# A Note on Nonproportional Damping

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**Abstract:** This note deals with three aspects of nonproportional damping in linear damped vibrating systems in which the stiffness and damping matrices are not restricted to being symmetric and positive definite. First, we give results on approximating a general damping matrix by one that commutes with the stiffness matrix when the stiffness matrix is a general diagonalizable matrix, and the damping and stiffness matrices do not commute. The criterion we use for carrying out this approximation is closeness in Euclidean norm between the actual damping matrix and its approximant. When the eigenvalues of the stiffness matrix are all distinct, the best approximant provides justification for the usual practice in structural analysis of disregarding the off-diagonal terms in the transformed damping matrix. However, when the eigenvalues of the stiffness matrix are not distinct, the best approximant to a general damping matrix turns out to be related to a block diagonal matrix, and the aforementioned approximation cannot be justified on the basis of the criterion used here. In this case, even when the damping and stiffness matrices commute, decoupling of the modes is not guaranteed. We show that for general matrices, even for symmetric ones, the response of the approximate system and the actual system can be widely different, in fact qualitatively so. Examples illustrating our results are provided. Second, we present some results related to the difficulty in handling general, nonproportionally damped systems, in which the damping matrix may be indefinite, by considering a simple example of a two degrees-of-freedom system. Last, we use this example to point out the nonintuitive response behavior of general nonproportionally damped systems when the damping matrix is indefinite. Our results point to the need for great caution in approximating nonproportionally damped systems by damping matrices that commute with the stiffness matrix, especially when considering general damping matrices. Such approximations could lead to qualitatively differing responses between the actual system and its proportionally damped approximation.

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## Introduction

We consider here the general system described by the linear matrix equation

$$M\ddot{x} + \tilde{C}\dot{x} + \tilde{K}x = f(t), \quad x(0) = x_0, \dot{x}(0) = \dot{x}_0 \tag{1}$$

where the $n$ by $n$ real matrix $M$ is assumed to be positive definite, and $x(t)$ is an $n$-vector. In structural analysis, the matrices $\tilde{K}$ and $\tilde{C}$ are usually assumed to be symmetric, positive definite, and real. Since $M$ is positive definite, we can scale the vector $x$ using the matrix $M^{1/2}$, so that for $y = M^{1/2}x$, Eq. (1) reduces to

$$\ddot{y} + C\dot{y} + Ky = M^{-1/2}f(t), \quad y(0) = M^{1/2}x_0, \dot{y}(0) = M^{1/2}\dot{x}_0 \tag{2}$$

where $C = M^{-1/2}\tilde{C}M^{-1/2}$ and $K = M^{-1/2}\tilde{K}M^{-1/2}$.

When the general matrices $C$ and $K$ are diagonalizable, the matrices $C$ and $K$ are simultaneously diagonalizable if and only if they commute (Horn and Johnson 1990). When the matrices $C$ and $K$ are symmetric matrices, they are both diagonalizable; furthermore, the matrix that simultaneously diagonalizes these symmetric (hemitian), commuting matrices, is orthogonal (unitary) (Horn and Johnson 1990).

Hence, if $C$ and $K$ are real, symmetric matrices that commute (see Caughey and O'Kelley 1960, 1965; Clough and Penzien 1993; and Rayleigh 1945), this allows us to make a coordinate transformation $z = U^T y$, where $U$ is an orthogonal matrix so that Eq. (2) now reduces to

$$\ddot{z} = \Lambda_C \dot{z} + \Lambda_K z = U^T M^{-1/2} f(t), z(0) = U^T M^{1/2} x_0, \dot{z}(0) = U^T M^{1/2} \dot{x}_0 \tag{3}$$

where $\Lambda_C = U^T C U$ and $\Lambda_K = U^T K U$ are each diagonal matrices.

Eq. (3) shows that the components of the $n$-vector $z(t)$ constitute a set of uncoupled ordinary differential equations that can be solved. We note that we do not need the matrices $K$ and/or $C$ to be positive definite for result of Eq. (3) to hold.

The solution $x(t)$ of Eq. (1) for a given set of initial conditions is then obtained, upon integrating the uncoupled Eq. (3) (analytically, and/or numerically) and is given by $x(t) = M^{-1/2}Uz(t) := Tz(t)$. This can be then rewritten as an expansion in the form

$$x(t) = \sum_{i=1}^{n} t_i z_i(t), \quad \text{with} \quad T^{-1}T = I, \tag{4}$$

where $t_i$ is the $i$-th column of the matrix $T$, and $z_i(t)$ is the $i$-th element of the $n$-vector $z(t)$.

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## General Damping and Stiffness Matrices That Do Not Commute

### Analytical Results

In what follows, we shall assume that the matrix $M$ is positive definite and so it will suffice for us to consider Eq. (2) as our starting point. We then have

$$\ddot{y} + D\dot{y} + Ky = M^{-1/2}f(t), y(0) = y_0, \dot{y}(0) = \dot{y}_0 \tag{5}$$

Furthermore, we shall no longer assume that the matrices $D$ and $K$ are real and symmetric, but only that the matrix $K$ is diagonalizable; that is, it has a full set of linearly independent eigenvectors.

Often, the two matrices $D$ and $K$ do not commute and in this paper we take up this case. The standard procedure usually adopted in structural analysis then is to simply use the similarity transformation $S$ so that $S^{-1}KS=\Lambda_k$, to obtain from [Eq. (5)] the equation

$$\ddot{z} + \Xi\dot{z} + \Lambda_k z = S^{-1}M^{-1/2}f(t), z(0) = S^{-1}y_0, \dot{z}(0) = S^{-1}\dot{y}_0 \tag{6}$$

by setting $y(t)=Sz(t)$. While the matrix $\Lambda_k$ in Eq. (6) is a diagonal matrix that has the eigenvalues of the matrix $K$ along its diagonal, the matrix $\Xi := [\varsigma_{ij}] = S^{-1}DS$ is now, in general, a full matrix. [We remind the reader that when $K$ is symmetric (hermitian), $S^{-1}=S^T(S^{-1}=S^*)$, and all the eigenvalues of $K$ are real.]

To contrast the work presented in this section with previous work on approximating a nonproportional damping matrix with one that commutes with $K$, we note that in this paper we allow the stiffness and damping matrices to have complex entries and to be nonsymmetric. Most previous work assumes that the mass, damping, and stiffness matrices are all real and positive definite (e.g., Caughey and O'Kelley 1965; Knowles 2006), oftentimes along with some additional restrictions on the properties of the damping matrix. For example, Shahruz and Ma (1988) deal with approximate decoupling of linear vibrating systems with $M$, $K$, $C>0$. The authors select a proportional damping matrix by choosing it so that the response of the proportionally damped system approximates as closely as possible (in $L^\infty$ norm) that of the nonproportionally damped system. They further restrict the class of damping and stiffness matrices to those in which neglect of the off diagonal terms (of the transformed proportional damping matrix) results in all the modes of the system being underdamped.

