

$$e_4 = \begin{bmatrix} 0 & 0.1665 & -0.0999 \\ -0.1665 & 0 & 0.2109 \\ 0.0999 & -0.2109 & 0 \end{bmatrix} \times 10^{-14} + \varepsilon \begin{bmatrix} 0 & 0.1144 & 0.0377 \\ -0.1144 & 0 & -0.0044 \\ -0.0377 & 0.0044 & 0 \end{bmatrix} \times 10^{-13}, \tag{40}$$

which shows that to computational accuracy G is a {4}-dual generalized inverse. We know from the Remark 3 that $G\hat{A} = I_n$, which computes to

$$G\hat{A} = \begin{bmatrix} 1.0000 & 0.0000 & -0.0000 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.0000 & -0.0000 & 1.0000 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.0133 & 0.1155 & 0.0355 \\ 0.0011 & -0.0178 & -0.0089 \\ -0.0022 & -0.0044 & 0.0133 \end{bmatrix} \times 10^{-13} \tag{41}$$

That G is not a {3}-dual generalized inverse is seen when $e_3 := [\hat{A}G - (\hat{A}G)^T]|_{\varepsilon^2=0}$ given Eq. (34) is computed. We find that

$$e_3 = \begin{bmatrix} 0 & 0.3442 & 0.0444 & 0.0666 \\ -0.3442 & 0 & -0.2665 & -0.3136 \\ -0.0444 & 0.2665 & 0 & -0.0167 \\ -0.0666 & 0.3136 & 0.0167 & 0 \end{bmatrix} \times 10^{-14} + \varepsilon \begin{bmatrix} 0 & 3.3571 & -4.9286 & 1.8036 \\ -3.3571 & 0 & 10.0714 & -6.7143 \\ 4.9286 & -10.0714 & 0 & 4.4464 \\ -1.8036 & 6.7143 & -4.4464 & 0 \end{bmatrix}. \tag{42}$$

As mentioned earlier the small numbers in the first matrix on the right hand side of Eq. (42) are a consequence of $G_r = A^+$. That the matrix $\hat{A}G$ is far from symmetric, and therefore does not satisfy the third dual generalized inverse condition, is seen from the non-zero second matrix on the right hand side.

Were we to consider the matrix $\hat{A} = A^T + \varepsilon B^T$, where A and B are given in Eq. (36) then the rank of A^T is 3. Since A^T is a matrix with $m = 3$ and $n = 4$, and its rank equals 3, Result 5 states that the matrix $G = (A^+)^T - \varepsilon (A^+)^T B^T (A^+)^T$ is a {1, 2, 3}-dual inverse of the matrix $\hat{A}$. This inverse is computed to be

$$G = \begin{bmatrix} 0.4643 & 0.3214 & -0.5714 \\ -2.0000 & -0.0000 & 1.0000 \\ 0.6071 & 0.0357 & -0.2857 \\ 0.6786 & -0.1071 & -0.1429 \end{bmatrix} + \varepsilon \begin{bmatrix} 2.1467 & -0.5523 & -0.1173 \\ -3.3214 & 0.3929 & 0.8571 \\ 0.5957 & -0.1645 & -0.0765 \\ -0.1798 & 0.0293 & -0.0561 \end{bmatrix}, \tag{43}$$

and it is a {2}-dual generalized inverse of $\hat{A}$. The errors in computing $e_1 = \hat{A}G\hat{A}|_{\varepsilon^2=0} - \hat{A}$ and $e_3 := \hat{A}G - (\hat{A}G)^T|_{\varepsilon^2=0}$, as before, are

$$\begin{aligned} e_1 = & \begin{bmatrix} 0.1332 & 0.4885 & 0.7105 & 0.9770 \\ -0.3553 & -0.4441 & -0.3553 & -0.3553 \\ -0.1332 & 0.0888 & 0.2665 & 0.3553 \end{bmatrix} \times 10^{-14} \\ & + \varepsilon \begin{bmatrix} -0.0033 & 0.0222 & 0.0533 & -0.0089 \\ -0.0533 & 0.0089 & 0.0355 & -0.1954 \\ -0.0178 & 0 & 0.0178 & -0.1421 \end{bmatrix} \times 10^{-13} \end{aligned} \tag{44}$$

and

$$e_3 = \begin{bmatrix} 0 & -0.2998 & -0.2220 \\ 0.2998 & 0 & -0.0111 \\ 0.2220 & 0.0111 & 0 \end{bmatrix} \times 10^{-14} + \varepsilon \begin{bmatrix} 0 & 0.1010 & 0.1799 \\ -0.1010 & 0 & 0.0133 \\ -0.1799 & -0.0133 & 0 \end{bmatrix} \times 10^{-13}, \tag{45}$$

corroborating Result 5, namely, that the matrix G in Eq. (38) is a {1, 2, 3}-dual generalized inverse of $\hat{A} = A^T + \varepsilon B^T$. The matrix $\hat{A}G$ is computationally very close to I_3.

