Nonclassically Damped Dynamic Systems: An Iterative Approach

Firdaus E. Udwadia
Professor of Civil Engineering,
Decision Systems,
and Mechanical Engineering.
University of Southern California,
Los Angeles, CA 90089-1114

Ramin S. Esfandiari
Assistant Professor of Mathematics.
Lecturer of Mechanical Engineering,
California State University,
Long Beach, CA 90840-5005

This paper presents a new, computationally efficient, iterative technique for determining the dynamic response of nonclassically damped, linear systems. Such systems often arise in structural and mechanical engineering applications. The technique proposed in this paper is heuristically motivated and iteratively obtains the solution of a coupled set of second-order differential equations in terms of the solution to an uncoupled set. Rigorous results regarding sufficient conditions for the convergence of the iterative technique have been provided. These conditions encompass a broad variety of situations which are commonly met in structural dynamics, thereby making the proposed iterative scheme widely applicable. The method also provides new physical insights concerning the decoupling procedure and shows why previous approximate approaches for uncoupling nonclassically damped systems have led to large inaccuracies. Numerical examples are presented to indicate that, even under perhaps the least ideal conditions, the technique converges rapidly to provide the exact time histories of response.

I Introduction

The analysis of structural and mechanical systems subjected to dynamic loads is an area of great interest to engineers so that safe and reliable designs can be generated. A large number of such systems are modeled by linear differential equations described by

$$M\ddot{x}(t)+C\dot{x}(t)+Kx(t)=a(t);$$
$$x(t_0)=x_0,\ \dot{x}(t_0)=\dot{x}_0,\ t\epsilon(t_0,\ T] \tag{1}$$

where, $x(t)$ is an N vector of displacements; $a(t)$ is an N vector of force each component of which is generally taken to be a continuous function of time, t; and, M, K, and C are the mass, the stiffness, and the damping matrices, respectively. The numerical solution of equation (1), when the matrices M, K, and C can be simultaneously diagonalized by a suitable transformation, is obtained by decoupling the system and solving for each "mode" of vibration separately. This modal superposition technique, besides being useful when the system response is dominated by a relatively few number of lower modes, provides a conceptual simplicity which lends itself to an enhanced intuitive understanding of the system's response. This conceptual decoupling is also pivotal in the use of the so-called spectrum methods which have gained considerable acceptance in various fields of application, like earthquake engineering and shock and vibration analysis.

In most physical systems, the matrix M is positive definite (being related to the inertial mass properties of the system) and symmetric. Also, the matrix K is real and symmetric, a consequence of the Betty-Maxwell reciprocity relations. Under such conditions we are guaranteed to find a transformation that simultaneously diagonalizes both M and K (Noble, 1969). However, to obtain a transformation that simultaneously diagonalizes M, K and the damping matrix C, requires that the matrix C have a special form (Caughey, 1960; Caughey and O'Kelly, 1963). For passive systems, the matrix C is positive definite.

Often our lack of knowledge about the damping mechanism in large, complex systems does not permit a detailed specification of the damping matrix C, and, based on experience, the analyst often prescribes percentages of critical damping related to the various modes, which he suspects control the system's response to a given sort of excitation. While this may be sufficient (and perhaps even the best one can do) in certain circumstances, there are other situations in which sufficient information may be available to describe the matrix C through more detailed experimental testing of materials and/or subcomponents and components. Thus, there arise situations (e.g., in the design of spacecraft components) in which a more sophisticated analysis may be warranted. It is these kinds of systems that this paper deals with.

In what follows we shall assume that the mass matrix, M, is positive definite and symmetric, and the stiffness matrix, K, is symmetric and real. When M, K, and C cannot be simultaneously diagonalized by a suitable matrix transformation, one is left with a coupled system of equations of the form

$$\ddot{z}(t)+F\dot{z}(t)+\Lambda z(t)=h(t);\ z(t_0)=z_0,\ \dot{z}(t_0)=\dot{z}_0,\ t\epsilon(t_0,\ T] \tag{2}$$

where the matrix F, in general, is now a full matrix, and

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$\Lambda = \text{Diag}\ \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is a diagonal matrix. For classically damped systems, the transformation $x = Tz$ will uncouple the system represented by equation (1) and cause F to be a diagonal matrix. Yet, in many practical applications, it turns out that such a complete decoupling of equation (1) cannot be accomplished. The loss of the decoupled form leads to two major problems. First, it causes an increased computational burden to obtain the solution of equation (1); secondly, and perhaps more importantly, it increases the conceptual complexity of the response of the system because it is no longer possible to think of the total response as a superposition in terms of the contributions from the real modes and real frequencies of vibration of the system (Clough and Mojtahedi, 1976; Foss, 1958). This has led to an intense activity among researchers in the fields of mechanical, civil, and aerospace engineering to develop approximate methods that can obtain the response of the coupled system through the integration of an "equivalent" uncoupled set of equations.

Two different approaches have so far been used to do this. The first approach conceives of equation (2) as a second-order matrix differential equation and attempts to use different methods to uncouple it. Most, if not all, of these techniques essentially revolve around replacing the general matrix F by an "equivalent" diagonal matrix, D (Thompson, Calkins, and Caravani, 1974). Several ways of arriving at the diagonal elements of D from the matrix F, have evolved. Among those more commonly employed are: (1) using the diagonal elements of F in the matrix D; (2) obtaining each of the diagonal elements of D by using suitable algebraic expressions involving the elements of the matrix F which belong to the row in which that diagonal element is found; and, (3) ascribing percentages of critical damping to various modes of vibration, thereby discarding, in large measure, the information provided by the damping matrix F. The most common of these methods is simply to ignore the off-diagonal elements of the matrix F which then naturally gives rise to errors in the calculation of the response quantities (Clough and Mojtahedi, 1976; Hasselman, 1976; Warburton and Soni, 1977). A good deal of literature has concentrated on the conditions under which these errors may be small (Foss, 1958; Warburton and Soni, 1977). Most such studies have centered around the development of criteria which principally involve the parameter values of the elements of Λ and F. They are generally concerned with the left-hand side of equation (2), either ignoring the forcing function on the right-hand side of equation (2) or assuming that the system is excited harmonically. Duncan and Taylor (1979) and Warburton and Soni (1977) appear to be some of the few people who have explicitly recognized that the error in such approximate response calculations is substantially affected by the nature of the forcing vector. Though they consider only harmonic excitations, they do treat the relative amplitudes of the components of the forcing vector in their formulations as a variable that may affect the error. Yet neither of them elaborates on this point in any detail. Hasselman (1976) provides a criterion for when modal coupling can be neglected. He shows that as two or more elements of the Λ matrix come closer to each other, substantial modal interaction may be anticipated and ignoring the off-diagonal terms of the matrix F would lead, in general, to substantial errors. Cronin (1976) develops an approach on the basis that the matrix F is almost diagonalizable by the transformation that diagonalizes M and K. Although this method may generate results of acceptable accuracy for some matrices C, its usefulness to general damping matrices is questionable.