Nonsymmetric stiffness and damping matrices can arise when structures are actively controlled, and in areas like microdynamics (see Caughey and Ma 1993). Nonsymmetric stiffness matrices can also occur in structural dynamics when collocation methods are used to discretize continuous systems to obtain computationally manageable structural models. Damping matrices adduced from experimental measurements often turn out to be nonsymmetric, and are often used in modeling complex systems. Matrices with complex entries can arise when dealing with modeling systems that have structural and viscoelastic damping. While Caughey and Ma (1993) deal with the conditions for the simultaneous triangularization of real mass, stiffness, and damping matrices, in this paper we look at approximating a general damping matrix so that it commutes with a general stiffness matrix and is closest to the given damping matrix in the Euclidean (Frobenius) norm. This norm has been used before by Knowles (2006), and he considers real symmetric $K$ and $C$ matrices. The results in this section generalize those in Knowles (2006) to general nonsymmetric matrices $K$ and $C$. We also show that for systems with general $K$ and $C$ matrices, though one can find a damping matrix that commutes with a stiffness matrix $K$ and that is closest in Euclidean norm to a given damping matrix $C$, when this damping matrix is used the equations of motion may still remain coupled.

A common procedure in structural engineering, is to retain only the diagonal elements of $\Xi$, zero out the off-diagonal elements, and thereby obtain a new damping matrix, $\Xi_d$, for which the system is now uncoupled. This procedure is also widely followed in the utilization of experimental data to arrive at analytical models in both civil and aerospace engineering, thereby permitting classical modal analysis to be carried out for complex building structures in the presence of soil structure interaction, and the analysis of aircraft and spacecraft structures. One might want to know how good this approximation of the matrix $\Xi$ solely by its diagonal elements might be. To measure "closeness" of two matrices we shall use the Euclidean (Frobenius) norm (Horn and Johnson 1990), which is defined for any $n$ by $n$ matrix $A$ as $\|A\|^2=\Sigma_{i,j=1}^n|a_{ij}|^2=\mathrm{Trace}(AA^*)$, where the star on $A$ denotes the complex-conjugate transpose of $A$. While other matrix norms could be used, this particular norm in addition to having a simple, identifiable meaning lends itself to considerable analytical ease.

We give a brief account of the four main results to follow in this section. The first result aims to find that damping matrix that is closest to the given general damping matrix of a system in Euclidean norm, and that commutes with a given general diagonalizable $K$ matrix whose eigenvalues are distinct. The second result does the same thing for systems when the eigenvalues of $K$ are not distinct. Here, we show that the system obtained by minimizing the Euclidean norm may not still be decoupleable, and the condition under which this happens is given. The third result deals with systems with general damping matrices whose stiffness matrices are hermitian and do not have distinct eigenvalues. The last result deals with the simplest situation wherein both the stiffness and damping matrices are hermitian. It is this last situation that has been dealt with in Knowles (2006).

*Result 1.* When all the eigenvalues of the matrix $K$ are distinct, of all the matrices that commute with $\Lambda_k$, the matrix, $\Xi_d$, that comes "closest" to the matrix $\Xi$ in Euclidean norm, is obtained by simply deleting all the off-diagonal terms of $\Xi$.

*Proof:* Take any matrix $\hat{C}$ that commutes with $\Lambda_k$. Hence, we must have $\Lambda_k\hat{C}=\hat{C}\Lambda_k$, or

$$\lambda_i\hat{C}_{ij} = \hat{C}_{ij}\lambda_j \tag{7}$$

where $\lambda_i$=eigenvalues of $K$ and $\hat{C}_{ij}=(i,j)$-th element of $\hat{C}$. Relation Eq. (7) simply requires that $(\lambda_i-\lambda_j)\hat{C}_{ij}=0$, from which we find that $\hat{C}_{ij}=0$ for $i\neq j$, since the eigenvalues of $K$ are all assumed distinct. Thus any matrix $\hat{C}$ that commutes with $\Lambda_k$ must be diagonal. We want to now find that matrix $\hat{C}$ which comes closest to the matrix $\Xi=S^{-1}DS$ in Euclidean norm, that is, we want to find the diagonal elements, $\hat{C}_{ii}$, of the diagonal matrix $\hat{C}$ which minimize

$$\|\Xi-\hat{C}\|^2 = \sum_{i=1}^{n}|\varsigma_{ii}-\hat{C}_{ii}|^2 + \sum_{i,j,i\neq j}|\varsigma_{ij}|^2. \tag{8}$$

The values of $\hat{C}_{ii}$, $i=1,2,\ldots,n$ that minimize Eq. (8) are

$$\hat{C}_{ii} = \varsigma_{ii}, \quad i=1,\ldots,n. \tag{9}$$

That this is a global minimum is clear by observing that, for any $k$, changing $\hat{C}_{kk}$ to $\hat{C}_{kk}=\varsigma_{kk}+z_k$, for any $z_k\neq 0$ always causes the right hand side of Eq. (8)) to increase, because $|z_k|>0$. Thus, the matrix $\Xi_d$ that commutes with $\Lambda_k$ and comes closest in Euclidean norm to the matrix $\Xi$ is given by $\Xi_d=\mathrm{Diag}(\varsigma_{11},\varsigma_{22},\ldots,\varsigma_{nn})$. The minimum distance between the two matrices $\Xi$ and $\Xi_d$ is then simply given by

$$\|\Xi-\Xi_d\|^2=\sum_{i,j,i\neq j}|\varsigma_{ij}|^2. \tag{10}$$

*Remark 1.* The above result appears to justify the practice in structural engineering of replacing the matrix $\Xi$ in Eq. (6) by one that contains only its diagonal elements (thereby ignoring the off-diagonal terms of the matrix $\Xi$) so as to obtain a diagonal damping matrix, which then allows the equation to be uncoupled. But we shall shortly see that using our criterion, this may not be always justifiable.

We point out that this is exactly the same optimal proportional damping matrix—based on the criterion of minimizing the norm of the error in the response—arrived at by Sharuz and Ma (1988) in which the authors make the assumptions that (1) $M$, $K$, $C>0$, and further (2) that the approximating damping matrix yields, along with the given stiffness matrix, a system all of whose modes are underdamped.

As we shall see below (see "Nonproportional Damping in Simple Systems with Two Degrees-of-Freedom"), for general $K$ and $C$ matrices, the determination of the proportional damping matrix in this fashion can lead to qualitatively (not just quantitatively) different behavior between the approximate system and the system being approximated.

What if the eigenvalues of $K$ are not distinct? Our matrix $\hat{C}$ in the proof above again is chosen to be one from among those that commute with $\Lambda_k$. Let us assume, without loss of generality, that any multiple eigenvalues of $K$ occur contiguously on the main diagonal of $\Lambda_k$. The condition $\Lambda_k\hat{C}=\hat{C}\Lambda_k$ translates as before to relation Eq. (7), and we conclude that $\hat{C}_{ij}=0$ whenever $\lambda_i\neq\lambda_j$. Noting the ordering of $\lambda_i$ terms, the matrix $\hat{C}$ is then a block-diagonal matrix of the form

$$\hat{C}=\begin{bmatrix}\hat{C}_1 & & & & \\ & \hat{C}_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \hat{C}_k\end{bmatrix} \tag{11}$$

where there is one block, $\hat{C}_i$, for each different eigenvalue of $K$. Each $\hat{C}_i$ is square and has a size equal to the multiplicity of the eigenvalue of $K$ to which it corresponds. Hence, for a matrix $\hat{C}$ to commute with $\Lambda_k$, we require the matrix $\hat{C}$ to have the block-diagonal structure shown in Eq. (11). We now need to find the elements of the matrix $\hat{C}$ so that it is as close as possible, in Euclidean norm, to the matrix $\Xi:=[\varsigma_{ij}]=S^{-1}DS$. To do this, it is convenient to partition the matrix $\Xi$ (which is a full matrix, in general) in a manner similar to $\hat{C}$ so that each block along the diagonal of $\Xi$ has the same size as the corresponding block of the matrix $\hat{C}$. We then obtain

$$\Xi=\begin{bmatrix}\Xi_1 & \Xi_{12} & \cdot & \cdot & \Xi_{1k}\\ \Xi_{21} & \Xi_2 & \cdot & \cdot & \cdot\\ \cdot & & \cdot & & \\ & & & \cdot & \\ \Xi_{k1} & & & & \Xi_k\end{bmatrix}. \tag{12}$$