We also see from the error $e_4 := G\hat{A}|_{\varepsilon^2=0} - (G\hat{A})^T|_{\varepsilon^2=0}$, which on computation yields

$$e_4 = \begin{bmatrix} 0 & 0.0422 & 0.0078 & 0.0125 \\ -0.0422 & 0 & -0.1205 & -0.1413 \\ -0.0078 & 0.1205 & 0 & 0.0094 \\ -0.0125 & 0.1413 & -0.0094 & 0 \end{bmatrix} \times 10^{-13} + \varepsilon \begin{bmatrix} 0 & -3.3571 & 4.9286 & -1.8036 \\ 3.3571 & 0 & -10.0714 & 6.7143 \\ -4.9286 & 10.0714 & 0 & -4.4464 \\ 1.8036 & -6.7143 & 4.4464 & 0 \end{bmatrix}, \tag{46}$$

that the matrix G in Eq. (43) is not a {4}-dual generalized inverse of $\hat{A}$.

We next provide an example where $m > n$ and the rank r of the matrix A is less than n.

3) Consider the 4 by 5 matrices

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 3 \\ 1 & 2 & 3 & 4 & 3 \\ 1 & 3 & 4 & 2 & 2 \\ 1 & 4 & 5 & -12.616795 & -1.523359 \end{bmatrix} \text{and } B = \begin{bmatrix} 1 & 5 & 10 & 2 & 4 \\ 2 & 6 & 12 & 4 & 8 \\ 3 & 7 & 14 & 6 & 12 \\ 4 & 8 & 16 & 8 & 16 \end{bmatrix} \tag{47}$$

and $\hat{A} = A + \varepsilon B$.

Here $m = 4$ and $n = 5$. The rank r of the matrix A is 3 and therefore $r < \min(m, n)$. Also, $rank(B) = rank(B^T) = 2$. The null space of the 5 by 4 matrix B^T is 2-dimensional and is spanned by the column vectors

$$\begin{bmatrix} 0.314460963948816 & 0.448457692711789 \\ -0.753541913910512 & -0.363558226395832 \\ 0.563700935974577 & -0.618256625343705 \\ -0.124619986012881 & 0.533357159027747 \end{bmatrix}, \tag{48}$$

and the null space of the matrix A^T is 1-dimensional and is spanned by the column vector

$$\begin{bmatrix} 0.314460963783024 \\ -0.753541913776108 \\ 0.563700936203141 \\ -0.124619986210058 \end{bmatrix} \tag{49}$$

so that $null(A^T) \subset null(B^T)$. Result 11 then informs us that the matrix G give in Eq. (1) is a {1, 2, 3}-dual inverse of the matrix $\hat{A}$. We note that since $r < m$, the matrix $(I_m - AA^+) \neq 0$ but since $null(A^T) \subset null(B^T)$, theoretically speaking, the matrix $(I_m - AA^+)B = 0$. These matrices are respectively computed as

$$I_m - AA^+ = \begin{bmatrix} 0.0989 & -0.2370 & 0.1773 & -0.0392 \\ -0.2370 & 0.5678 & -0.4248 & 0.0939 \\ 0.1773 & -0.4248 & 0.3178 & -0.0702 \\ -0.0392 & 0.0939 & -0.0702 & 0.0155 \end{bmatrix} \tag{50}$$

and

$$(I_m - AA^+)B = \begin{bmatrix} -0.0333 & -0.1443 & -0.2887 & -0.0666 & -0.1332 \\ 0.0500 & 0.1443 & 0.2887 & 0.0999 & 0.1776 \\ 0.0944 & 0.2998 & 0.5995 & 0.1887 & 0.3553 \\ 0.1055 & -0.1055 & -0.2109 & 0.2109 & 0.4219 \end{bmatrix} \times 10^{-14}. \tag{51}$$

The computational results are as expected.

Furthermore, the null space of A has 2 dimensions and is spanned by the 2 column vectors,

$$null(A) = \begin{bmatrix} 0.9207 & -0.2713 \\ -0.2436 & -0.7080 \\ 0.0690 & 0.6425 \\ 0.0582 & 0.0218 \\ -0.2911 & -0.1092 \end{bmatrix} \tag{52}$$

while the null space of B has three dimensions and is spanned by 3 column vectors

$$null(B) = \begin{bmatrix} -0.6879 & 0.6922 & 0.0000 \\ 0.6344 & 0.6305 & 0.0000 \\ -0.3172 & -0.3152 & -0.0000 \\ 0.0688 & -0.0692 & -0.8944 \\ 0.1376 & -0.1384 & 0.4472 \end{bmatrix} \tag{53}$$