The second category of approaches conceives of the system of equations (2) to be a system of $2N$ first-order differential equations. The dynamic response is then obtained by direct integration or through the determination of the complex eigenvalues and complex mode shapes. Although the generalized modal superposition method of Foss (1958) is thoroughly

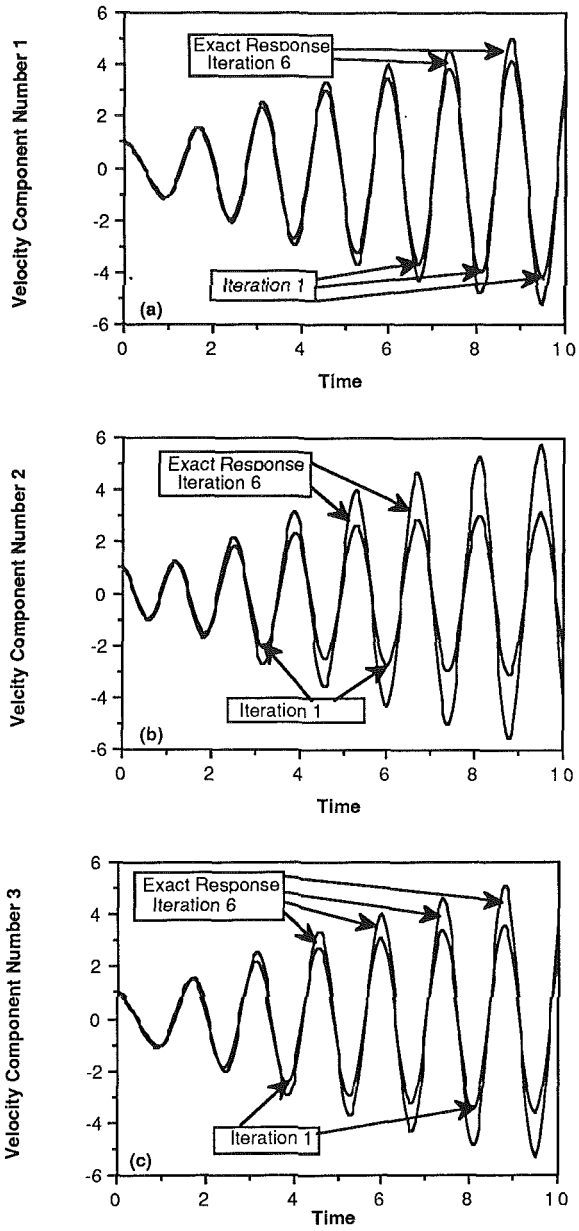

Fig. 1 Variations of the three velocity components corresponding to the exact response, first, and sixth iterations (example 1) relative to time

studied and well established, (*a*) it is computationally intensive, and (*b*) the physical significance of the various elements of the method is not as well understood as for the classical modal method. In a recent effort to clarify this significance, Veletsos and Ventura (1986) use the complex mode shape and complex frequency approach. They show that the corresponding transient displacement of a nonclassically damped multi-degree-of-freedom system can be expressed as a linear combination of displacements and the true relative velocities of a series of single degree-of-freedom systems subjected to similar excitations. Different approaches including modal superposition, complex mode shapes, direct integration, and weighted damping ratios have been compared in (Clough and Mojtahedi, 1976) where direct integration has been indicated as the preferred method. A recursive step-by-step approach in the time domain, again requiring information about the complex mode shapes and complex frequencies, has been introduced in (Singh and Ghafory-Ashtiany, 1976). P. D. Spanos, et al. (1988) deal with the decoupled analysis of classically

damped, large linear systems through an iterative predictor-corrector type scheme using substructuring.

All the approximate methods which use an uncoupled set of equations to represent the coupled set have at least two important shortfalls. Each of these techniques leads to inaccuracies, that, in general, increase as two or more of the elements of Λ approach each other. Thus, situations where multiple eigenvalues (or closely clustered eigenvalues) of Λ exist, lead to substantial errors (Hasselman, 1976). Further, it is difficult to assess, in general, the error that is caused in the computation of the response for arbitrary forcing functions $h(t)$ and matrices Λ thereby leaving the analyst ignorant of the extent of error caused by his approximation. In particular, as pointed out in (Duncan and Taylor, 1979), the extent of error generated by these currently available approximate techniques is extremely difficult to assess for large multi-degree-of-freedom systems.

In this paper we present a computationally efficient, iterative scheme for the numerical solution of coupled differential equations. Extensive numerical testing shows that the scheme works well. It is capable of handling arbitrary forcing vectors $h(t)$ and provides error bounds on the accuracy of the results obtained. More importantly, the method throws light on the uncoupling process and indicates that any coupled system of the form (2) can be replaced by an uncoupled system if and only if the forcing functions in the two systems are appropriately modified. *Thus the replacement of the matrix* F *by a diagonal matrix* D *requires concomitant changes in the forcing function* h(t) *if the responses obtained from the coupled and the uncoupled systems are to be identical.* The technique not only provides a simple computational method for handling coupled differential equations which cannot be represented by classical normal modes, but yields insight into the physics of the response of such systems. It indicates why previous efforts which involved uncoupling by concentrating on the properties of the system as represented in the left-hand side of equation (1) without explicit recognition of the forcing terms on the right-hand side, were bound to yield, in general, inaccurate results.