We thus need to find the submatrices $\hat{C}_i$ such that the Euclidean norm

$$\|\Xi-\hat{C}\|^2=\sum_{i=1}^{k}\|\Xi_i-\hat{C}_i\|^2+\sum_{i=1}^{k}\sum_{\substack{j=1\\ i\neq j}}^{k}\|\Xi_{ij}\|^2 \tag{13}$$

is a minimum. The choice of the matrices $\hat{C}_i$ to minimize the right hand side of Eq. (13) is given by

$$\hat{C}_i=\Xi_i,\quad i=1,\ldots,k. \tag{14}$$

Let us denote the matrix $\hat{C}$—with elements described by relation Eq. (14) which then gives the minimal Euclidean norm shown in Eq. (13)—by the block-diagonal matrix

$$\Xi_m=\begin{bmatrix}\Xi_1 & & & & \\ & \Xi_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \Xi_k\end{bmatrix}. \tag{15}$$

Note that the matrix $\Xi_m$ is obtained by including only those block-matrices $\Xi_i$ that lie along the diagonal of the matrix $\Xi$, each of whose size corresponds to the multiplicity of the corresponding eigenvalue of $K$. We thus find that the matrix that commutes with $\Lambda_k$ and that is closest in Euclidean norm to $\Xi$ is the matrix $\Xi_m$ (the subscript $m$ is used to remind one that we are now dealing with the case when $K$ has multiple eigenvalues) and that the Euclidean distance between these matrices is given simply by

$$\|\Xi-\Xi_m\|^2=\sum_{i=1}^{k}\sum_{\substack{j=1,\\ i\neq j}}^{k}\|\Xi_{ij}\|^2 \tag{16}$$

We then have the following result:

*Result 2.* When all the eigenvalues of the matrix $K$ are not distinct (but not *all* equal), of all the matrices that commute with $\Lambda_k$, the block-diagonal matrix, $\Xi_m$ described by Eq. (15) above comes closest in Euclidean norm to the matrix $\Xi:=[\varsigma_{ij}]=S^{-1}DS$. The size of each subblock $\Xi_i$ along the diagonal of the matrix $\Xi_m$ equals the multiplicity of the corresponding eigenvalue of $K$.

*Remark 2.* We note that the matrix $\hat{C}$ that comes closest in Euclidean norm to the matrix $\Xi$ and that commutes with the diagonal matrix $\Lambda_k$ is no longer a diagonal matrix when the eigenvalues of the matrix $K$ are repeated. Were we then to approximate the matrix $\Xi$ by the matrix $\Xi_m$ in Eq. (6), we would still have a coupled system of differential equations (see section "Numerical Examples"). If in addition, the matrix $\Xi_m$ turns out to be diagonalizable, then since, the matrices $\Xi_m$ and $\Lambda_k$ commute, they can be simultaneously diagonalized by a similarity transformation (Horn and Johnson 1990). Yet, as we shall show in "Numerical Examples," the response of the actual system may be widely different from that of the approximate system so found.

Our next two results are restricted to stiffness matrices, $K$, that are hermitian; hence, they are guaranteed to be diagonalizable, and have real eigenvalues.

*Result 3.* Let the matrix $K$ in Eq. (5) be hermitian, and let $D$ and $K$ not commute. If the eigenvalues of the matrix $K$ are all distinct, then of all the matrices that commute with the matrix $K$, the one that is closest in Euclidean norm to $D$ is given by

$$\hat{D} = U\Xi_d U^*, \tag{17}$$

where the unitary matrix $U$ is such that $\Lambda_k = U^*KU$ is a diagonal matrix, $\Xi := [\varsigma_{ij}] = U^*DU$, and $\Xi_d$ is obtained from $\Xi = U^*DU$ by deleting all the off diagonal elements of $\Xi$.

*Proof:* Since the matrix $K$ is hermitian, it is diagonalizable by a unitary matrix $U$. But since $D$ and $K$ do not commute, they cannot be simultaneously diagonalized by the same unitary matrix. Using Eq. (5), and setting $y(t) = Uz(t)$, we can diagonalize the matrix $K$ using the unitary matrix $U$, to get

$$\ddot{z} + \Xi\dot{z} + \Lambda_k z = U^* M^{-1/2} f(t), z(0) = U^* y_0, \dot{z}(0) = U^* \dot{y}_0,$$

where $\Lambda_k = U^*KU$, and $\Xi = U^*DU$.

Now let us consider any matrix $E$ that commutes with $K$. Since $E$ and $K$ commute, we find that $\Lambda_k$ must commute with $\hat{C} = U^*EU$, and vice versa. The remainder of the proof, using this $\hat{C}$ follows the proof of *Result 1*, until we conclude that the matrix $\Xi_d = \mathrm{Diag}(\varsigma_{11}, \varsigma_{22}, \ldots, \varsigma_{nn})$ minimizes the norm $\|\Xi - \hat{C}\|^2$ among all matrices $\hat{C}$ that commute with $\Lambda_k$. But

$$\begin{aligned} \|\Xi - \hat{C}\|^2 &= \mathrm{Trace}[(\Xi - \hat{C})(\Xi - \hat{C})^*] \\ &= \mathrm{Trace}[U(\Xi - \hat{C})U^* U(\Xi - \hat{C})^* U^*] \\ &= \mathrm{Trace}[(D - E)(D - E)^*] \end{aligned} \tag{18}$$

Now the minimum of the left hand side is obtained when $\hat{C} = \Xi_d$, as mentioned. And since, by Eq. (18), the minimum of the right-hand side must equal the minimum of the left-hand side (which occurs when $\hat{C} = \Xi_d$), the minimum of the right-hand side occurs when $E = U\Xi_d U^* := \hat{D}$.

*Remark 3.* We have therefore proved that when the matrix $K$ is hermitian with no multiple eigenvalues, the matrix that is closest to the matrix $D$ in Euclidean norm and that commutes with $K$ is the matrix $\hat{D} = U\Xi_d U^*$, where $\Xi_d$ is the diagonal matrix obtained by suppressing all the off-diagonal terms of the matrix $\Xi = U^*DU$. There may thus be justifiable grounds to approximate Eq. (5) by

$$\ddot{y} + \hat{D}\dot{y} + Ky = M^{-1/2} f(t), y(0) = y_0, \dot{y}(0) = \dot{y}_0 \tag{19}$$

The transformation $y(t) = Uz(t)$ where $U$ diagonalizes the matrix $K$, will yield an uncoupled system of equations

$$\ddot{z} + \Xi_d \dot{z} + \Lambda_k z = U^* M^{-1/2} f(t), z(0) = U^* y_0, \dot{z}(0) = U^* \dot{y}_0 \tag{20}$$

in which the matrices $\Lambda_k$ and $\Xi_d$ are diagonal.

