Nonideal Constraints and Lagrangian Dynamics

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Abstract: This paper deals with mechanical systems subjected to a general class of non-ideal equality constraints. It provides the explicit equations of motion for such systems when subjected to such nonideal, holonomic and/or nonholonomic, constraints. It bases Lagrangian dynamics on a new and more general principle, of which D'Alembert's principle then becomes a special case applicable only when the constraints become ideal. By expanding its perview, it allows Lagrangian dynamics to be directly applicable to many situations of practical importance where non-ideal constraints arise, such as when there is sliding Coulomb friction.

INTRODUCTION

One of the central problems in the field of mechanics is the determination of the equations of motion pertinent to constrained systems. The problem dates at least as far back as Lagrange (1787), who devised the method of Lagrange multipliers specifically to handle constrained motion. Realizing that this approach is suitable to problem-specific situations, the basic problem of constrained motion has since been worked on intensively by numerous scientists, including Volterra, Boltzmann, Hamel, Novozhilov, Whittaker, and Synge, to name a few.

About 100 years after Lagrange, Gibbs (1879) and Appell (1899) independently devised what is today known as the Gibbs-Appell method for obtaining the equations of motion for constrained mechanical systems with nonintegrable equality constraints. The method relies on a felicitous choice of quasicoordinates and, like the Lagrange multiplier method, is amenable to problem-specific situations. The Gibbs-Appell approach relies on choosing certain quasicoordinates and eliminating others, thereby falling under the general category of elimination methods (Udwadia and Kalaba 1996). The central idea behind these elimination methods was again first developed by Lagrange when he introduced the concept of generalized coordinates. Yet, despite their discovery more than a century ago, the Gibbs-Appell equations were considered by many, up until very recently, to be at the pinnacle of our understanding of constrained motion; they have been referred to by Pars (1979) in his opus on analytical dynamics as ''probably the simplest and most comprehensive equations of motion so far discovered.''

Dirac considered Hamiltonian systems with constraints that were not explicitly dependent on time; he once more attacked the problem of determining the Lagrange multipliers of the Hamiltonian corresponding to the constrained dynamical system. By ingeniously extending the concept of Poisson brackets, he developed a method for determining these multipliers in a systematic manner through the repeated use of the consistency conditions (Dirac 1964; Sudarshan and Mukunda 1974). More recently, an explicit equation describing constrained motion of both conservative and nonconservative dynamical systems within the confines of classical mechanics was developed by Udwadia and Kalaba (1992). They used as their starting point Gauss's principle (1829) and considered general bilateral constraints that could be both nonlinear in the generalized velocities and displacements and explicitly dependent on time. Furthermore, their result does not require the constraints to be functionally independent.

All the above-mentioned methods for obtaining the equations of motion for constrained systems deal with *ideal* constraints, wherein the constraint forces do no work under virtual displacements. The motion of an unconstrained system is, in general, altered by the imposition of constraints; this alteration in the motion of the unconstrained system can be viewed at as being caused by the creation of additional ''forces of constraint'' brought into play through the imposition of these constraints. One view of the main task of analytical dynamics is that it gives a prescription for (uniquely) determining the accelerations of particles at any instant of time, given their masses, positions, and velocities; the kinematic constraints they need to satisfy; and the ''given'' (impressed) forces acting on them, at that instant. The properties of the constraint forces that are generated depend on the physical situation; these properties need to be provided in order to determine the particle accelerations. Usually, they come from experiments. The principle of D'Alembert, which was first stated in its generality by Lagrange (1787), assumes that the constraints are such that the forces of constraint do no work under virtual displacements. Such constraints are often referred to as ideal constraints and seem to work well in many practical situations. As pointed out by Lagrange, they provide a significant simplification, which enables a relatively easy description of the accelerations of the constrained system. This simplification arises because under this assumption only the ''given forces'' do work under virtual displacements; the total work done by the constraint forces is zero, and hence no forces of constraint appear in the relation dealing with the total work done on the system under virtual displacements. Additionally, from an algebraic standpoint, the assumption of ideal constraints happens to provide just the right amount of information for the accelerations of the constrained system to be *uniquely* determined (Udwadia and Kalaba 1996)—that is, the problem of finding the particle accelerations in the presence of ideal constraints is neither overdetermined nor underdetermined.

However, the assumption of ideal constraints excludes situations that often arise in practice. Indeed, such occurrences are commonplace in physics and engineering. Typically, the inclusion of nonideal constraint forces that do work under virtual displacements causes considerable difficulties in Lagrangian formulations; consequently, Lagrangian formulations of analytical dynamics *exclude* these sorts of constraints. Exactly how general, nonideal, equality constraints might be included within the framework of Lagrangian mechanics thus remains an open question today.