Section II of the paper presents the iterative approach on the basis of heuristic reasoning. For purposes of clarity in understanding and implementation, the algorithm is presented using pseudo-code. This is followed in Section III by a rigorous analysis of the sufficient conditions under which the iterative scheme is convergent. It is shown analytically that for a relatively large class of problems that are encountered in the analysis of structural and mechanical systems, the iterative scheme converges to the correct result. Section IV provides some numerical examples to show the efficacy of the proposed method. Among these examples are those with which previous investigators have found some measure of difficulty when using the usual uncoupling techniques described previously (Duncan and Taylor, 1979; Hasselman, 1976). We have chosen systems with multiple, undamped natural frequencies, and forcing functions whose frequencies coincide with these natural frequencies, thus providing rather stringent numerical tests of the technique proposed. Section V provides a discussion of the method and compares it with some of the methods proposed hereto.

II The Iterative Approach

As stated previously, under the assumption that M is positive definite and symmetric, and K is a real, symmetric matrix, equation (1) can always be cast into the form of equation (2). It will therefore suffice to begin with equation (2) in which F, Λ, $z(t_0)$, and $\dot{z}(t_0)$ are taken to be known. Let equation (2) be replaced by the system

$$\ddot{u}(t)+D\dot{u}(t)+\Lambda u(t)=f(t),\ u(t_0)=z_0,\ \dot{u}(t_0)=\dot{z}_0, \qquad (3)$$

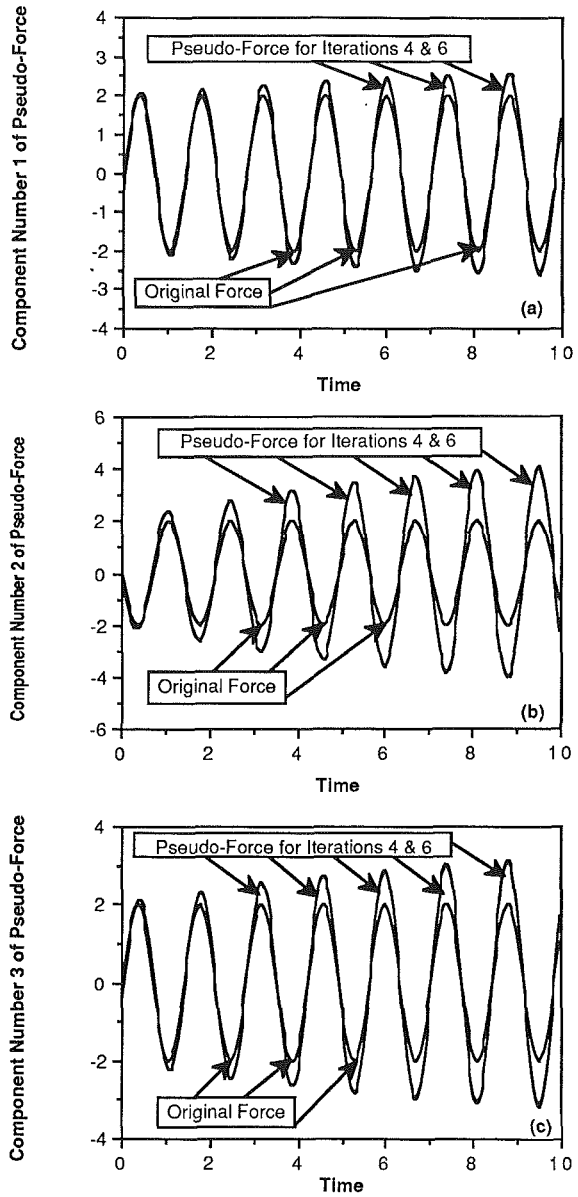

Fig. 2 Variations of the three velocity components of the pseudo-force corresponding to the first, fourth, and sixth iterations (example 1) as a *function of time*

where $D=\text{Diag}\ \{d_1, d_2, \ldots, d_N\}$ is the diagonal matrix obtained by taking the diagonal elements of F, and the function $f(t)$ is as yet unknown. Denoting the error in the response between equations (2) and (3) by the N-vector $\delta(t)=z(t)-u(t)$ and subtracting equation (3) from equation (2), we get

$$\ddot{\delta}(t)+D\dot{\delta}(t)+\Lambda\delta(t)$$
$$=h(t)-f(t)-(F-D)\dot{z},\ \delta(t)=\dot{\delta}(t)=0. \qquad (4)$$

Since (4) is a linear system of differential equations with zero initial conditions,

$$\delta(t)=0,\ t\in[t_0, T],\ \text{if and only if}\ f(t)=h(t)-(F-D)\dot{z}(t). \qquad (5)$$

This implies that the solutions of equations (2) and (3) will be identical, i.e., $z(t)\equiv u(t)$ if and only if the right-hand side of equation (3) is taken to be $f(t)$ as defined in equation (5). The only snag in performing such a replacement, is that the response $z(t)$ is not known, and in fact is obtained through a solution of equation (2), which is what we want to solve for in the first place. To circumvent this problem, we consider the following iterative scheme which uses successive approxima-

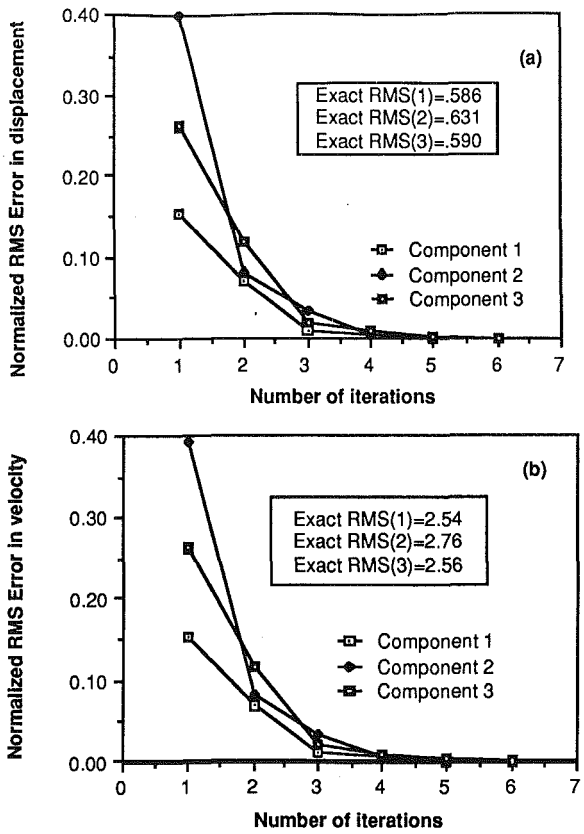

Fig. 3 RMS error of the three components of displacement (*a*) and velocity (*b*) (example 1) versus number of iterations