To compute whether $null(A) \subseteq null(B)$ we compute the rank of the matrix $[null(A) \quad null(B)]$, which turns out to be 5. Hence, $null(A) \not\subseteq null(B)$. Since the rank of A is less than n, $(I_n - A^+A) \neq 0$ and since $null(A) \subseteq null(B)$, we know that $B(I_n - A^+A) \neq 0$. These matrices are computed as

$$I_n - A^+A = \begin{bmatrix} 0.9214 & -0.0322 & -0.1108 & 0.0477 & -0.2384 \\ -0.0322 & 0.5606 & -0.4716 & -0.0296 & 0.1482 \\ -0.1108 & -0.4716 & 0.4175 & 0.0180 & -0.0902 \\ 0.0477 & -0.0296 & 0.0180 & 0.0039 & -0.0193 \\ -0.2384 & 0.1482 & -0.0902 & -0.0193 & 0.0966 \end{bmatrix} \tag{54}$$

and

$$B(I_n - A^+A) = \begin{bmatrix} -1.2062 & -1.4124 & 1.3814 & 0.0103 & -0.0515 \\ -1.3969 & -1.2938 & 1.3093 & -0.0052 & 0.0258 \\ -1.5876 & -1.1753 & 1.2371 & -0.0206 & 0.1031 \\ -1.7784 & -1.0567 & 1.1649 & -0.0361 & 0.1804 \end{bmatrix}, \tag{55}$$

confirming our analytical results. The {1, 2, 3}-dual generalized inverse is computed as

$$G = \begin{bmatrix} 0.1010 & 0.0101 & -0.0410 & 0.0085 \\ -0.1802 & 0.0375 & 0.1551 & 0.0198 \\ -0.0792 & 0.0476 & 0.1141 & 0.0283 \\ -0.1349 & 0.0367 & 0.1109 & -0.0604 \\ 0.4246 & 0.0191 & -0.2102 & 0.0049 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.0681 & -0.0512 & -0.1116 & -0.0235 \\ -0.4796 & -0.1458 & 0.0721 & -0.0027 \\ -0.4115 & -0.1969 & -0.0395 & -0.0262 \\ -0.0999 & 0.0166 & 0.0813 & 0.0151 \\ 0.4994 & -0.0832 & -0.4065 & -0.0754 \end{bmatrix} \tag{56}$$

The value of the matrix $e_1 = \hat{A}G\hat{A}|_{\varepsilon^2=0} - \hat{A}$ computes to

$$e_1 \times 10^{14} = \begin{bmatrix} 0.0222 & 0.0444 & 0.0444 & -0.1332 & 0.0888 \\ -0.0222 & -0.0666 & -0.0444 & -0.2665 & -0.0888 \\ -0.0444 & -0.0888 & -0.1332 & -0.2665 & -0.1776 \\ 0.0444 & -0.0888 & 0 & 0 & 0.2442 \end{bmatrix} + \varepsilon \begin{bmatrix} -0.0333 & 0 & 0.1776 & 0 & 0 \\ -0.0888 & 0 & -0.1776 & -0.1776 & -0.2665 \\ 0.0888 & 0.0888 & -0.1776 & 0.0888 & 0 \\ 0 & -0.0888 & 0.3553 & 0 & 0 \end{bmatrix}, \tag{57}$$

showing that, computationally speaking, G is a {1}-dual generalized inverse. As shown in Result 9 and Remark 6, the matrix product $\hat{A}G|_{\varepsilon^2=0}$ should compute to AA^+, andnottoI_m. The matrix $\hat{A}G|_{\varepsilon^2=0}$ computes to

$$\hat{A}G\Big|_{\varepsilon^2=0} = \begin{bmatrix} 0.9011 & 0.2370 & -0.1773 & 0.0392 \\ 0.2370 & 0.4322 & 0.4248 & -0.0939 \\ -0.1773 & 0.4248 & 0.6822 & 0.0702 \\ 0.0392 & -0.0939 & 0.0702 & 0.9845 \end{bmatrix} + \varepsilon \begin{bmatrix} -0.0278 & -0.0222 & -0.0222 & -0.0056 \\ -0.0888 & 0.0222 & 0.0444 & 0.0111 \\ 0 & 0.0444 & 0.0999 & 0.0111 \\ 0.2220 & 0 & -0.1277 & -0.0167 \end{bmatrix} \times 10^{-14} \tag{58}$$

and the first matrix on the right hand side of Eq. (58) differs from AA^+ by zero with the computational accuracy of Matlab. This shows that G is indeed a {3}-dual generalized inverse, as expected.

That G may not be a {4}-dual generalized inverse, is corroborated by computing

$$G\hat{A}\Big|_{\varepsilon^2=0} - (G\hat{A})^T\Big|_{\varepsilon^2=0} = \begin{bmatrix} 0 & 0.6939 & 0.6661 & 0.3886 & -0.5551 \\ -0.6939 & 0 & -0.0555 & -0.2776 & -0.4163 \\ -0.6661 & 0.0555 & 0 & -0.2776 & -0.5274 \\ -0.3886 & 0.2776 & 0.2776 & 0 & -0.3296 \\ 0.5551 & 0.4163 & 0.5274 & 0.3296 & 0 \end{bmatrix} \times 10^{-15} + \varepsilon \begin{bmatrix} 0 & 0.0000 & 0.3144 & -0.0412 & 0.2062 \\ -0.0000 & 0 & 0.1289 & -0.0825 & 0.4124 \\ -0.3144 & -0.1289 & 0 & 0.0670 & -0.3351 \\ 0.0412 & 0.0825 & -0.0670 & 0 & -0.0000 \\ -0.2062 & -0.4124 & 0.3351 & 0.00000 & \end{bmatrix}, \tag{59}$$

which is clearly not zero. Recall that the smallness of the numbers in the first matrix on the right hand side of Eq. (59) is due to the fact that $G_r = A^+$, and is a result of the properties of the MP generalized inverse of A. It is the second matrix on the right hand side that shows errors in the satisfaction of the fourth dual generalized inverse condition, and it is not zero.