For example, the empirical sliding friction law suggested by Coulomb has been found to be useful in modeling many mechanical systems; such forces of sliding friction constitute constraint forces that indeed do work under virtual displacements. The special case of Coulomb friction can be handled in the Lagrangian framework, though in a roundabout way that resembles more Newtonian mechanics than Lagrangian mechan-

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ics (Rosenberg 1972). It requires a reformation of the Lagrangian approach by positing that the ''given'' forces are known functions of the constraint forces. Even after all this, as stated by Rosenberg (1972), ''Lagrangian mechanics is not a convenient vehicle for dealing with [friction forces].'' Goldstein (1972), in his treatment of Lagrangian dynamics, asserts that ''this [total work done by constraint forces equal to zero] is no longer true if sliding friction forces are present, and we must exclude such systems from our [Lagrangian] formulation.'' Moreover, as mentioned before, it leaves open the question of how one might handle within the Lagrangian framework *more general* forces of constraint that indeed do work under virtual displacements. In the 200-year history of analytical dynamics, this problem has resisted a direct assault so far as we know, because, unlike the ideal constraints situation, now the constraint forces must appear in the relations dealing with virtual work. More importantly, a major stumbling block has been the question of what sort of nonideal constraints yield a unique set of accelerations for a given unconstrained system.

In this paper, the writers obtain the explicit equations of motion for general, conservative and nonconservative, dynamical systems under the influence of a general class of nonideal bilateral constraints. We show that such nonideal constraints can be brought with simplicity and ease within the general framework of Lagrangian mechanics and prove that, like ideal constraints, they uniquely determine the accelerations of the constrained system of particles. By deriving the explicit equations of motion for nonideal equality constraints, we expand the perview of Lagrangian mechanics to include a much wider variety of situations that often arise in practice, including sliding Coulomb friction. Instead of D'Alembert's principle, Lagrangian mechanics now becomes rooted in a new principle, of which D'Alembert's principle becomes a special case. Three simple examples dealing with sliding friction and frictional drag are provided to illustrate the main result.

EQUATIONS OF MOTION

Consider first an unconstrained system of particles, each particle having a constant mass. By ''unconstrained,'' we mean that the number of generalized coordinates, n, needed to describe the configuration of the system at any time, t, equals the number of degrees of freedom of the system. The equation of motion for such a system can be written in the form

$$\mathbf{M}(q, t)\ddot{\mathbf{q}} = \mathbf{Q}(q, \dot{q}, t); \quad \mathbf{q}(0) = \mathbf{q}_0, \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \tag{1}$$

where $\mathbf{q}(t)$ is the n-vector (i.e., n by 1 vector) of generalized coordinates; $\mathbf{M}$ is an n by n symmetric, positive-definite matrix; $\mathbf{Q}$ is the known n-vector of impressed forces; and the dots refer to differentiation with respect to time. The acceleration, $\mathbf{a}$, of the unconstrained system at any time t is then given by the relation $\mathbf{a}(q, \dot{q}, t) = \mathbf{M}^{-1}\mathbf{Q}$. We shall assume that both $\mathbf{a}(t)$ and the displacement $\mathbf{q}(t)$ of the unconstrained system described by (1) are locally unique.

We shall assume that this system is subjected to a set of $m = h + s$ consistent constraints of the form

$$\boldsymbol{\varphi}(q, t) = 0 \tag{2}$$

and

$$\boldsymbol{\psi}(q, \dot{q}, t) = 0 \tag{3}$$

where $\boldsymbol{\varphi}$ is an h-vector and $\boldsymbol{\psi}$ is an s-vector. Furthermore, we shall assume that the initial conditions $\mathbf{q}_0$ and $\dot{\mathbf{q}}_0$ satisfy these constraint equations at time $t = 0$.

Assuming that (2) and (3) are sufficiently smooth, we differentiate (2) twice with respect to time, and (3) once with respect to time, to obtain the equation

$$\mathbf{A}(q, \dot{q}, t)\ddot{\mathbf{q}} = \mathbf{b}(q, \dot{q}, t) \tag{4}$$

where the matrix $\mathbf{A}$ is m by n; and $\mathbf{b}$ is a suitably defined m-vector that results from carrying out the differentiations. This set of constraint equations includes, among others, the usual holonomic, nonholonomic, scleronomic, rheonomic, catastatic, and acatastatic varieties of constraints; combinations of such constraints may also be permitted in (4). In the presence of the constraints, the number of degrees of freedom of the system is less than n. We shall resist the temptation to eliminate the redundant coordinates (and/or quasi-coordinates), a strategy that has customarily been used for the last 200 years. Instead, the underlying theme of our approach will be to determine an explicit equation for the acceleration vector $\ddot{\mathbf{q}}(t)$ of the constrained system at time t, given the vectors of generalized displacement, $\mathbf{q}(t)$; generalized velocity, $\dot{\mathbf{q}}(t)$; the given force, $\mathbf{Q}(q(t), \dot{q}(t), t)$; and the nature of the constraints described by (4) at time t.