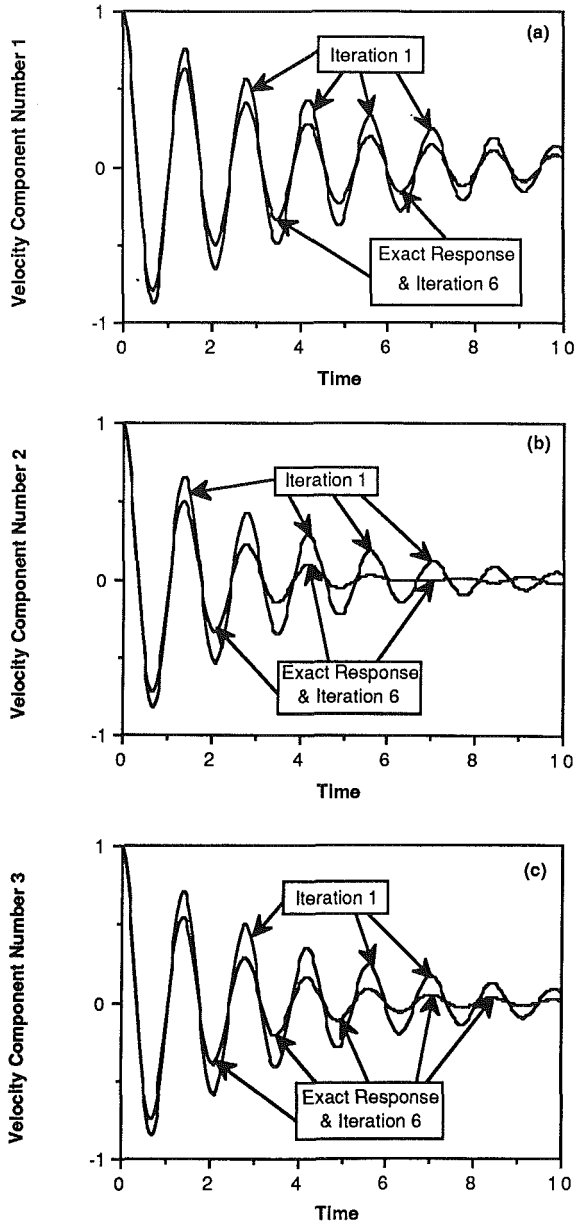

Fig. 4 Variations of the three velocity components corresponding to the exact response, first, and sixth iterations (example 2) versus time

tions for $z(t)$. The scheme can best be described in the following algorithmic form where the superscript n is used to denote quantities related to the nth iteration.

Step 1: Set $t_i = t_0,\ t_e = T$

$$u(t_i) = z_0,\ \dot{u}(t_i) = \dot{z}_0$$

Procedure I.

Step 2: Set $n = 1$;

$$\hat{f}^{(0)}(t) = h(t), t\epsilon(t_i,\ t_e]$$

Step 3: Solve the *uncoupled* system of equations:

$$\ddot{u}^{(n)}(t) + D\dot{u}^{(n)}(t) + \Lambda u^{(n)}(t) = \hat{f}^{(n-1)}(t), t\epsilon(t_i,\ t_e] \qquad (6)$$

$$u(t_i) = z_0,\ \dot{u}(t_i) = \dot{z}_0$$

Obtain:

$$\dot{u}^{(n)}(t),\ t\epsilon(t_i,\ t_e)$$

Set 4A: Set $\hat{z}^{(n)}(t) = \dot{u}^{(n)}(t),\ t\epsilon(t_i,\ t_e]$

Step 4B: Set $\hat{f}^{(n)}(t) = h(t) - (F - D)\hat{z}^{(n)}(t),\ t\epsilon(t_i,\ t_e]$

Set $n = n + 1$

Step 5: If:

$\hat{f}^{(n)}(t)$ converges {or $u^{(n)}(t)$ converges}

Then:

$$z(t) \cong \hat{z}^{(n)}(t),\ \dot{z}(t) \cong \hat{\dot{z}}^{(n)}(t),$$
$$f(t) \cong \hat{f}^{(n)}(t);\ t\epsilon(t_i,\ t_e]$$

Stop.

Else:

Go to Step 3.

The following important elements of the iterative scheme should be noted.

1 At each iteration a set of uncoupled equations (as per Step 3) are solved.

2 In Steps 4A and 4B of Procedure I we have explicitly indicated the iterative approximations of:

(*a*) $\hat{\dot{z}}(t)$, the solution of coupled system of equations (2), and,

(*b*) $\hat{f}^{(n)}(t)$, the right-hand side of equation (3), which we shall refer to henceforth as *the pseudo-force.*

Perhaps the most significant contribution of such an iterative scheme is the conceptual understanding that the coupled system represented by equation (2), could be thought of in terms of an uncoupled system with an altered (and obviously diagonal) damping matrix *provided* that appropriate adjustments can be made to the vector, $h(t)$, on the right-hand side. Past efforts to replace the coupled system by a set of uncoupled equations have concentrated mainly on obtaining "equivalent" diagonal forms D corresponding to the

general matrix F without making appropriate alterations to the forcing function, $h(t)$ on the right-hand side of equation (2). Without such an adjustment, it is little wonder that previous approximate methods which were used to uncouple equation (2) led to solutions (responses) which were, at times, significantly different from those of the coupled system. We then obtain the following conceptual picture:

$$\begin{bmatrix}\text{coupled system with}\\ \text{full matrix } F \text{ and}\\ \text{force vector } h(t)\end{bmatrix} \text{ may be made equivalent to } \begin{bmatrix}\text{uncoupled system with}\\ \text{diagonal matrix } D \text{ and}\\ \text{pseudo-force vector } f(t)\end{bmatrix}.$$

Thus, our heuristically-developed scheme shows that it may be possible to think of the response of the system represented by equation (2) as being separable into different "modes" *provided* that it is subjected to the pseudo-force vector $f(t)$ rather than the actual forcing vector $h(t)$. Having developed the plausibility of the algorithm on heuristic grounds, we next investigate the conditions under which such a conceptual picture is guaranteed to be valid. In the following section, sufficient conditions for convergence of this iterative method are investigated, namely, the conditions under which the error $\delta^{(n)}(t) = z(t) - u^{(n)}(t) \to 0$ as $n \to \infty$.