4) This example illustrates Corollary 1 presented in Section 1. Consider the dual matrix $\hat{A} = A + \varepsilon B$ where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -4 \end{bmatrix} \tag{60}$$

with $\hat{A} = A + \varepsilon B$.

Here $m = 3$ and $n = 2$, The columns of the matrix A are independent, its rank is 2, which equals n. The sufficient conditions given in Result 6 for the matrix $G = A^+ - \varepsilon A^+BA^+$ to be a {1,2,4}-dual generalized inverse of the matrix $\hat{A}$ are therefore satisfied. Hence G must be at least a {1,2,4}-dual generalized inverse of $\hat{A}$. In fact, Result 6 states a bit more, namely, for the specific matrix A given in Eq. (60), the matrix G is a {1,2,4}-generalized inverse for *any and all* 3 by 2 matrices B. However, for the specific matrix B given in Eq. (60) more can be said.

We find that the null space of A^T is spanned by the vector $u = [1, -2, 1]^T$ and for this specific matrix B we find that

$$B^T u = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = 0 \tag{61}$$

Hence, the vector u belongs to the $null(B^T)$, and by Corollary 1, G is a {1,2,3,4}-dual generalized inverse of $\hat{A}$.

For brevity, we present the computational results for $\varepsilon = 3$. Errors in the satisfaction of the 4 generalized inverse conditions when computed show that the matrices e_1, e_2, e_3, e_4 have numbers of $O(10^{-13})$ or smaller. For brevity, they have not been shown here.

5) Lastly, we consider the two diagonal matrices

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{62}$$

and consider the generalized inverse of the dual matrix $\hat{A} = A + \varepsilon B$. This example relates to Remark 12 in Sction1.

Both A and B have rank 2, and they are rank deficient. The matrix $G = A^+ - A^+BA^+$ computes to

$$G = G_r + \varepsilon G_d = \begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{63}$$

As seen by inspection, the null spaces of the matrix AB^T and A^T are spanned respectively by the columns vectors of the matrices

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{64}$$

Clearly, $null(A^T) \subseteq null(AB^T)$, and hence by Result 9, the matrix G is a {3}-dual generalized inverse of $\hat{A}$.

Similarly, the null spaces of A^TB and A are spanned respectively by the columns of the matrices

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{65}$$

and since $null(A) \subseteq null(A^TB)$, by Result 10, the matrix G is a {4} dual inverse of $\hat{A}$.

Since we know that G is a {2}-dual inverse of all dual matrices, all that remains to be checked is if G satisfies the first dual generalized inverse condition. One can see by inspection that $null(A) \not\subset null(B)$ and $null(A^T) \not\subset null(B^T)$. So it is likely that G may not be a {1}-dual generalized inverse. That this is indeed true is borne out by the fact that on computation we find that

$$(I_m - AA^+)B(I_n - A^+A) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{66}$$

which is not zero. Thus the matrix G given in Eq. (63) is a {2,3,4}-dual inverse of the matrix $\hat{A}$ with A and B given in Eq. (62). For brevity, the numerical computations showing errors in the satisfaction of the second, third, and fourth, dual generalized inverse conditions are not shown. They are all zero to within machine precision, and corroborate the analytical results. In fact, the matrix $\hat{A}$ has no {1}-dual generalized inverse and what is important here is the illustration that G is not only (i) not a {1,2,3,4}-dual generalized inverse of $\hat{A}$ but that (ii) no {1,2,3,4}-dual generalized inverse exists for the matrix $\hat{A}$.

4. Kinematic applications of the analytical results

To demonstrate the usefulness and versatility of the dual generalized inverse, the solution of some meaningful kinematic problems will be discussed in this section.[1] In particular, the following problems will be considered:

I. Rigid body finite screw motion parameters from redundant point positions measurements.
II. Screw motion parameters from redundant homologous point and line features.
III. Rigid body infinitesimal screw motion parameters from point velocities measurements.

Solutions to these problems have been proposed in thoughtful investigations [9–21]. However, as it will be shown, the availability of the dual generalized inverse allows a concise and unified treatment of different problem typologies.

4.1. Rigid body finite motion

The unknown body motion parameters are:

- the Plücker coordinates of the finite screw axis or the corresponding unit dual vector

$$\hat{u} = u + \varepsilon u^0 \tag{67}$$

where u is the unit vector representing the direction of the axis and $u^0 = r \times u$ the moment of u about the origin of the Cartesian reference system.

- the angle of rotation ϑ about the axis;

[1] The authors thank the anonymous reviewer that suggested the development of some of these examples.

- the translation s along the axis.