Consider any instant of time t. When constraints are imposed at that instant of time on the unconstrained system, the motion of the unconstrained system is, in general, altered from what it might have been (at that instant of time) in the absence of these constraints. We view this alteration in the motion of the unconstrained system as being caused by an additional set of forces, called the ''forces of constraint,'' acting on the system at that instant of time. Since we shall be dealing with nonideal constraints, we shall refrain from defining this additional set of forces—called the forces of constraint—as the set of forces that do no work under virtual displacements, as has been the common practice in analytical dynamics (see, for example, Rosenberg 1972). The equation of motion of the constrained system can then be expressed as

$$\mathbf{M}(q, t)\ddot{\mathbf{q}} = \mathbf{Q}(q, \dot{q}, t) + \mathbf{Q}^c(q, \dot{q}, t), \mathbf{q}(0) = \mathbf{q}_0, \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \tag{5}$$

where the additional ''constraint force'' $\mathbf{Q}^c(q, \dot{q}, t)$ arises by virtue of the constraints (2) and (3) imposed on the unconstrained system, which is described by (1). Our aim is to determine $\mathbf{Q}^c$ explicitly at time t in terms of the known quantities $\mathbf{M}$, $\mathbf{Q}$, $\mathbf{A}$, and $\mathbf{b}$ and information about the nonideal nature of the constraint force, at time t.

Starting from the extended D'Alembert's principle (Udwadia et al. 1997), we shall obtain the constraint force $\mathbf{Q}^c$ explicitly, using ideal bilateral constraints; then, on the basis of a new principle, an explicit equation for a class of nonideal bilateral constraints will be obtained. In what follows, we will usually omit for the sake of brevity the explicit arguments of the various matrices and vectors.

A generalized virtual displacement at time instant t is defined as any nonzero infinitesimal n-vector, $\mathbf{v}$, which satisfies the relation (Udwadia and Kalaba 1996)

$$\mathbf{A}\mathbf{v} = 0 \tag{6}$$

Defining $\mathbf{u} = \mathbf{M}^{1/2}\mathbf{v}$, (6) is equivalent to the equation

$$\mathbf{B}\mathbf{u} = 0 \tag{7}$$

where the m by n matrix $\mathbf{B} = \mathbf{A}\mathbf{M}^{-1/2}$. Denoting $\ddot{\mathbf{r}} = \mathbf{M}^{1/2}\ddot{\mathbf{q}}$, (4) can be expressed as

$$\mathbf{B}\ddot{\mathbf{r}} = \mathbf{b} \tag{8}$$

The general solution of (8) is

$$\ddot{\mathbf{r}} = \mathbf{B}^+\mathbf{b} + (\mathbf{I} - \mathbf{B}^+\mathbf{B})\mathbf{w} \tag{9}$$

where the n by m matrix $\mathbf{B}^+$ is the Moore-Penrose (1955) inverse of the matrix $\mathbf{B}$; and $\mathbf{w}$ is any arbitrary n-vector. The first term on the right-hand side of (8) is known because both the vector $\mathbf{b}$ and the matrix $\mathbf{B}$ are known at time t; it therefore remains to determine the vector $(\mathbf{I} - \mathbf{B}^+\mathbf{B})\mathbf{w}$, based on the principles of mechanics.

Ideal Constraints

By ideal constraints we mean that the constrained system evolves in such a way that at each instant of time, the total work done by the constraint forces under virtual displacements is zero. This is an alternative statement of D'Alembert's principle; it forms the foundation of Lagrangian dynamics as we know it today. Using (4) and (5), this requires that for all nonzero vectors $\mathbf{v}$ such that $\mathbf{Av} = 0$ (Udwadia et al. 1997):

$$\mathbf{v}^T\mathbf{Q}^c = \mathbf{v}^T[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}] = 0 \tag{10}$$

Expressing (10) in terms of the previously defined vectors $\mathbf{u}$ and $\ddot{\mathbf{r}}$, and noting the equivalence between relations (6) and (7), we find that the acceleration $\ddot{\mathbf{r}}$ must be such that for all nonzero vectors $\mathbf{u}$ that satisfy the relation $\mathbf{Bu} = 0$, we must have

$$\mathbf{u}^T\mathbf{M}^{-1/2}\mathbf{Q}^c = \mathbf{u}^T\mathbf{M}^{-1/2}[\mathbf{M}^{1/2}\ddot{\mathbf{r}} - \mathbf{Q}] = \mathbf{u}^T[\ddot{\mathbf{r}} - \mathbf{M}^{-1/2}\mathbf{Q}] = 0 \tag{11}$$

Furthermore, the acceleration $\ddot{\mathbf{r}}$ must satisfy the constraints and must therefore satisfy (8); hence, it must be of the form given by (9). Thus, from (11) it follows that for all nonzero vectors $\mathbf{u}$ that satisfy the relation $\mathbf{Bu} = 0$, we require

$$\mathbf{u}^T[\mathbf{B}^+\mathbf{b} + (\mathbf{I} - \mathbf{B}^+\mathbf{B})\mathbf{w} - \mathbf{M}^{-1/2}\mathbf{Q}] = 0 \tag{12}$$