III Convergence of the Iterative Scheme

The algorithm that was explained in step-wise form to expose the underlying heuristics can now be represented as

$$\ddot{u}^{(n)}(t) + D\dot{u}^{(n)}(t) + \Lambda u^{(n)}(t) = h(t) + (D-F)\dot{u}^{(n-1)}(t), \quad (7a)$$

with

$$u^{(n)}(t_0) = z_0,\ n = 0, 1, 2, \ldots, \quad (7b)$$

$$\dot{u}^{(n)}(t_0) = \dot{z}_0,\ n = 1, 2, 3, \ldots, \text{ and,} \quad (7c)$$

$$\dot{u}^{(0)}(t) = 0,\ t \epsilon (t_0, T]. \quad (7d)$$

Also, equation (2) can be rewritten as

$$\ddot{z}(t) + D\dot{z}(t) + \Lambda z(t) = h(t) + (D-F)\dot{z}(t),\ t \epsilon (t_0, T] \quad (8a)$$

$$\text{with } \dot{z}(t_0) = \dot{z}_0, \text{ and } z(t_0) = z_0. \quad (8b)$$

Subtracting equation (7) from equation (8), and noting that $\delta^{(n)}(t) = z(t) - u^{(n)}(t)$, we obtain

$$\ddot{\delta}^{(n)}(t) + D\dot{\delta}^{(n)}(t) + \Lambda\delta^{(n)}(t) = -(F-D)\dot{\delta}^{(n-1)}(t),$$

$$n = 1, 2, \ldots \quad (9a)$$

$$\text{with } \dot{\delta}^{(0)}(t) = \dot{z}(t) - \dot{u}^{(0)}(t),\ t \epsilon (t_0, T]; \text{ and,} \quad (9b)$$

$$\dot{\delta}^{(n)}(t_0) = \delta^{(n)}(t_0) = 0,\ n = 1, 2, \ldots. \quad (9c)$$

Denoting Fourier Transforms with tildes so that

$$\tilde{\delta}(\omega) = \int_0^\infty \delta(t) e^{i\omega t} dt, \quad (10)$$

we obtain, by (9),

$$[-\omega D + i(\omega^2 I - \Lambda)]\tilde{\delta}^{(n)}(\omega) = \omega B \tilde{\delta}^{(n-1)}(\omega) \quad (11)$$

where $B = [b_{ij}] = F - D$. This then yields the recursion

$$\tilde{\delta}^{(n)}(\omega) = [S(\omega)]^n\, \tilde{\delta}^{(0)}(\omega), \quad (12)$$

where

$$S(\omega) = [\omega(\omega A)^{-1} B], \quad (13)$$

and

$$\omega A = -\omega D + i\omega G \quad (14)$$

is a diagonal matrix, with

$$D = \text{Diag}\{d_1, d_2, d_3, \ldots, d_N\} \quad (15)$$

and

$$\omega G = \text{Diag}\{\omega^2 - \lambda_1, \omega^2 - \lambda_2, \omega^2 - \lambda_3, \ldots, \omega^2 - \lambda_N\}. \quad (16)$$

Noting equation (9), the vector $\delta^{(0)}(t)$ corresponds to the error in the initial iterate. We are now ready to present the following convergence results.

Theorem 1: $u^{(n)}(t) \to z(t)$ almost everywhere (a.e.) if and only if $S(\omega)$ is quasi-nilpotent, i.e., $\{S(\omega)\}^n \to 0$, for all $\omega \epsilon \mathbf{R}$, as $n \to \infty$. (We denote the real line by $\mathbf{R}$).

This condition is equivalent to the spectral radius of $S(\omega)$, denoted $\rho_s(\omega)$, being less than unity for all $\omega \epsilon \mathbf{R}$ (Noble, 1969).

Proof: $u^{(n)}(t) \to z(t)$ almost everywhere implies that $\tilde{\delta}^{(n)}(\omega) \to 0$ for all ω, and for arbitrary $\tilde{\delta}^{(0)}(\omega)$. Noting equation (12) the result follows. Also if $S^n(\omega) \to 0$, then $\tilde{\delta}^{(n)}(\omega) \to 0$.

Since the determination of the spectral radius of $S(\omega)$ for all $\omega \epsilon \mathbf{R}$ is difficult, especially for large matrices, we present

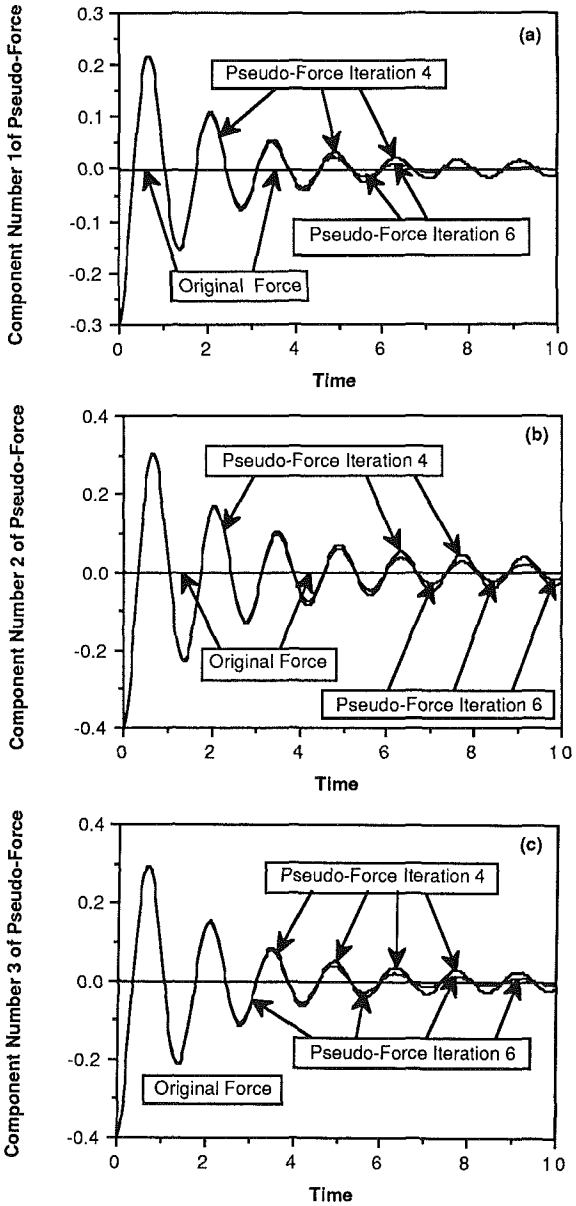

Fig. 5 Variations of the three components of the pseudo-force generated at the first, fourth, and sixth iterations (example 2) versus time