We may distinguish two different data typologies:

- Of a discrete set of $n \geq 3$ non collinear points on the body, the initial and final position vectors, ${}^{i}r_k$ and ${}^{f}r_k$ $(k = 1, 2, \ldots, n)$, respectively.
- Initial and final positions of mixed set of n points and m lines. In particular, the initial and final positions of these lines are specified by means of the dual unit line vectors

$$ {}^{p}\hat{h}_j = {}^{p}h_j + \varepsilon {}^{p}h_j^o (j = 1, 2, \ldots, m) \tag{68} $$

where the left superscript $p = i, f$ distinguishes initial and final positions.

The barycenter of the points set is defined as

$$ {}^{p}g = \frac{\sum_{k=1}^{n} {}^{p}r_k}{n} \quad (p = i, f) \tag{69} $$

whereas the homologous lines positions are represented with the dual vectors

$$ {}^{p}\hat{p}_j = {}^{p}p_j + \varepsilon {}^{p}p_j^o \quad (j = 1, 2, \ldots, n) \tag{70} $$

where

$$ {}^{p}p_j = {}^{p}r_k - {}^{p}g \tag{71} $$

$$ {}^{p}p_j^o = {}^{p}r_k \times ({}^{p}r_k - {}^{p}g) \quad (p = i, f) \tag{72} $$

Since the unknown dual displacement matrix $\hat{A}$ transforms ${}^{i}\hat{h}_j$ into ${}^{f}\hat{h}_j$, the following relation holds:

$$ {}^{f}\hat{h}_j = \hat{A}{}^{i}\hat{h}_j \tag{73} $$

Similarly, ${}^{i}\hat{p}_j$ is transformed into ${}^{f}\hat{p}_j$ by means of

$$ {}^{f}\hat{p}_j = \hat{A}{}^{i}\hat{p}_j \tag{74} $$

Introduced the dual matrices

$$ {}^{m}\hat{H} = \begin{bmatrix} {}^{p}\hat{h}_1 & {}^{p}\hat{h}_2 & \ldots & {}^{p}\hat{h}_m \end{bmatrix} \quad (p = i, f) \tag{75} $$

and

$$ {}^{m}\hat{P} = \begin{bmatrix} {}^{p}\hat{p}_1 & {}^{p}\hat{p}_2 & \ldots & {}^{p}\hat{p}_n \end{bmatrix} \quad (p = i, f) \tag{76} $$

from (7) and (8) respectively follows

$$ {}^{f}\hat{H} = \hat{A}{}^{i}\hat{H} \tag{77a} $$

$$ {}^{f}\hat{P} = \hat{A}{}^{i}\hat{P} \tag{77b} $$

These conditions form an overdetermined system of equations where the elements of A are the unknowns.

Since,

$$ {}^{i}\hat{H}^{+}{}^{i}\hat{H} = I \tag{78} $$

the dual displacement matrix can be computed by means of the expression

$$ \hat{A} = {}^{f}\hat{H}\, {}^{i}\hat{H}^{+}, \tag{79} $$

when the lines features are prescribed or

$$ \hat{A} = {}^{f}\hat{P}\, {}^{i}\hat{P}^{+}, \tag{80} $$

when the points features are prescribed.

When both points and lines features are simultaneously prescribed (e.g. [17,18]), it is convenient to partition S matrix in submatrices ${}^{p}\hat{P}$ and ${}^{p}\hat{H}$

$$ {}^{p}\hat{S} = \begin{bmatrix} {}^{p}\hat{P} & {}^{p}\hat{H} \end{bmatrix} \quad (p = i, f) \tag{81} $$

and obtain the dual displacement matrix by means of

$$ \hat{A} = {}^{f}\hat{S}\, {}^{i}\hat{S}^{+} \tag{82} $$

The dual rotation angle $\hat{\vartheta} = \vartheta + \varepsilon s$ and the dual screw axis $\hat{u}$ are extracted from $\hat{A}$ by means of the algorithm outlined in [2].

If the prescribed features in the initial and final body configurations are not consistent with the hypothesis of rigidity, then the computed dual displacement matrix will be not orthogonal. In this case, to avoid iterative procedures as the one in [18], it is recommended to apply the dual QR decomposition [3]. Since $\hat{A} = \hat{Q}\hat{R}$ with $\hat{Q}$ dual orthogonal matrix and $\hat{R}$ upper dual triangular matrix, the screw parameters are extracted from $\hat{Q}$ instead of $\hat{A}$.

4.2. Rigid body motion: infinitesimally separated positions

In this case the initial and final positions are separated by an infinitesimal screw motion.

Let r_k and $\dot{r}_k$ $(k = 1, 2, \ldots, n)$ be the prescribed position and velocity vectors of a set n points of a rigid body, respectively. Differentiating Eq. (69) with respect to time t follows

$$\dot{g} = \frac{\sum_{k=1}^{n} \dot{r}_k}{n} \tag{83}$$

If we denote with

$$\hat{p}_k = (r_k - g) + \varepsilon g \times (r_k - g) = \begin{bmatrix} \hat{p}_{kx} \\ \hat{p}_{ky} \\ \hat{p}_{kz} \end{bmatrix} (k = 1, 2, \ldots, n) \tag{84}$$

then the dual velocity vectors representing the relative point velocities with respect to barycenter are [3]

$$\hat{v}_k = \frac{d\hat{p}_k}{dt} = (\dot{r}_k - \dot{g}) + \varepsilon \dot{g} \times (r_k - g) + \varepsilon g \times (\dot{r}_k - \dot{g})(k = 1, 2, \ldots, n) \tag{85}$$