However, $\mathbf{Bu} = 0$ implies $\mathbf{u}^+\mathbf{B}^+ = 0$, which in turn implies $\mathbf{u}^T\mathbf{B}^+ = 0$, since $\mathbf{u} \neq 0$. Requirement (12) can then be rephrased as requiring that the vector $\mathbf{w}$ be such that

$$\mathbf{u}^T[\mathbf{w} - \mathbf{M}^{-1/2}\mathbf{Q}] = 0,$$

$$\forall \text{ nonzero vectors } \mathbf{u} \text{ that satisfy the relation } \mathbf{u}^T\mathbf{B}^T = 0 \tag{13}$$

This implies that $\mathbf{w}$ must be given by the relation

$$\mathbf{w} = \mathbf{M}^{-1/2}\mathbf{Q} + \mathbf{B}^T\mathbf{z} \tag{14}$$

where $\mathbf{z}$ is an arbitrary m-vector. Replacing $\mathbf{w}$ from (14) into (9) now yields

$$\ddot{\mathbf{r}} = \mathbf{B}^+\mathbf{b} + (\mathbf{I} - \mathbf{B}^+\mathbf{B})(\mathbf{M}^{-1/2}\mathbf{Q} + \mathbf{B}^T\mathbf{z}) \tag{15}$$

But the matrix $(\mathbf{I} - \mathbf{B}^+\mathbf{B})$ is symmetric; hence

$$[(\mathbf{I} - \mathbf{B}^+\mathbf{B})\mathbf{B}^T\mathbf{z}]^T = \mathbf{z}^T\mathbf{B}(\mathbf{I} - \mathbf{B}^+\mathbf{B}) = \mathbf{z}^T(\mathbf{B} - \mathbf{B}\mathbf{B}^+\mathbf{B})$$
$$= \mathbf{z}^T(\mathbf{B} - \mathbf{B}) = 0 \tag{16}$$

so that (15) reduces to

$$\ddot{\mathbf{r}} = \mathbf{M}^{-1/2}\mathbf{Q} + \mathbf{B}^+(\mathbf{b} - \mathbf{B}\mathbf{M}^{-1/2}\mathbf{Q}) \tag{17}$$

Noting that $\ddot{\mathbf{r}} = \mathbf{M}^{1/2}\ddot{\mathbf{q}}$ and $\mathbf{B} = \mathbf{AM}^{-1/2}$, (17) yields the equation of motion for the constrained system as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q} + \mathbf{M}^{1/2}(\mathbf{AM}^{-1/2})^+(\mathbf{b} - \mathbf{Aa}) \tag{18}$$

where $\mathbf{a}$ is the acceleration of the unconstrained system and is defined as $\mathbf{a} = \mathbf{M}^{-1}\mathbf{Q}$. We have thus obtained explicitly the constraint force at time t when the constraints are ideal as

$$\mathbf{Q}_i^c(q, \dot{q}, t) = \mathbf{M}^{1/2}(q, t)\{\mathbf{A}(q, \dot{q}, t)\mathbf{M}^{-1/2}(q, t)\}^+$$
$$\cdot\{\mathbf{b}(q, \dot{q}, t) - \mathbf{A}(q, \dot{q}, t)\mathbf{a}(q, \dot{q}, t)\} \tag{19}$$

The subscript i on $\mathbf{Q}^c$ on the left-hand side is used to explicitly indicate that the constraint force given by (19) is obtained under the assumption of ideal constraints. Eq. (19) was first derived by starting from Gauss's principle of least constraint (see Udwadia and Kalaba 1992).

Nonideal Constraints

We shall consider here a class of nonideal constraints. For such constraints we need to postulate a new principle that reduces to D'Alembert's principle when the constraints become ideal. We now state this principle as follows: The total work done at time t by the constraint forces $\mathbf{Q}^c$ under virtual displacements at time t is given by

$$\mathbf{v}^T\mathbf{Q}^c = \mathbf{v}^T\mathbf{C}(q, \dot{q}, t) \tag{20}$$

where $\mathbf{C}(q, \dot{q}, t)$ is a known, prescribed, sufficiently smooth n-vector (it needs only to be $\mathcal{C}^1$), and $\mathbf{v}$ is the virtual displacement n-vector at time t. When $\mathbf{C} \equiv 0$, this principle reduces to D'Alembert's principle, and all the constraints are ideal.

We thus require that for all nonzero vectors $\mathbf{v}$ such that $\mathbf{Av} = 0$:

$$\mathbf{v}^T\mathbf{Q}^c = \mathbf{v}^T[\mathbf{M}\ddot{\mathbf{q}} - \mathbf{Q}] = \mathbf{v}^T\mathbf{C}(q, \dot{q}, t) \tag{21}$$

As before, this reduces to the requirement that for all nonzero vectors $\mathbf{u} = \mathbf{M}^{1/2}\mathbf{v}$ such that $\mathbf{Bu} = 0$

$$\mathbf{u}^T[\ddot{\mathbf{r}} - \mathbf{M}^{-1/2}\mathbf{Q}] = \mathbf{u}^T[\mathbf{B}^+\mathbf{b} + (\mathbf{I} - \mathbf{B}^+\mathbf{B})\mathbf{w} - \mathbf{M}^{-1/2}\mathbf{Q}]$$
$$= \mathbf{u}^T\mathbf{M}^{-1/2}\mathbf{C}(q, \dot{q}, t) \tag{22}$$