*Result 4.* Let the matrix $K$ in Eq. (5) be hermitian, and let $D$ and $K$ not commute. If the eigenvalues of the matrix $K$ are not all distinct (and not all equal), then: of all the matrices that commute with the matrix $K$, the one that is closest in Euclidean norm to $D$ is given by

$$\hat{D} = U\Xi_m U^*, \tag{21}$$

where the unitary matrix $U$ is such that $\Lambda_k = U^*KU$ is a diagonal matrix, $\Xi = U^*DU$, and $\Xi_m$ is the block diagonal matrix obtained from $\Xi$ by partitioning it so that each diagonal block $\Xi_i$ [see Eq. (15)] is the size of the multiplicity of the corresponding eigenvalue of $K$, in the manner described in proving *Result 2*.

*Proof:* Take any matrix $E$ that belongs to the set that commutes with the matrix $K$. Using the unitary matrix $U$ to diagonalize $K$, we get, as before, $\Lambda_k = U^*KU$, where, without loss of generality, we assume that any multiple eigenvalues of $K$ occur contiguously on the main diagonal of $\Lambda_k$. Since $K$ and $E$ commute, $\Lambda_k$ and $\hat{C} = U^*EU$ must commute, (and vice versa) so that $\Lambda_k\hat{C} = \hat{C}\Lambda_k$. We now follow the same line of reasoning presented in proving *Result 2*, and we conclude that, of all matrices that commute with $\Lambda_k$ the one that is closest in Euclidean norm to the matrix $\Xi = U^*DU$ is given by the matrix $\Xi_m$ shown in Eq. (15). And again because

$$\|\Xi - \hat{C}\|^2 = \|D - E\|^2 \tag{22}$$

The minimum of the left-hand side is obtained when $\hat{C} = \Xi_m$, which causes $\hat{D} = U\Xi_m U^*$.

*Remark 4.* When the matrix $K$ is hermitian and has multiple eigenvalues, the matrix closest to the matrix $D$ in Euclidean norm that commutes with $K$ is the matrix $\hat{D} = U\Xi_m U^*$, where $\Xi_m$ is block-diagonal. $\Xi_m$ is obtained by considering the blocks along the diagonal of the matrix $U^*DU$, that correspond to the multiplicity of each of the eigenvalues of $K$ (see *Result 2*). There may thus be justifiable grounds to approximate Eq. (5) by

$$\ddot{y} + \hat{D}\dot{y} + Ky = M^{-1/2} f(t), y(0) = y_0, \dot{y}(0) = \dot{y}_0 \tag{23}$$

with $\hat{D} = U\Xi_m U^*$. The transformation $y(t) = Uz(t)$ where $U$ is unitary and diagonalizes the matrix $K$, will then yield the coupled system of equations

$$\ddot{z} + \Xi_m \dot{z} + \Lambda_k z = U^* M^{-1/2} f(t), z(0) = U^* y_0, \dot{z}(0) = U^* \dot{y}_0 \tag{24}$$

The matrices $\Xi_m$ and $\Lambda_k$ commute; therefore, if the matrix $\Xi_m$ is diagonalizable (see Example 3, Numerical Examples), then we can always uncouple the equations by a similarity transformation $S_1$ that simultaneously diagonalizes both $\Xi_m$ and $\Lambda_k$ (Horn and Johnson 1990). Assuming the diagonalizability of $\Xi_m$, we set $z(t) = S_1 w$ in the above equation, which then becomes

$$\ddot{w} + \Lambda_c \dot{w} + \Lambda_k w = S_1^{-1} U^* M^{-1/2} f(t), w(0) = S_1^{-1} U^* y_0, \dot{w}(0) = S_1^{-1} U^* \dot{y}_0 \tag{25}$$

where $\Lambda_c = S_1^{-1}\Xi_m S_1$. The transformation $S_1$ can be explicitly obtained as

$$S_1 = \begin{bmatrix} T_1 & & & \\ & T_2 & & \\ & & . & \\ & & & T_k \end{bmatrix} \tag{26}$$

where $\Xi_i T_i = T_i \Omega_i$, $i = 1, 2, \ldots, k$. The matrix $T_i$ corresponds to the diagonal-block $\Xi_i$ in the matrix $\Xi_m$ shown in Eq. (15), and the matrices $\Omega_i$ are diagonal. The diagonalizability of the matrix $\Xi_m$, guarantees the diagionalizability of each of its constituent diagonal blocks. Also, $S_1^{-1}\Lambda_k S = \Lambda_k$. Note that the transformation $y(t) = US_1 w(t)$ that takes us from the coupled system Eq. (5) to the uncoupled, approximate system Eq. (25) is not, in general, unitary.

*Remark 5.* When both the matrices $D$ and $K$ are hermitian, then the matrix $\Xi = U^*DU$ is hermitian, and hence so is $\Xi_m$. Since $\Xi_m$ is hermitian, it is guaranteed to be diagonalizable. Now we see that in Eq. (24) we have the two matrices $\Xi_m$ and $\Lambda_k$

and since both are hermitian (actually, the matrix $\Lambda_k$ is a real diagonal matrix), we can find a unitary matrix that simultaneously diagonalizes both of them. Thus the matrix $S_1$ in Eq. (25), which we shall now call $V$, becomes a unitary matrix, and, with $z(t)=Vw(t)$, Eq. (25) yields the uncoupled equation

$$\ddot{w} + \Lambda_c \dot{w} + \Lambda_k w = V^* U^* M^{-1/2} f(t), w(0) = V^* U^* y_0, \dot{w}(0) = V^* U^* \dot{y}_0 \tag{27}$$

The way we obtain the unitary transformation $V$ is similar to the way we found the matrix $S_1$. Since each block submatrix $\Xi_i$ in Eq. (15) is now hermitian, each block can be diagonalized by a unitary matrix $V_i$. An equation similar to Eq. (26) would therefore accrue where we replace $S_1$ by the unitary matrix $V$, and the diagonal blocks $T_i$ by the unitary blocks $V_i$ that diagonalize the block-hermitian submatrices $\Xi_i$, $i=1,2,\ldots,k$. The diagonal matrices $\Omega_i$ are, naturally, real now.

*Remark 6.* If the symmetric matrix $D$ is sign indefinite (that is, has one or more negative eigenvalues along with some positive eigenvalues), the matrix $\hat{D}$, closest in Euclidean norm to $D$, may turn out to be positive definite, so that the approximation obtained of $D$ would yield a different qualitative response. Hence, the use of such an approximation needs to be done with considerable care when general damping matrices are involved (see also "Nonproportional Damping in Simple Systems with Two Degrees-of-Freedom"). However, when the damping matrix $D$ is positive definite, the matrix $\hat{D}$ that is closest in Euclidean norm will also be positive definite, as can be shown by simply using the properties defining positive definite matrices.

### Numerical Examples

We present four simple examples illustrating the results obtained above.