Introducing the dual angular velocity vector $\hat{\omega}$, which characterizes the entire velocity field of body, the extension of Poisson's relation to dual vectors gives

$$\hat{v}_k = \hat{\omega} \times \hat{p}_k (k = 1, 2, \ldots, n) \tag{86}$$

The Eqs. (20) forms an overdetermined system of dual equations where the $\hat{\omega}$ components of are unknown. The solution of such system can be obtained using the dual generalized inverse herein discussed. In particular, the dual angular speed vector is simultaneously computed by means of the equation

$$\hat{\omega} = -\hat{T}^{+} \cdot \hat{v} \tag{87}$$

where $\hat{T}^{+}$ is the dual generalized inverse of matrix of

$$\hat{T} = \begin{bmatrix} \tilde{\hat{p}}_1 \\ \tilde{\hat{p}}_2 \\ \vdots \\ \tilde{\hat{p}}_n \end{bmatrix} \tag{88}$$

and

$$\tilde{\hat{p}}_k = \begin{bmatrix} 0 & -\hat{p}_{kz} & \hat{p}_{ky} \\ \hat{p}_{kz} & 0 & -\hat{p}_{kx} \\ -\hat{p}_{ky} & \hat{p}_{kx} & 0 \end{bmatrix} \tag{89}$$

skew symmetric matrix associated to dual vector $\hat{p}_k$,

$$\hat{v} = \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \vdots \\ \hat{v}_n \end{bmatrix} \tag{90}$$

4.3. Numerical example

4.3.1. Example 1

Let us assume that

$${}^{i}r_1 = [1 \quad 0 \quad 0]^T \quad {}^{i}r_2 = [0 \quad 1 \quad 0]^T \quad {}^{i}r_3 = [0 \quad 1 \quad 1]^T \quad {}^{i}r_4 = [0 \quad 0 \quad 1]^T$$

are the initial position vectors of points and

$${}^{f}r_1 = [2 \quad 0 \quad 0]^T \quad {}^{f}r_2 = [1 \quad 0 \quad 1]^T \quad {}^{f}r_3 = [1 \quad -1 \quad 1]^T \quad {}^{f}r_4 = [1 \quad -1 \quad 0]^T$$

the final position vectors of points.

With the previous numerical data, follows

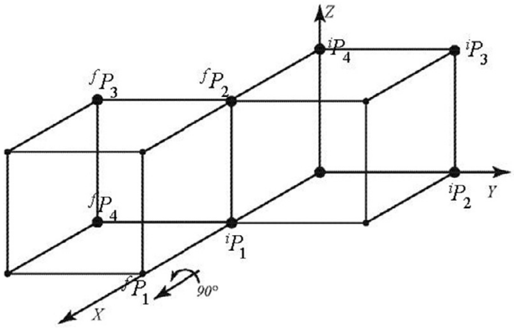

Fig. 1. Screw motion of a cube.

$$
{}^{i}\hat{P}^{+} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}
$$

and

$$
{}^{f}\hat{P} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & -\frac{3}{4} & -\frac{3}{4} & \frac{1}{2} \\ 1 & \frac{1}{2} & -\frac{3}{4} & -\frac{3}{4} \end{bmatrix}
$$

From (14), the final screw displacement dual matrix is deduced

$$
\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

Therefore, the dual unit vector of the screw axis is

$$
\hat{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

with dual angle

$$
\vartheta = \frac{\pi}{2} + \varepsilon
$$

The screw motion of the rigid body motion is shown in Fig. 1. Since matrix ${}^{i}\hat{P}$ is a 4×3 full rank matrix, consistently with Result 6, the formula expressed by Eq. (1) gives a {1,2,4} dual generalized inverse.

4.3.2. Example 2

Given three points vectors of a rigid body

$$
r_1 = [1 \quad 1 \quad 7]^T \quad r_2 = [4 \quad 7 \quad 1]^T \quad r_3 = [7 \quad 10 \quad 10]^T
$$

along with their velocity vectors

$$
v_1 = [7 \quad -5 \quad 1]^T \quad v_2 = [-5 \quad 4 \quad 4]^T \quad v_3 = [1 \quad -2 \quad 4]^T
$$

can be obtained the infinitesimal screw axis and the dual angular velocity. The numerical data of this problem are from Angeles [20]. They have been also used by Baroon & Ravani [21] to test their method based on a spatial extension of Reuleaux's construction.

With the given data, Eqs. (22) and (24) can be specialized, respectively, as follows:

$$
\hat{T} = \begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 3 \\ 5 & -3 & 0 \\ 0 & 5 & 1 \\ -5 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -4 & 4 \\ 4 & 0 & -3 \\ -4 & 3 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & 2 & -22 \\ -2 & 0 & -36 \\ 22 & 36 & 0 \\ 0 & -4 & 20 \\ 4 & 0 & 36 \\ -20 & -36 & 0 \\ 0 & 2 & 2 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \quad \hat{v} = \begin{bmatrix} 6 \\ -4 \\ -2 \\ -6 \\ 5 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 26 \\ 34 \\ -60 \\ -22 \\ -35 \\ 57 \\ -4 \\ 1 \\ 3 \end{bmatrix}
$$

Table 1
The position data of six points and six lines at two distinct configurations. (adapted from [18]).