Again noting that since $\mathbf{u} \neq 0$, $\mathbf{Bu} = 0$, implies $\mathbf{u}^T\mathbf{B}^+ = 0$, we obtain the requirement that

$$\mathbf{u}^T[\mathbf{w} - \mathbf{M}^{-1/2}\mathbf{Q} - \mathbf{M}^{-1/2}\mathbf{C}(q, \dot{q}, t)] = 0 \tag{23}$$

for all nonzero vectors $\mathbf{u}$ that satisfy the relation $\mathbf{u}^T\mathbf{B}^T = 0$. Hence, $\mathbf{w}$ is of the form

$$\mathbf{w} = \mathbf{M}^{-1/2}\mathbf{Q} + \mathbf{M}^{-1/2}\mathbf{C}(q, \dot{q}, t) + \mathbf{B}^T\mathbf{z} \tag{24}$$

Substitution of this $\mathbf{w}$ in (9) then gives, along the same lines as before:

$$\ddot{\mathbf{r}} = \mathbf{M}^{-1/2}\mathbf{Q} + \mathbf{B}^+(\mathbf{b} - \mathbf{BM}^{-1/2}\mathbf{Q}) + (\mathbf{I} - \mathbf{B}^+\mathbf{B})\mathbf{M}^{-1/2}\mathbf{C} \tag{25}$$

from which the equation of constrained motion is obtained as

$$\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q} + \mathbf{M}^{1/2}(\mathbf{AM}^{-1/2})^+(\mathbf{b} - \mathbf{Aa})$$
$$+ \mathbf{M}^{1/2}\{\mathbf{I} - (\mathbf{AM}^{-1/2})^+(\mathbf{AM}^{-1/2})\}\mathbf{M}^{-1/2}\mathbf{C} \tag{26}$$

Eq. (26) thus provides the explicit equation of motion for a dynamical system subjected to bilateral, holonomic and/or nonholonomic equality constraints that are nonideal. When $\mathbf{C} \equiv 0$, all the constraints become ideal and the third member on the right-hand side of (26) disappears, yielding (18).

We have also obtained explicitly the constraint force at time t when one or more of the constraints are nonideal. Noting (19), the total constraint force can be expressed as

$$\mathbf{Q}^c = \mathbf{Q}_i^c + \mathbf{Q}_{ni}^c \tag{27}$$

where the n-vector $\mathbf{Q}_{ni}^c$ is the contribution to the total constraint force from the presence of nonideal constraints; it is given by

$$\mathbf{Q}_{ni}^c = \mathbf{M}^{1/2}\{\mathbf{I} - (\mathbf{AM}^{-1/2})^+(\mathbf{AM}^{-1/2})\}\mathbf{M}^{-1/2}\mathbf{C}$$
$$= \mathbf{C} - \mathbf{M}^{1/2}(\mathbf{AM}^{-1/2})^+\mathbf{AM}^{-1}\mathbf{C} \tag{28}$$

We note that for all nonideal constraints of the form given by (20), we obtain the particle accelerations *uniquely*.

Eq. (27) shows that the total constraint force $\mathbf{Q}^c$ can be decomposed into the sum of two constraint force n-vectors, $\mathbf{Q}_i^c$ and $\mathbf{Q}_{ni}^c$. *The first of these is the constraint force that would have existed had all the constraints been ideal; the second may be thought of as a "correction term" to account for the presence of nonideal constraints.* In the presence of constraints, the first term $\mathbf{Q}_i^c$ in (27) is, in general, ever-present; the second appears only when one or more of the constraints is nonideal.

Example 1

Consider a bead of mass m moving under gravity on a straight wire that is inclined to the horizontal at an angle θ, $0 < \theta < (\pi/2)$. The unconstrained motion of the bead (in the absence of the constraint imposed on its motion by the wire) is given by

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} \tag{29}$$

where the x-direction is taken along the horizontal and the y-direction is taken pointing upwards. The vector on the right-hand side of (29) represents the given force $\mathbf{Q}$. The wire constraint can be described by the equation

$$y = x \tan \theta \tag{30}$$

which, upon two differentiations with respect to time, yields

$$\ddot{y} = \ddot{x} \tan \theta \tag{31}$$

Since this can be written as

$$[-\tan \theta \quad 1] \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = 0 \tag{32}$$

the matrix $\mathbf{A} = [-\tan \theta \quad 1]$, and

$$(\mathbf{AM}^{-1/2})^+ = m^{1/2} \cos^2\theta \begin{bmatrix} -\tan \theta \\ 1 \end{bmatrix}$$

and the scalar $\mathbf{b} = 0$.