*Example 1:* Let the damping and stiffness matrices be

$$D_1 = \begin{bmatrix} 0.2 & -0.05 & -0.01 \\ -0.05 & 0.3 & -0.02 \\ -0.05 & -0.1 & 0.3 \end{bmatrix} \text{ and } K_1 = \begin{bmatrix} 10 & -1 & -2 \\ -1 & 20 & -3 \\ -3 & -1 & 30 \end{bmatrix} \tag{28}$$

The matrix $K_1$ is diagonalizable (has linearly independent eigenvectors), and $K_1$ and $D_1$ do not commute. The (distinct) eigenvalues of $K_1$ are: 9.5531, 19.9182, and 30.5288; the corresponding eigenvectors that form the three columns of $S$ [see Eq. (6)] are: $[0.9791, 0.1369, 0.1504]^T$, $[0.113, -0.9915, -0.0647]^T$, and $[0.0806, 0.266, -0.9606]^T$.

The matrices

$$\Xi = S^{-1} D_1 S = \begin{bmatrix} 0.1714 & 0.0473 & -0.0144 \\ 0.0728 & 0.2901 & -0.007 \\ 0.1421 & -0.0775 & 0.3385 \end{bmatrix}$$

and

$$\Lambda_k = \begin{bmatrix} 9.5331 & & \\ & 19.9182 & \\ & & 30.5288 \end{bmatrix} \tag{29}$$

and so, by *Result 1*, the matrix that is closest to $\Xi$ in Euclidean norm and that commutes with $\Lambda_k$ is the matrix $\Xi_d$ that is obtained by suppressing all the off-diagonal terms of $\Xi$.

If the matrix $\Xi$ in Eq. (6) is approximated by $\Xi_d$, the uncoupled equation becomes

$$\ddot{z} + \Xi_d \dot{z} + \Lambda_k z = S^{-1} M^{-1/2} f(t), z(0) = S^{-1} y_0, \dot{z}(0) = S^{-1} \dot{y}_0 \tag{30}$$

*Example 2:* Consider the stiffness and damping matrices $K_2$ and $D_2$ given, respectively, by

$$K_2 = \begin{bmatrix} 9.8776 & -1.0701 & -1.1391 \\ -1.4039 & 19.7686 & -0.1593 \\ -1.5418 & -0.1643 & 19.7433 \end{bmatrix} \text{ and } D_2 = D_1 \tag{31}$$

where $D_1$ is defined in Eq. (28). The matrix $\Lambda_k$ in Eq. (6) is now given by

$$\Lambda_k = \begin{bmatrix} 9.5331 & & \\ & 19.9182 & \\ & & 19.9182 \end{bmatrix} \tag{32}$$

so $K_2$ has multiple eigenvalues which are 9.5531, 19.9182, and 19.9182; its eigenvectors are the same as those of matrix $K_1$ in the previous example. The matrix

$$\Xi = S^{-1} D S = \begin{bmatrix} 0.1714 & 0.0473 & -0.0144 \\ 0.0728 & 0.2901 & -0.007 \\ 0.1421 & -0.0775 & 0.3385 \end{bmatrix} \tag{33}$$

so the equation of motion Eq. (6), takes the form

$$\ddot{z} + \Xi \dot{z} + \Lambda_k z = S^{-1} M^{-1/2} f(t), z(0) = S^{-1} y_0, \dot{z}(0) = S^{-1} \dot{y}_0 \tag{34}$$

with $\Lambda_k$ and $\Xi$ given by Eqs. (32) and (33), respectively.

The matrix $\Xi_m$ that is closest to $\Xi$ in Euclidean norm and that commutes with $\Lambda_k$ is given, using *Result 2*, by

$$\Xi_m = \begin{bmatrix} 0.1714 & 0 & 0 \\ 0 & 0.2901 & -0.007 \\ 0 & -0.0775 & 0.3385 \end{bmatrix} \tag{35}$$

Note that now because $K_2$ has multiple eigenvalues, the matrix $\Xi_m$ is no longer obtained by simply ignoring all but the diagonal elements of $\Xi$. The matrix $\Xi_m$ is a block diagonal matrix, as described in Eq. (15). The approximate equation of motion obtained by using the matrix $\Xi_m$ to replace $\Xi$ then becomes

$$\ddot{z} + \Xi_m \dot{z} + \Lambda_k z = S^{-1} M^{-1/2} f(t), z(0) = S^{-1} y_0, \dot{z}(0) = S^{-1} \dot{y}_0 \tag{36}$$

which is a coupled system of equations. Since the matrix $\Xi_m$ given in Eq. (35) is diagonalizable, and the matrices $\Xi_m$ and $\Lambda_k$ commute, they can be simultaneously diagonalized (Horn and Johnson 1990). The matrix that simultaneously diagonalizes them can be found reasoning as follows (see also *Remark 4*). Since $\Xi_m$ can be diagonalized, its lower, right-hand-corner two by two submatrix can also be diagonalized. The matrix

$$T = \begin{bmatrix} -0.5974 & 0.1198 \\ -0.8020 & -0.9928 \end{bmatrix} \tag{37}$$

diagonalizes this submatrix. Hence, the matrix that simultaneously diagonalizes $\Xi_m$ and $\Lambda_k$ is then the block diagonal matrix

$$S_1 = \begin{bmatrix} 1 & \\ & T \end{bmatrix} \tag{38}$$

and setting $z(t)=S_1 w(t)$ we get the uncoupled equations

$$\begin{gathered} \ddot{w} + \Lambda_c \dot{w} + \Lambda_k w = S_1^{-1} S^{-1} M^{-1/2} f(t) \\ w(0) = S_1^{-1} S^{-1} y_0, \dot{w}(0) = S_1^{-1} S^{-1} \dot{y}_0 \end{gathered} \tag{39}$$

where $\Lambda_c = S_1^{-1} \Xi_m S_1 = \mathrm{Diag}(0.1714, 0.2808, 0.3479)$. Note that $\Lambda_k = S_1^{-1} \Lambda_k S_1$, which is given by Eq. (32).

Note that, in general, when the stiffness matrix is not hermitian, the eigenvalues and eigenvectors of $K$ may be complex, and we will need to use complex arithmetic.

*Example 3:* Let the stiffness matrix $K_3$ be the symmetric matrix

$$K_3 = \begin{bmatrix} 10.9927 & -2.6316 & -2.4339 \\ -2.6316 & 19.1423 & -0.7176 \\ -2.4339 & -0.7176 & 19.2545 \end{bmatrix} \quad (40)$$

and

$$D_3 = \begin{bmatrix} 0.16551 & -0.01632 & 0.02330 \\ -0.02896 & 0.297676 & 0.10561 \\ 0.02441 & -0.10614 & 0.10822 \end{bmatrix} \quad (41)$$

The eigenvalues of $K_3$ are again 9.5531, 19.9182, and 19.9182. The eigenvectors are orthonormal and the corresponding columns of the orthogonal matrix $U$ that diagonalizes $K_3$ are: $[0.9280, 0.2736, 0.2530]^T$, $[-0.00778, 0.69306, -0.72083]^T$, and $[0.3726, 0.6669, 0.64527]^T$. The matrix

$$\Xi = U^T D_3 U = \begin{bmatrix} 0.1714 & -0.03 & 0.02 \\ 0.04 & 0.2 & 0.2 \\ 0.01 & 0 & 0.2 \end{bmatrix} \quad (42)$$

and $\Lambda_k = U^T K_3 U$ is the same as that given in Eq. (32). Because the matrix $K$ has repeated eigenvalues, the matrix $\Xi_m$ that commutes with $\Lambda_k$ and that is closest in Euclidean norm to the matrix $\Xi$ is then given by

$$\Xi_m = U^T D_3 U = \begin{bmatrix} 0.1714 & 0 & 0 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix} \quad (43)$$