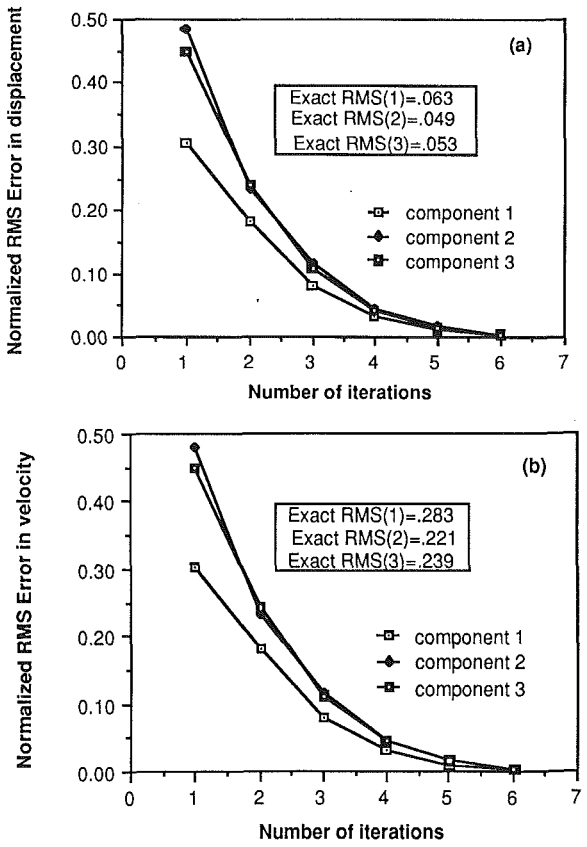

Fig. 6 RMS error of the components of displacement (*a*) and velocity (*b*) (example 2) versus number of iterations

here some sufficient conditions related to the matrix F which ensure convergence of the iterative scheme (9).

Lemma 1: The spectral radius of matrix $S(\omega)$ has the following property:

$$\rho_s(\omega) \le \max_{1\le i\le N}\left\{\frac{\omega}{|\alpha_i|}\sum_{j=1}^{N}|b_{ij}|\right\},\ \omega\epsilon\mathbf{R}, \tag{17}$$

where,

$$|\alpha_i| = \sqrt{\omega^2 d_i^2 + (\lambda_i - \omega^2)^2}. \tag{18}$$

Proof: We have (Noble, 1969)

$$\rho_s(\omega) \le \|\omega(A\omega)^{-1}B\|_\infty. \tag{19}$$

Noting that the matrix denoted by (ωA) is diagonal and that the matrix B has zeros along its diagonal, the result follows.

Lemma 2: If the matrix $\Lambda = \text{Diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is nonsingular, i.e., $\lambda_i \neq 0$, $i\epsilon\ (1,N)$, then

(*a*) $\rho_s(0) = 0$, and

(*b*) $\rho_s(\pm\infty) = 0$.

Proof: Using relation (17), and taking the appropriate limits, the result follows.

As a consequence, $\tilde{\delta}^{(n)}(0) \to 0$ as $n \to \infty$.

Theorem 2: If the matrix F is strictly diagonally dominant, then $u^{(n)}(t) \to z(t)$ almost everywhere.

Proof: We have, by Lemma 1,

$$\max_{\omega\epsilon\mathbf{R}}\{\rho_s(\omega)\} = \max_{\omega\epsilon\mathbf{R}}\left\{\max_{1\le i\le N}\frac{\omega}{|\alpha_i|}\sum_{j=1}^{j=N}|b_{ij}|\right\}. \tag{20}$$

However, the elements of the matrix B do not depend on ω. Also, using (18), we get

$$\left\{\frac{\omega}{|\alpha_i|}\right\} \le \frac{1}{|d_i|},\ i=1, 2, 3, \ldots, N;\ \omega\epsilon\mathbf{R}, \tag{21a}$$

with,

$$\max_{\omega\epsilon\mathbf{R}}\left\{\frac{\omega}{|\alpha_i|}\right\} = \frac{1}{|d_i|},\ i=1, 2, 3, \ldots, N. \tag{21b}$$

The maxima in (21*b*) occur at

$$\omega^2 = \lambda_i,\ i=1, 2, 3, \ldots, N. \tag{22}$$

Thus, we obtain

$$\max_{\omega\epsilon\mathbf{R}}\{\rho_s(\omega)\} \le \max_{1\le i\le N}\left\{\frac{1}{|d_i|}\sum_{j=1}^{j=N}|b_{ij}|\right\}. \tag{23}$$

We note that $[b_{ii}] = 0$, $[b_{ij}] = [f_{ij}]$ for $i \neq j$, and $d_i = [f_{ii}]$. Strict diagonal dominance of matrix F therefore requires that the right-hand side of (23) be less than unity. We thus obtain $\rho_s(\omega) < 1$, for all $\omega\epsilon\mathbf{R}$. By Theorem 1, the result then follows.

We next prove that if F is a symmetric positive definite matrix an additional set of sufficient conditions can be obtained to ensure convergence of the iterative scheme. In view of Lemma 2, we shall concentrate on the case $\omega \neq 0$.

Lemma 3: Let F be a positive definite matrix. Then

$$\rho_s(\omega) = \rho_{A^{-1}B}(\omega) \le \rho_{D^{-1}B}. \tag{24}$$

Proof: Let $|\lambda_0|$ be the spectral radius of $A^{-1}B$. Thus,

$$Bx = \lambda_0 Ax = \lambda_0(-D+iG)x \tag{25}$$

where x is the eigenvector corresponding to λ_0. Therefore, we have

$$|\lambda_0| = \frac{|x^H Bx|}{|x^H(-D+iG)x|} = \frac{|x^H Bx|}{\left|\sum_{k=1}^{k=N}(-d_k + ig_k)\,|x_k|^2\right|}$$

$$\le \frac{|x^H Bx|}{\left|\left\{\sum_{k=1}^{k=N} -d_k\,|x_k|^2\right\}\right|}. \tag{26}$$

Denoting the spectral radius of $D^{-1}B$ by $|\mu_0|$, we then get

$$|\lambda_0| \le \max_{\forall y}\left\{\frac{|y^H By|}{\left|\left\{\sum_{k=1}^{k=N} -d_k\,|y_k|^2\right\}\right|}\right\} = |\mu_0|, \tag{27}$$

and hence the result.