Were the constraint represented by (30) assumed to be ideal, the equation of motion for the system, by (18), would be

$$m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -mg \end{bmatrix} + mg \begin{bmatrix} -\sin \theta \cos \theta \\ \cos^2\theta \end{bmatrix} \tag{33}$$

The second member on the right-hand side indicates explicitly the constraint force $\mathbf{Q}_i^c$ generated by the ideal constraint represented by (30). The magnitude of this constraint force is $mg \cos \theta$.

Were we to include Coulomb friction along the inclined wire with a coefficient of friction μ, the constraint will no longer be ideal; the work done by the constraint force under any virtual displacement $\mathbf{v}$ can then be represented as

$$\mathbf{v}^T\mathbf{Q}^c = \mathbf{v}^T\mathbf{C} \equiv -\mathbf{v}^T \frac{\mu}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} mg \cos \theta \tag{34}$$

Relation (34) states that the frictional force acts along the constraint, in a direction opposing the motion, and has a magnitude of $\mu|\mathbf{Q}_i^c|$. We note that by virtue of (30), $\dot{y} = \dot{x} \tan \theta$, so that the vector

$$\mathbf{C} = -\mu mg \cos \theta \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \operatorname{sgn}(\dot{x}) \tag{35}$$

The nonideal constraint given by (34) now provides an additional constraint force given by

$$\begin{aligned} \mathbf{Q}_{ni}^c &= \mathbf{M}^{1/2}\{\mathbf{I} - (\mathbf{AM}^{-1/2})^+(\mathbf{AM}^{-1/2})\}\mathbf{M}^{-1/2}\mathbf{C} \\ &= \left\{\mathbf{I} - \begin{bmatrix} \sin^2\theta & -\sin \theta \cos \theta \\ -\sin \theta \cos \theta & \cos^2\theta \end{bmatrix}\right\} \mathbf{C} \\ &= -\begin{bmatrix} \mu mg \cos^2\theta \\ \mu mg \cos \theta \sin \theta \end{bmatrix} \operatorname{sgn}(\dot{x}) \end{aligned} \tag{36}$$

so that the constrained equation of motion is given by

$$\begin{aligned} m \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} &= \begin{bmatrix} 0 \\ -mg \end{bmatrix} + mg \begin{bmatrix} -\sin \theta \cos \theta \\ \cos^2\theta \end{bmatrix} \\ &\quad - \mu mg \begin{bmatrix} \cos^2\theta \\ \cos \theta \sin \theta \end{bmatrix} \operatorname{sgn}(\dot{x}) \end{aligned} \tag{37}$$

where we have explicitly shown on the right-hand side the three different constituents of the forces acting on the system. The first term corresponds to the given forces $\mathbf{Q}$; the second corresponds to the force $\mathbf{Q}_i^c$ generated by the presence of the constraint given by (30), were it an ideal constraint; and the third corresponds to the additional force $\mathbf{Q}_{ni}^c$ generated by the presence of the nonideal constraint given by (30), whose nature is further described by (34). Equations of motion using more general descriptions of the frictional forces can be obtained in a similar manner.

It should be pointed out that $\mathbf{Q}_{ni}^c$ and the equation of motion for the constrained system could have been directly written down without the simplifications presented in (35) by using (28) and (26), wherein the vector $\mathbf{C}$ is given by

$$\mathbf{C} = -\frac{\mu}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} mg \cos \theta$$

Example 2

Consider a particle of unit mass constrained to move in a circle in the vertical plane on a circular ring of radius R under the action of gravity. The unconstrained motion of the particle is given by

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} \tag{38}$$

and the constraint is represented by $x^2 + y^2 = R^2$, which, upon two differentiations with respect to time, becomes

$$[x \quad y] \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -(\dot{x}^2 + \dot{y}^2) \tag{39}$$

so that $\mathbf{A} = [x \quad y]$, and

$$\mathbf{A}^+ = \frac{1}{R^2} \begin{bmatrix} x \\ y \end{bmatrix}$$

Were this constraint to be ideal, the force of constraint $\mathbf{Q}_i^c$ would be given by (18) so that

$$\mathbf{Q}_i^c = -\begin{bmatrix} x/R \\ y/R \end{bmatrix} \frac{(\dot{x}^2 + \dot{y}^2 - gy)}{R} \tag{40}$$

and the equation of motion of the constrained system becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} - \begin{bmatrix} x/R \\ y/R \end{bmatrix} \frac{(\dot{x}^2 + \dot{y}^2 - gy)}{R} \tag{41}$$

The magnitude of this constraint force, had the constraint been ideal, is given by

$$|\mathbf{Q}_i^c| = \left| \frac{(\dot{x}^2 + \dot{y}^2 - gy)}{R} \right|$$

Let the nature of the nonideal constraint generated by sliding friction between the ring and the mass be described by

$$\mathbf{v}^T\mathbf{Q}^c = \mathbf{v}^T\mathbf{C} \equiv -\mathbf{v}^T \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \frac{\mu|\mathbf{Q}_i^c|}{\sqrt{(\dot{x}^2 + \dot{y}^2)}} \tag{42}$$