The matrices $\Lambda_k$ and $\Xi_m$ commute with one another, and the closest approximation in the Euclidean norm to the matrix $D_3$ that commutes with $K_3$ is given by

$$\hat{D} = U \Xi_m U^T = \begin{bmatrix} 0.1759 & -0.0083 & -0.00772 \\ -0.0589 & 0.2903 & 0.0875 \\ 0.0470 & -0.0981 & 0.10514 \end{bmatrix} \quad (44)$$

As seen from Eq. (43), the lower two by two right corner block in the matrix $\Xi_m$ shows us that it cannot be diagonalized since it is in Jordan form, and hence the matrix $\Xi_m$ is not diagonalizable. Thus, though our matrix $\Xi_m$ commutes with $\Lambda_k$ (and therefore $\hat{D}$ commutes with $K_3$) the system with the damping matrix $\hat{D}$ that is closest in Euclidean norm to the damping matrix $D_3$ still cannot be decoupled leaving us with the *coupled* set of equations

$$\ddot{z} + \Xi_m \dot{z} + \Lambda_k z = U^T M^{-1/2} f(t), z(0) = U^T y_0, \dot{z}(0) = U^T \dot{y}_0 \quad (45)$$

*Example 4:* As our last example we consider the simplest case when the stiffness matrix and the damping matrix are both symmetric. Let $K_4 = K_3$, where $K_3$ is given in Eq. (40). Let us take

$$D_4 = \begin{bmatrix} 0.2 & -0.05 & -0.15 \\ -0.05 & 0.3 & -0.02 \\ -0.15 & -0.02 & 0.3 \end{bmatrix} \quad (46)$$

Since $K_4 = K_3$ the eigenvalues of $K_4$ and the corresponding eigenvectors are the same as those given in Example 3. The symmetric matrix

$$\Xi = U^T D_4 U = \begin{bmatrix} 0.1153 & -0.0760 & 0.0674 \\ -0.0760 & 0.3675 & -0.0245 \\ 0.0674 & -0.0245 & 0.3173 \end{bmatrix} \quad (47)$$

Of all the matrices that commute with $\Lambda_k$ given in Eq. (32) the one that commutes and is closest in Euclidean norm to $\Xi$ is given by

$$\Xi_m = \begin{bmatrix} 0.1153 & 0 & 0 \\ 0 & 0.3675 & -0.0245 \\ 0 & -0.0245 & 0.3173 \end{bmatrix} \quad (48)$$

Note that the matrix $\Xi_m$ is now symmetric (see *Remark 5*), and hence diagonalizable. Since the symmetric matrices $\Xi_m$ and $\Lambda_k$ commute, we can find an orthogonal transformation that diagonalizes both of them. The orthogonal matrix that diagonalizes the right hand corner, two by two submatrix, in Eq. (48) is (see *Remark 5*)

$$V_1 = \begin{bmatrix} 0.3768 & 0.9263 \\ 0.9263 & -0.3768 \end{bmatrix} \quad (49)$$

Accordingly, the orthogonal matrix that diagonalizes both $\Xi_m$ and $\Lambda_k$ simultaneously is therefore given by

$$V_1 = \begin{bmatrix} 1 & \\ & V_1 \end{bmatrix} \quad (50)$$

Using the transformation $z(t) = Vw(t)$, the system of equations

$$\ddot{w} + \Lambda_c \dot{w} + \Lambda_k w = V^T U^T M^{-1/2} f(t), w(0) = V^T U^T y_0, \dot{w}(0) = V^T U^T \dot{y}_0 \quad (51)$$

becomes uncoupled with the diagonal matrix $\Lambda_c = \mathrm{Diag}(0.1153, 0.3073, 0.3774)$ and $\Lambda_k = \mathrm{Diag}(9.5531, 19.9182, 19.9182)$, as in Eq. (32).

Furthermore, since the matrix $K_4$ is real and symmetric, *Result 4* tells us that the matrix $\hat{D}$ that commutes with the matrix $K_4$ and is closest to the matrix $D_4$ in Euclidean norm is given by

$$\hat{D} = U E_m U^T = \begin{bmatrix} 0.1499 & -0.0565 & -0.0660 \\ -0.0565 & 0.3005 & 0.0070 \\ -0.0660 & 0.0070 & 0.3496 \end{bmatrix} \quad (52)$$

## Nonproportional Damping in Simple Systems with Two Degrees-of-Freedom

Structural analysis has traditionally concerned itself with systems for which the matrices $D$ and $K$ in Eq. (5) are both taken to be (symmetric and) positive definite, and when they both commute—the so-called classically damped situation. Such classically damped linear systems have been studied extensively over the years, and several useful results regarding their stability and the determination of their response, both analytically and computationally, have been established (Caughey and O'Kelley 1965; Udwadia 1992; Udwadia and Esfandiary 1990; Udwadia 1993; Knowles 2006). However, less attention has been paid to systems in which the matrices $D$ and $K$ are more general matrices. With the advent and technological feasibility of structural control, structural systems may not contain solely passive elements. Systems that are actively controlled, can, in many instances be interpreted as containing active elements that could feed energy into the system. Such systems appear more difficult to understand, and in this section we turn to their somewhat nonintuitive behavior by considering a simple two degrees-of-freedom system in which the damping matrix can be indefinite. We consider a simple example of the homogeneous system Eq. (5) described by

$$\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & -d \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_2 \\ -k_2 & k_3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (53)$$

where $d$, $k_1 > 0$. We purposely take the damping matrix, $D$, to be simply diagonal in order to expose the difficulties in intuitively explaining the response of this system. When $k_1 = k_2$, the system can be interpreted as the model of a two storey structure, except that the damping related to the lower storey mass is negative, presumably caused by, say, a positive velocity-feedback control force applied to it. In Eq. (53), the matrices $D$ and $K$ are easily identified by comparison with Eq. (5). We shall denote for convenience, $k_3 := k_1 + \alpha$, and $d_1 := d + \beta$; the latter makes Trace$(D) = \beta$. Despite the simplicity of this system, its response appears to be nonintuitive at several levels.

First, it may appear that since the damping related to the $y_2$ coordinate (because of the lower diagonal term in the damping matrix) is negative, the system should be unstable; this is incorrect, as we shall shortly see. Second, having realized this—that the system is not necessarily unstable—one might reorganize one's thinking to interpret the stability of the system as being plausibly caused by the possible presence of a sufficiently large and positive upper diagonal element, $d_1$, in the diagonal damping matrix $D$. Such a large enough positive value of $d_1$ in the damping matrix, one might think, would indeed affect the overall 'effective' damping in the system, because, after all, the system is a coupled system of equations, and the effect of the positive damping on the motion, $y_1$, of the top storey mass of the structure must naturally pervade throughout the entire structure. While this interpretation appears plausible, it too is not entirely correct. For this reasoning would imply that the more positive the value of $d_1$ is made, the more stable the coupled system will become; and again, this is incorrect.