At the initial configuration Lines ${}^{i}h$		
Point ${}^{i}r$	Lines ${}^{i}h$	Lines ${}^{i}h^{0}$
(8.0, 6.0, 2.0)	(0.1972, 0.1267, 0.9721)	(4.9874, −3.1136, −0.6058)
(4.0, 0.0, 4.0)	(0.2248, 0.5846, 0.7795)	(−8.3797, 0.4647, 2.0687)
(2.0, 11.0, −6.0)	(0.5765, 0.1627, 0.8006)	(−2.4020, −4.8040, 2.7061)
(5.0, 6.0, 7.0)	(0.0609, 0.0000, 0.9981)	(9.9813, −0.1829, −0.6099)
(4.0, 7.0, 9.0)	(−0.1937, −0.1549, 0.9687)	(4.998649, 9.493558, 2.518699)
(4.4, 3.0, 0.0)	(0.8899, −0.1001, −0.4449)	(2.0023, −21.8033, −0.9010)
At the initial configuration lines ${}^{i}h$		
Point ${}^{f}r$	Lines ${}^{f}h$	Lines ${}^{f}h^{0}$
(−7.3801, −14.8362, −5.6198)	(0.7386, 0.1980, −0.6443)	(−6.0525, −7.3061, −9.1842)
(−8.2420, 8.3324, −9.2195)	(0.8591, 0.4543, −0.2353)	(5.4624, −13.7875, −6.6742)
(−8.5317, 12.3697, 5.2240)	(0.5169, 0.5520, −0.6542)	(−0.6225, −13.6783, 12.0351)
(−2.7462, 12.0418, −7.7921)	(0.7240, 0.0162, −0.6895)	(−6.9688, −1.8993, −7.3627)
(−0.3589, 11.5897, −8.1025)	(0.6876, −0.2805, −0.6697)	(1.2463, −1.2115, 1.7870)
(−9.5559, 10.3030, −4.7979)	(−0.6616, 0.7414, −0.1119)	(14.8335, 14.6844, 9.5872)

Table 2
The imperfect position data of six points and six lines at the displaced configuration (adapted from [18]).

At the final configuration with 10% of random error		
Point ${}^{f}r$	Lines ${}^{f}h$	Lines ${}^{f}h^{0}$
(−7.0906, 14.6219, −5.9719)	(0.7442, 0.2015, −0.6367)	(−5.9570, −7.2802, −9.2668)
(−7.6305, 7.6276, −10.0261)	(0.8607, 0.4553, −0.2274)	(5.3797, −13.7610, 6.6875)
(−7.9875, 13.1092, 5.3780)	(0.4758, 0.5808, −0.6604)	(−0.4700, −13.4988, −12.2117)
(−2.5652, 13.1537, −8.2065)	(0.7236, 0.0194, −0.6898)	(−6.9683, −1.8993, −7.3631)
(−0.3259, 12.4703, −7.6607)	(0.6974, −0.2970, −0.6521)	(1.2022, −1.1932, 1.8290)
(−9.1206, 9.6892, −5.1000)	(−0.6743, 0.7303, −0.1088)	(14.6192, −14.9200, −9.5499)

From (21), making use of the formula for computing the dual generalized inverse, the dual angular velocity is readily obtained

$$\hat{\omega} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The corresponding screw parameters are as follows:

- $\hat{u} = [\begin{matrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{matrix}] + \varepsilon [\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}]$, unit dual vector of the infinitesimal screw axis;
- $\hat{\omega} = \dot{\vartheta} + \varepsilon \dot{s} = \sqrt{3} + \varepsilon\sqrt{3}$ norm of the dual angular velocity.

The results herein obtained match those of Angeles [20] and Baroon & Ravani [21].

4.3.3. Example 3

The coordinates of six points and six lines at the two finitely separated positions of a rigid body are prescribed. The screw parameters of the finite motion need to be computed.

The numerical data of this example, consistent with the hypothesis of rigid body, are summarized in Table 1 and taken from the work of Ge & Ravani [18].

By means of Eqs. (9) and (10), the dual matrices ${}^{p}\hat{H}$ and, respectively, are evaluated. Then, by means of (15), the matrices ${}^{p}\hat{S}(p = i, f)$ are assembled. Table 2.

Finally, the application of (16) gives the dual displacement matrix:

$$\hat{A} = \begin{bmatrix} -0.3045 & 0.5947 & 0.7440 \\ 0.8700 & 0.4917 & -0.0369 \\ -0.3877 & 0.6361 & -0.6671 \end{bmatrix} + \varepsilon \begin{bmatrix} 2.4113 & 5.6387 & -3.5199 \\ -2.3548 & 3.3873 & -10.3913 \\ -7.1809 & -7.8943 & -3.3562 \end{bmatrix}$$

Applying the procedure outlined in [3], the finite screw parameters can be extracted from A:

- the screw axis has direction $u = [0.5003 \quad 0.8413 \quad 0.2047]^T$;

- a point on the axis has coordinates $[-5.5573 \quad 3.3539 \quad -0.2034]^T$
- the rotation angle is $\vartheta = 2.4039 rad$ and the displacement along the screw axis is $s = -1.8210$

The correctness of the results can be verified by means of conditions (7) and (8).