Along the circular trajectory of the particle, $x\dot{x} = -y\dot{y}$, and we get

$$\mathbf{C} = -\frac{\mu|\mathbf{Q}_i^c|}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\frac{\mu|\mathbf{Q}_i^c|}{R} \begin{bmatrix} -y \operatorname{sgn}(x) \\ |x| \end{bmatrix} \operatorname{sgn}(\dot{y}) \tag{43}$$

The contribution to the constraint force provided by this nonideal constraint is then given, by using (28) (note: $\mathbf{M} = \mathbf{I}_2$) as

$$\begin{aligned} \mathbf{Q}_{ni}^c &= \{\mathbf{I} - \mathbf{A}^+\mathbf{A}\}\mathbf{C} = -\frac{\mu|\mathbf{Q}_i^c|}{R} \left\{ I - \frac{1}{R^2} \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \right\} \\ &\quad \cdot \begin{bmatrix} -y \operatorname{sgn}(x) \\ |x| \end{bmatrix} \operatorname{sgn}(\dot{y}) = -\mu|\mathbf{Q}_i^c| \begin{bmatrix} -y \operatorname{sgn}(x)/R \\ |x|/R \end{bmatrix} \operatorname{sgn}(\dot{y}) \end{aligned} \tag{44}$$

The equation of motion for the particle then becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} - \begin{bmatrix} x/R \\ y/R \end{bmatrix} \frac{(\dot{x}^2 + \dot{y}^2 - gy)}{R} - \mu|\mathbf{Q}_i^c| \begin{bmatrix} -y\ \mathrm{sgn}(x)/R \\ x\ \mathrm{sgn}(x)/R \end{bmatrix} \mathrm{sgn}(\dot{y}) \quad (45)$$

As before, we could have directly used the relation for **C** [given by the first equality in (43)] in (26) to obtain the equation of motion of the nonideally constrained system.

Example 3

We consider next a particle of unit mass, moving with respect to an inertial frame of reference, subjected to the given forces $f_x(x, y, z, t)$, $f_y(x, y, z, t)$, and $f_z(x, y, z, t)$ acting on it in the x-, y-, and z-directions, respectively. The particle is subjected to the nonholonomic, nonideal, constraint

$$\dot{y} = z\dot{x} \quad (46)$$

where the work done by the constraint force $\mathbf{Q}^c$ in a virtual displacement $\mathbf{v}$ is given by

$$\mathbf{v}^T\mathbf{Q}^c = -\mathbf{v}^T \left(a_0 v^2 \frac{v}{|v|} \right) \quad (47)$$

Here, v is the velocity of the particle as it executes its constrained motion; and a_0 is a given constant. Note that the nonideal constraint force $\mathbf{Q}^c$ is brought into play only because the particle is required to satisfy the nonholonomic, kinematical constraint given by (46). We shall assume that the initial position and velocity of the particle is provided and that it satisfies the constraint equation (46). Our aim is to determine the equation of motion for the particle in the presence of the nonideal, nonholonomic constraint described by (46) and (47).

The unconstrained equation of motion of the particle is then given by ($\mathbf{M} = \mathbf{I}$)

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \mathbf{Q} \quad (48)$$

Differentiating (46), (4) yields

$$\mathbf{A} = [-z \quad 1 \quad 0], \ \mathbf{b} = \dot{x}\dot{z} \quad (49)$$

and (19) gives the contribution to the total constraint force, were the nonholonomic constraint to be ideal, as

$$\mathbf{Q}_i^c = [-z \quad 1 \quad 0]^T \frac{(\dot{x}\dot{z} + z f_x - f_y)}{(1 + z^2)} \quad (50)$$

Additionally, the contribution to the total constraint force provided by virtue of the nonholonomic constraint being nonideal is given by (28) as

$$\mathbf{Q}_{ni}^c = -a_0[\dot{x} + z\dot{y} \quad z\dot{x} + z^2\dot{y} \quad \dot{z}(1 + z^2)]^T \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}}{(1 + z^2)} \quad (51)$$

Finally, the equation that describes the motion of the particle with the nonholonomic, nonideal constraint [as described by (46) and (47)] is then simply given by

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \mathbf{Q} + \mathbf{Q}_i^c + \mathbf{Q}_{ni}^c = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} + \frac{(\dot{x}\dot{z} + z f_x - f_y)}{(1 + z^2)} \begin{bmatrix} -z \\ 1 \\ 0 \end{bmatrix} - a_0 \begin{bmatrix} \dot{x} + z\dot{y} \\ z\dot{x} + z^2\dot{y} \\ \dot{z}(1 + z^2) \end{bmatrix} \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}}{(1 + z^2)} \quad (52)$$

For this nonideal, nonholonomic system, we thus obtain an easy decomposition of the right-hand side in terms of the given force (vector), $\mathbf{Q}$; the force of constraint that would have been generated were the nonholonomic constraint ideal, $\mathbf{Q}_i^c$; and the addition force, $\mathbf{Q}_{ni}^c$, engendered because the constraint is not ideal. Such a decomposition of the acceleration of the particle often assists in understanding the physics of the problem.