Before we go further, we may note that if this dynamical system is to have an asymptotically stable equilibrium point, it must be dissipative (Pars 1972). That is, phase volumes must shrink as the system evolves in time. Expressing the general, homogeneous Eq. (5) in phase space, as

$$\begin{bmatrix} \dot{y} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}\begin{bmatrix} y \\ v \end{bmatrix}, \quad (54)$$

the time rate of change of the phase volume $\boldsymbol{\Delta}$ of this dynamical system is found to be given by the relation

$$\dot{\boldsymbol{\Delta}} = \mu \boldsymbol{\Delta} \quad (55)$$

where $\mu = -$Trace$(D)$. Then, for phase volumes to shrink as the dynamical system evolves, we must require Trace$(D) > 0$. Hence, a necessary condition for the homogeneous system Eq. (5) to have an asymptotically stable equilibrium point is that Trace$(D) > 0$.

Specializing this result to our example system described by Eq. (53), we must then have $\beta > 0$. Thus, the trace of the matrix $D$ appears to represent our intuitive notion of the overall 'effective' damping in the system, and yet, as we shall soon show, this is not the entire, or even the correct, picture of the system's response.

Were we to investigate sufficient (and necessary) conditions for the asymptotic stability of system Eq. (53), using the Routh-Hurwitz criterion, we would find that the following four conditions must be met:

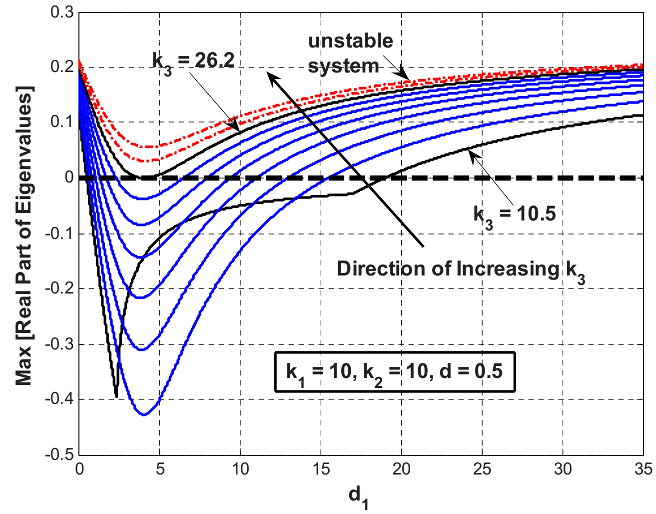


**Fig. 1.** Plot of the maximum of the real part of all the eigenvalues of system Eq. (53) with $k_1 = k_2 = 10$ and $d = 0.5$. Each curve represents a fixed value of $k_3$. The values of $k_3$ shown in the figure range from 10.5–30.71 in equal increments. For values of $d_1$ for which a curve goes below the dashed, dark line, the system is stable. For those values of $d_1$ for which a curve is above the dashed, dark line, the system is unstable. The two uppermost dash-dotted lines show behavior that is unstable for all values of $d_1$.

$$K > 0$$

$$\beta > 0$$

$$\gamma_1 := k_1 - d\left(\beta + \frac{\alpha}{\beta} + d\right) > 0$$

$$\gamma_2 := k_1 - \frac{\det(K)}{\gamma_1} + \frac{\alpha}{\beta} d + \alpha > 0 \quad (56)$$

where $\det(K)$ stands for determinant of the matrix $K$, which is given in Eq. (53). The failure of one or more of these conditions guarantees that the system will not be asymptotically stable. While the first two relations above might be physically interpreted as requiring, respectively, that the stiffness matrix be positive definite and the overall 'effective' damping (the trace of the damping matrix) be positive (as provisionally reasoned above), the remaining two conditions appear more difficult to interpret physically.

One can best observe what these two conditions represent by looking at results from a numerical simulation. We fix $d = 0.5$ and $k_1 = k_2 = 10$. For different values of $k_3$, we plot in Fig. 1 the maximum of the real part of all the eigenvalues of the system described by Eq. (53) as a function of $d_1$ ($x$-axis). Each curve is for a different value of $k_3$. Consider one such curve for some fixed value of $k_3$ that intersects the dark, dashed line. We observe that for values of $d_1$ that are positive and small, the system is unstable since the maximum value of the real part among all the eigenvalues is positive; this is as predicted by condition 2 of the Routh-Hurwitz criterion above [see Eq. (56)]. Furthermore, when the value of $d_1$ increases so that $\beta > 0$, the system can still be unstable, since $\beta > 0$ is only a necessary condition for asymptotic stability, as proved earlier. Eventually, as $d_1$ increases further, the system becomes stable, but it remains stable only over an interval of values of $d_1$. Each curve that intersects the dark, dashed line

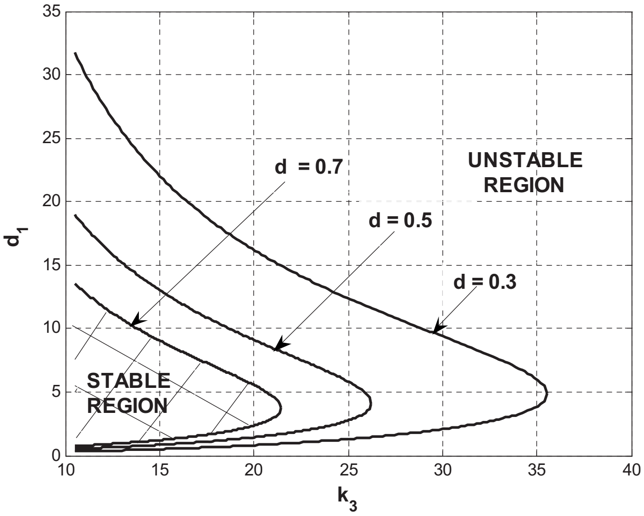


**Fig. 2.** Plot of $d_1$ versus $k_3$ using the same values of $k_1$ and $k_2$ as in Fig. 1, the regions of stability for three different values of $d$ are shown. The region to the right of each curve is the unstable region. The interval of $d_1$ over which stability exists reduces as the value of $d$ increases. The hatched region shows the stability zone for $d=0.7$.

shows that for large enough values of $d_1$ the system again becomes unstable! In fact, as seen from the two dash-dotted lines in the figure, when $k_3 \geqslant 26.22$ the system is unstable for all values of $d_1$. Our intuitive reasoning that increasing the value of $d_1$ pervasively increases the overall effective damping in the system, and thereby stabilizes the system, is thus incorrect, because large enough values of the damping $d_1$ also make the system unstable, as do very small values of $d_1$. In fact, as we saw, when $k_3 \geqslant 26.22$ no positive value of $d_1$ can render the system stable.

Fig. 2 shows the region of stability for the system for three different values of $d$. As the value of $d$ increases, for a given value of $k_3$ the interval of values of $d_1$ over which the system remains stable reduces. Notice that for each value of $d$, there is a value of $k_3$ beyond which the system is unstable, no matter what value of $d_1$ is chosen. Thus for $d=0.7$, and $k_3 \geqslant 21.5$ the system is unstable for all values of the damping $d_1$. Furthermore, we see from the figure that the system with $d=0.7$ is unstable when $k_3=20$ and $d_1=8$; to make the system stable, we need to decrease the damping (not increase it, as might be naively intuited!) so we enter the stable hatched region shown in the figure.