4.3.4. Example 4

In this numerical example, whose data have been taken from the work of Ge & Ravani [18], some errors in the position data of the final positions of points and lines have been introduced.

Following the same procedure outlined in the previous example, one obtains the following tentative dual displacement matrix

$$\hat{A}^* = \begin{bmatrix} -0.3295 & 0.5561 & 0.7087 \\ 0.7718 & 0.6882 & 0.0565 \\ -0.4433 & 0.7340 & -0.6384 \end{bmatrix} + \varepsilon \begin{bmatrix} 1.2744 & 8.1531 & -2.8684 \\ -2.5173 & 3.0853 & -8.5439 \\ -6.1824 & -8.8202 & -2.8842 \end{bmatrix}$$

Since $\hat{A}^*$ does not satisfy the orthogonality condition, the dual QR decomposition is applied

$$\hat{A}^* = \hat{Q}\,\hat{R}$$

with

$$\hat{Q} = \begin{bmatrix} -0.3472 & -0.4910 & -0.7990 \\ 0.8132 & -0.5820 & 0.0043 \\ -0.4671 & -0.6482 & 0.6013 \end{bmatrix} + \varepsilon \begin{bmatrix} 1.4884 & -6.3504 & 3.2557 \\ -2.9934 & -4.1098 & 9.9842 \\ -6.3181 & 8.4995 & 4.2551 \end{bmatrix} \tag{91}$$

and

$$\hat{R} = \begin{bmatrix} 0.9491 & 0.0237 & 0.0981 \\ 0 & -11.494 & 0.0330 \\ 0 & 0 & -0.9499 \end{bmatrix} + \varepsilon \begin{bmatrix} 0.3981 & -20.722 & 0.3146 \\ -0.0000 & -0.2023 & -1.9087 \\ 0.0000 & 0.0000 & 0.6758 \end{bmatrix} \tag{92}$$

Assuming $\hat{A} = \hat{Q}$, the following finite parameters can be extracted from $\hat{A}$:

- the screw axis has direction $u = [-0.4363 \quad -0.2219 \quad 0.8720]^T$;
- a point on the axis has coordinates $[-6.0801 \quad 0.1136 \quad -3.0131]^T$;
- the rotation angle is $\vartheta = 2.2968 rad$ and the displacement along the screw axis is $s = -1.0923$.

The error in the rotation angle is about 4.5% whereas the one on displacement along the screw axis is about 50%. The results reported by Ge & Ravani [18] are affected by a much less error. However, in the computations numerical weights have been tuned to refine results accuracy.

5. Conclusions

Generalized inverses of dual matrices $\hat{A} = A + \varepsilon B$, with A an m by n matrix, have been investigated through a study of the properties of the matrix $G = A^+ - \varepsilon A^+ B A^+$ [2]. The n by m matrix A^+ is the Moore-Penrose generalized inverse of the matrix A. Sufficient conditions for the matrix G to be a dual generalized inverse (of various types) of the matrix $\hat{A}$ have been developed both when A is full rank and when it is rank deficient. The major contributions of this paper are the following.

1. The n by m matrix G is a {2}-dual generalized inverse of $\hat{A}$ for all m by n dual matrices $\hat{A}$.
2. When $m > n$ and the matrix A has full rank (i.e., rank r of A equals n), then the matrix G is a {1,2,4}-dual generalized inverse of $\hat{A}$ for all m by n matrices B. When $m < n$ and the matrix A has full rank (i.e., rank r of A equals m), then the matrix G is a {1,2,3}-dual generalized inverse of $\hat{A}$ for all m by n matrices B. For a square matrix, A, with full rank, G gives the {1,2,3,4}-dual generalized inverse of the dual matrix $\hat{A}$ for all m by n matrices B.
3. For general matrices A, when $null(A^T) \subseteq null(AB^T)$ the matrix G is a {3}-dual generalized inverse of the matrix $\hat{A}$. When $null(A) \subseteq null(A^T B)$ the matrix G is a {4}-dual generalized inverse of the matrix $\hat{A}$.
4. For general matrices A, when $null(A) \subseteq null(B)$, then the matrix G is a {1,2,4}-dual generalized inverse of the matrix $\hat{A}$. When the $null(A^T) \subseteq null(B^T)$, then the matrix G is a {1,2,3}-dual generalized inverse of the matrix $\hat{A}$. When $null(A) \subseteq null(B)$ and $null(A^T) \subseteq null(B^T)$, the matrix G is a {1,2,3,4}-dual generalized inverse of $\hat{A}$.

Several kinds of numerical examples to corroborate and test the analytical results are provided. Applications of the dual matrix generalized inverse to different kinematic analysis problems have been discussed.

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