CONCLUSIONS

This paper deals with determining the explicit equation of motion for a constrained dynamic system where the forces of constraint satisfy a more general principle than that first enunciated by D'Alembert and formalized by Lagrange (1787). We state this principle as follows.

Consider an unconstrained system with n degrees of freedom. Let the system be subjected to general holonomic and nonholonomic constraints. At each instant of time t, the virtual work, $\mathbf{v}^T\mathbf{Q}^c$, done by the force-of-constraint n-vector, $\mathbf{Q}^c$, under any virtual displacement n-vector, $\mathbf{v}$, is given by $\mathbf{v}^T\mathbf{C}(q, \dot{q}, t)$, where $\mathbf{C}(q, \dot{q}, t)$ is a sufficiently smooth (at least $\mathbf{C}^1$) n-vector function of its arguments.

The following points may be noted relevant to the equation of motion obtained, and to the above-mentioned principle:

1. The principle generalizes D'Alembert's principle for bilateral constraints to situations when the constraint forces do work under virtual displacements. It encompasses situations such as sliding Coulomb friction, and many other types of constraint forces.
2. When the function $\mathbf{C}(q, \dot{q}, t)$ is identically zero, the principle stated above reduces to D'Alembert's principle, and the equation of motion of the constrained system reverts back to the equation known earlier (Udwadia and Kalaba 1996).
3. A deeper understanding of the underlying physics is obtained. The total constraint force $\mathbf{Q}^c$ [see (26)–(28)] is seen to be made up of two additive contributions. The first contribution, $\mathbf{Q}_i^c$, to the total constraint force comes from consideration of the constraints as though they were ideal; the second term comes from the fact that one or more of the constraints are not ideal and $\mathbf{C}(q, \dot{q}, t)$ is not identically zero.
4. We have obtained a simple, explicit equation of motion for a general conservative or nonconservative system subjected to holonomic and/or nonholonomic bilateral constraints that may be nonideal.
5. For any given sufficiently smooth n-vector $\mathbf{C}(q, \dot{q}, t)$, the acceleration vector of the constrained system is uniquely determined and the trajectory is locally unique.
6. No elimination of coordinates or quasi-coordinates (as required by the Gibbs-Appell approach) is undertaken. The equations of motion pertinent to the constrained system with nonideal holonomic and/or nonholonomic bilateral constraints are obtained in the same set of coordinates which are used to describe the unconstrained system, thereby showing simply and explicitly the effect of the addition of constraints on the equations of motion of the unconstrained system.

Since its inception, Lagrangian mechanics has been built upon the underlying principle of D'Alembert. This principle makes the confining assumption that all constraints are ideal constraints for which the sum total of the work done by the forces of constraint under virtual displacements is zero. Though often applicable, experiments show that this assumption may be invalid in many practical situations, such as when sliding friction is important. This paper releases Lagrangian mechanics from this confinement and obtains the explicit equations of motion allowing for holonomic and/or nonholonomic bilateral constraints that are nonideal. The explicit equations of motion obtained here are accordingly based on a more general principle, which then includes D'Alembert's principle as a special case when the constraints are ideal.

APPENDIX. REFERENCES

Appell, P. (1899). ''Sur une forme generale des equations de la dynamique.'' *C. R. Acad. Sci.*, Paris, 129, 459–460 (in French).

Dirac, P. A. M. (1964). *Lectures in quantum mechanics*. Yashiva University, New York.

Gauss, C. (1829). ''Uber ein neues allgemeines grundgesetz der mechanik.'' *J. Reine Angewandte Mathematik*, 4, 232–235 (in German).

Gibbs, W. (1879). ''On the fundamental formulae of dynamics.'' *Am. J. Math.*, 2, 49–64.

Goldstein, H. (1981). *Classical mechanics*. Addison-Wesley, Reading, Mass.

Lagrange, J. L. (1787). *Mecanique analytique*. Mme Ve Courcier, Paris (in French).

Pars, L. A. (1979). *A treatise on analytical dynamics.* Oxbow Press, Woodbridge, Conn.

Penrose, R. (1955). ''A generalized inverse of matrices.'' *Proc., Cambridge Phil. Soc.*, Cambridge, U.K., 51, 406–413.

Rosenberg, R. (1972). *Analytical dynamics of discrete systems.* Plenum Press, New York.

Sudarshan, E. C. G., and Mukunda, N. (1974). *Classical dynamics: a modern perspective.* Wiley, New York.

Udwadia, F. E., and Kalaba, R. E. (1992). ''A new perspective on constrained motion.'' *Proc., Roy. Soc. Lon.*, London, 439, 407–410.

Udwadia, F. E., and Kalaba, R. E. (1996). *Analytical dynamics: a new approach.* Cambridge University Press, New York.

Udwadia, F. E., Kalaba, R. E., and Hee-Chang, E. (1997). ''Equations of motion for mechanical systems and the extended D'Alembert's principle.'' *Quarterly of Appl. Math.*, 55(2), 321–331.