Lemma 4: If F is positive definite and symmetric, then the eigenvalues of $-D^{-1}B$ are all less than unity.

Proof: Let λ be an eigenvalue of $-D^{-1}B$. Then

$$-D^{-1}(F-D)x = \lambda x. \tag{28}$$

Let $D^{-1/2}y = x$. Then

$$-D^{1/2}(F-D)D^{-1/2}y = \lambda y, \text{ and}$$

$$-y^T[D^{-1/2}F\,D^{-1/2}]y + y^T y = \lambda y^T y.$$

Normalizing $y^T y = 1$ and noting that F is positive definite, we get

$$\lambda < 1. \tag{29}$$

Hence, the result.

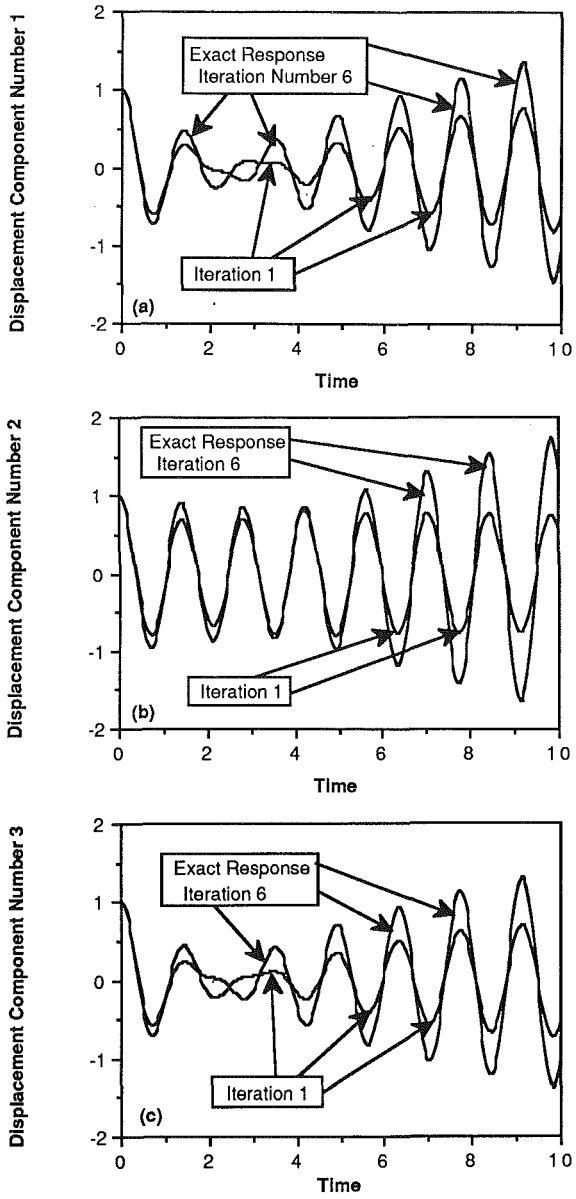

Fig. 7 Variations of the displacement components for the exact response, first, and sixth iterations (example 3) as a function of time

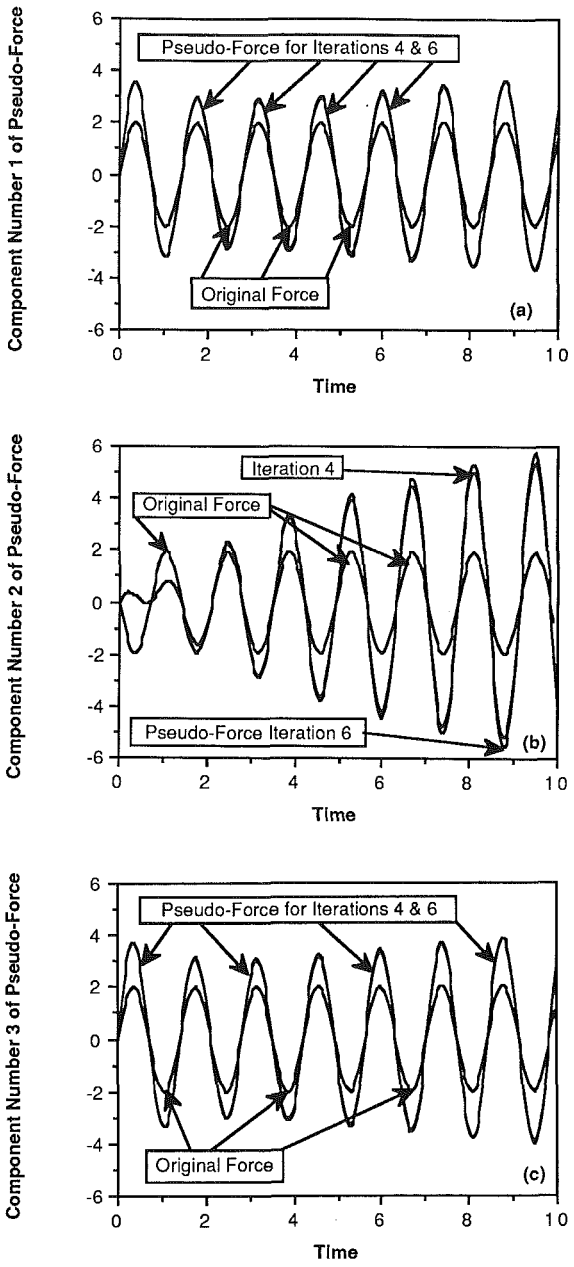

Fig. 8 Variations of the components of the pseudo-force generated at the first, fourth, and sixth iterations (example 3) versus time

Lemma 5: If $\lambda_k(D^{-1}F) < 2$, $k = 1, 2, \ldots, N$, then $\rho_{D^{-1}B} < 1$.