One way of understanding the somewhat interesting behavior of the system shown in Fig. 2 is to look at the energy input to the lower storey mass, and compare it with the energy dissipated by the damping at the upper storey mass of the structure. The input energy at time $t$ is proportional to $d\int_0^t \dot{y}_2^2 dt$, while the dissipated energy is proportional to $d_1\int_0^t \dot{y}_1^2 dt$. When $d_1$ is very small, the input energy exceeds the dissipated energy, and the system becomes unstable. As $d_1$ increases the dissipated energy increases and when it exceeds the input energy, the system becomes asymptotically stable, as shown by the second of the Routh-Hurwitz conditions in Eq. (56). However, when $d_1$ is further increased and becomes large enough, then $\dot{y}_1$ (which is a function of $d_1$, and this is the critical point!) becomes small, again causing the input energy to exceed the dissipated energy, and the system loses its stability. Similarly, as the strength of the lower storey is increased and its stiffness, $k_3$, increases, the frequency of oscillation of the system increases, and with it, $\dot{y}_2$. This eventually causes the input energy to exceed the energy dissipated, leading to instability, as seen in Fig. 1. And when $k_3$ is high enough, as seen in Fig. 2, no amount of damping in the upper storey can render the system stable. Thus increasing the stiffness $k_3$ of the lower storey beyond a point would lead to structural failure!

Lastly, we consider what might happen if we approximated the damping matrix $D$ in Eq. (53) by another matrix $\hat{D}$ that can be simultaneously diagonalized along with the stiffness matrix, and that is as close as possible in Euclidean norm to the diagonal (but simultaneously-undiagonalizable) damping matrix $D$. While at first this might appear to be an imminently reasonable thing to do, we show that it leads to a completely incorrect understanding of the system's behavior. We choose the parameters $d=0.3$, $d_1=16.4$, $k_1=k_2=10$, and $k_3=20$. For these values of the parameters, the system is unstable, as can be determined using the Routh-Hurwitz criterion (see Fig. 2). The eigenvalues of the stiffness matrix are distinct, and the proportional damping matrix that is closest in Euclidean norm to the damping matrix $D=\mathrm{Diag}(16.4,-0.3)$, is found, using the method in "General Damping and Stiffness Matrices That Do Not Commute," to be

$$\hat{D}=\begin{bmatrix} 9.6 & 3.3 \\ 3.3 & 6.3 \end{bmatrix} \tag{57}$$

which turns out to be a positive definite matrix!

Figs. 3(a and b) show the time histories of the response of the nonproportionally damped system Eq. (53) and its approximation using the proportional damping matrix $\hat{D}$. The initial conditions are taken to be $y_1(0)=3$, $y_2(0)=5$, and $\dot{y}_1(0)=\dot{y}_2(0)=0$. Fig. 3(a) confirms that, for the parameters chosen, the original nonproportionally damped system is unstable. However, we find that our approximated, proportionally damped system is stable!

While it has been known for some time now that the replacement of a nonproportionally damped system by one that is proportionally damped can lead to significant deviations in estimating the system's response (e.g., Udwadia and Esfandiary 1990), here we have a situation that appears somewhat more severe; for even the qualitative behavior of the two systems is different—the proportionally damped system that is closest in Euclidean norm to the nonproportionally damped system is stable, while the nonproportionally damped system that is being approximated is not.

## Conclusions and Remarks

In this note, we have looked at general damped linear systems with positive definite mass matrices in which the damping and stiffness matrices can be complex and nonsymmetric. Such matrices often arise in actively controlled structural systems. We obtain the approximation of a given damping matrix that cannot be simultaneously diagonalized along with a stiffness matrix, by one that is closest in Euclidean norm to the given damping matrix and that commutes with the stiffness matrix. Our results are applicable to general diagonalizable stiffness matrices.

We show that in arriving at this approximation, there is an essential distinction between when the eigenvalues of the stiffness matrix are distinct and when they are not. The usual approach used in structural engineering practice in both civil and aerospace engineering of neglecting the off diagonal terms obtained from a symmetric positive definite damping matrix can be supported, as shown, by the underlying idea that the resulting diagonal matrix is the closest matrix (in Euclidean norm) to the given damping matrix that can be simultaneously diagonalized with the stiffness

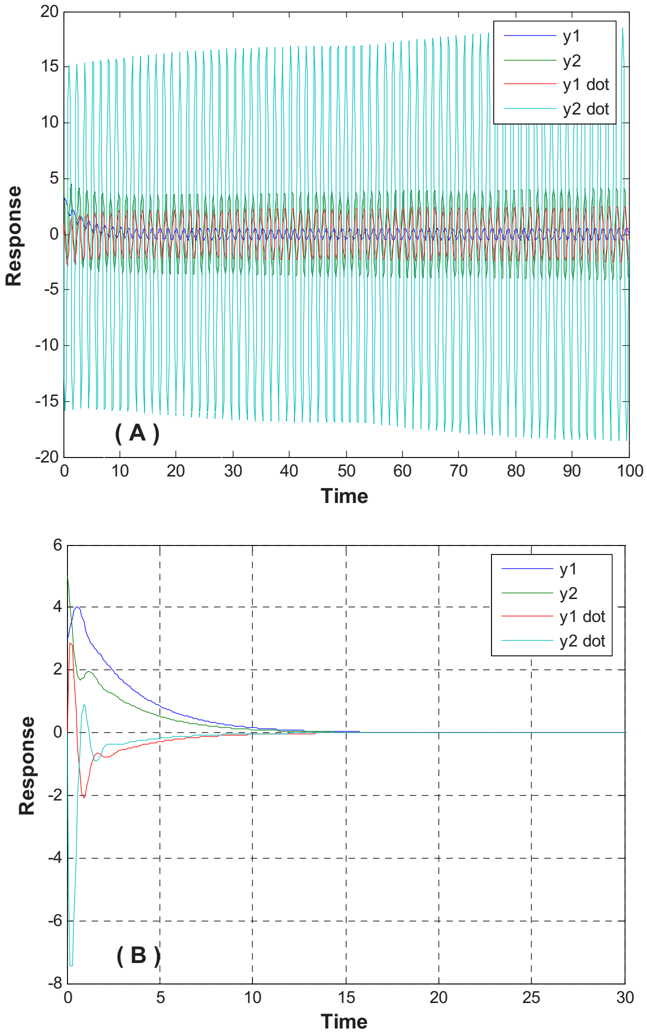


**Fig. 3.** (a) Response of the nonproportionally damped system showing unstable response; (b) response of the proportionally damped system using the damping matrix $\hat{D}$ given in Eq. (57) showing stable behavior

matrix. However, as seen in this paper, this conclusion does not necessarily follow when the eigenvalues of the stiffness matrix are multiple and when we have general stiffness and damping matrices. Interestingly, for systems with general damping and stiffness matrices approximate damping matrices that commute with the stiffness matrix can still leave the system of equations of motion coupled.

As pointed out in "Nonproportional Damping in Simple Systems with Two Degrees-of-Freedom," the approximate system for which the stiffness and damping matrices commute and whose damping matrix is closest in Euclidean norm to a given damping matrix can have a *qualitatively* different response from the actual system for general damping matrices, even symmetric ones. We specifically point out, by considering a simple two degrees-of-freedom system, the nonintuitive nature of the response of nonproportionally damped systems when we get away from our standard assumptions that the damping and stiffness matrices be both positive definite. This is especially shown to be so when such general systems may contain active elements, as commonly arise in the active control of structures. We show that their stability is more difficult to physically interpret and their approximation by damping matrices that commute with the stiffness matrices needs to be carried out with considerable care and caution.

As we move to the deployment of actively controlled structural systems, it is apparent that there is a great need for an improved understanding of the dynamics of linear systems with general stiffness and damping matrices.

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