Proof: From the previous result, we get

$$\lambda = 1 - y^T D^{-1/2} F D^{-1/2} y. \tag{30}$$

Noting that $D^{-1}F$ and $D^{-1/2}F D^{-1/2}$ have the same eigenvalues, the result follows.

Theorem 3: If F is symmetric and positive definite and if all the eigenvalues of $D^{-1}F$ are less than 2, then the iterative scheme (9) converges.

Proof: The result follows from Lemmas 3 and 5.

IV Numerical Results

In this section we present some numerical results on three nonclassically damped systems. The three successive systems considered in this section correspond, roughly speaking, to increasing levels of generality of the nature of the damping matrix F. In each case, the matrix F is far from a diagonal matrix, the off-diagonal terms being appreciable compared to those on the diagonal. These examples have been specifically chosen so that the system responses are strongly coupled through the damping terms. Customary uncoupling methods in N-space fail miserably in these situations (Duncan and Taylor, 1979; Hasselman, 1976). We find that in all the examples studied, the iterated results, after only a few iterations, are almost the same as those obtained from using the fourth-order Runge-Kutta Integration Scheme (Press et al., 1986).

Direct use of the fourth-order Runge-Kutta procedure to obtain the response of the linear system given by equation (1) requires that it be first cast in $2N$-order form. The number of multiplications involved for each time-step in this direct approach is then approximately $(16N^2 + 18N)$. The iterative technique developed in this paper utilizes the Nigam-Jennings algorithm (Nigam and Jennings, 1968) for numerical computations. This requires, for each time-step, a total of

$(8N+N^2)I$ multiplications, where I is the number of iterations. Thus, for large N, when convergence occurs within 16 iterations, the iterative method becomes computationally superior. The numerical work reported here was performed on a MacIntosh microcomputer. Throughout this section it shall be assumed that the various parameter values are provided in consistent physical units.

Example 1: Consider a three degree-of-freedom system, which has the following parameter values:

$$F = \begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix}, \text{ and, } \Lambda = \text{Diag}\{20.0,\ 20.0,\ 20.0\}. \tag{31}$$

The matrix F is thus *nonsymmetric* but diagonally dominant. (It should be noted that the off-diagonal terms cannot be thought of as "small" compared to the diagonal elements). It is also positive definite, and its eigenvalues are $\mu_1 = 0.914$, $\mu_2 = 0.334$, and $\mu_3 = 0.252$. Its spectral norm is therefore less than unity. Theorem 2 of Section III thus ensures that the iterative scheme developed herein will converge.

The parameters t_0 and T are taken to be zero and 10 units (seconds), respectively. The initial displacements and velocities are taken to be

$$z_i(0) = 0,\ i = 1,\ 2,\ 3;\ \text{and,}\ \dot{z}_i(0) = 1.0,\ i = 1,\ 2,\ 3. \tag{32}$$

The system is subjected to the force vector $h(t)$ given by

$$h_1(t) = 2\sin\sqrt{20}t,\ h_2(t) = -2\sin\sqrt{20}t,$$
$$\text{and } h_3(t) = 2\sin\sqrt{20}t,\ t\epsilon(0,\ 10]. \tag{33}$$

The diagonal elements of the matrix Λ are all equal, and had there been no damping in the system (i.e., if F were zero), the three undamped frequencies of the uncoupled system would all be equal and have a value of $\sqrt{20}$ units. The identical values of these diagonal elements (Hasselman, 1976) show that intense interaction is to be expected through the coupling created by the matrix F. This would be all the more prominent because the excitation is also taken to have a frequency of $\sqrt{20}$ units. Standard decoupling methods used to date have been known to provide totally erroneous results in this situation (Hasselman, 1976).

The technique that was described in Section II is utilized to obtain the response of the coupled system. At each step the uncoupled system is integrated using the algorithm first presented by Nigam and Jennings (1968) which requires only eight multiplications for each time-step. At each iteration the pseudo-forces are obtained and are then used to obtain the response of the system at the next iteration, as described by equation (7). The matrix D is comprised of the diagonal elements of the matrix F. For the first iteration, the pseudo-force vector is taken to be the same as the given forcing vector, namely $h(t)$. Results from the first iteration therefore provide a measure of the extent of error in determining the system's response had all the off-diagonal elements of the matrix F been ignored. Along with the results obtained by the iterative scheme, we also provide the results obtained by using the fourth-order Runge-Kutta method with an error tolerance of 10^{-3}. In this sequel we shall refer to the results obtained from this Runge-Kutta Integration as "Exact Response," for short.

Figures 1(a), 1(b), and 1(c) show a comparison of the three components of the velocity of the system as calculated by the Runge-Kutta method and the iterative scheme developed. The differences in the responses obtained at the first iteration and the sixth iteration using the uncoupled system are substantial and therefore noteworthy. The error at the first iteration would be identical to that obtained by ignoring all the off-diagonal elements of the matrix F. Convergence to the response calculated using the Runge-Kutta scheme results within about six iterations. Figure 2 shows the components of the original forcing vector and the pseudo-forces used to determine the response of the uncoupled system at the fourth and sixth iterations. *We thus note that the response of the coupled system can be correctly obtained through an integration of the uncoupled equations if proper modification of the right-hand sides of these uncoupled equations is made.* The lack of such an appropriate modification of the forcing vector on the right-hand side of equation (2) is the cause of the poor accuracies arrived at by methods used by several previous investigators. As expected, the pseudo-forces at the fourth and sixth iterations are almost identical.

Figures 3(a) and 3(b) show the rate of convergence of each component of the displacement and velocity response as a function of the iteration number. The normalized root mean square (RMS) error, e_i, in each displacement component is defined by the relation

$$\text{normalized RMS error at iteration } n \text{ in component } i = \frac{\text{RMS of } \{u_i^{(n)} - z_i^{RK}\}}{\text{RMS of } \{z_i^{RK}\}} \tag{34}$$

where $u_i^{(n)}$ is the nth iterate of component i, and z_i^{RK} is the ith component of the response of equation (2) calculated using the direct Runge-Kutta Method. A similar relation is used for the normalized RMS error in the velocity components. The RMS responses obtained using the Runge-Kutta method are also provided in the figures.

Example 2: Next, the same system described in equations (31) and (32) is used, except that the forcing vector $h(t)$ is set to zero. Thus the coupled system's response to an initial velocity is sought. Figures 4(a)-4(c) again show the responses obtained by the Runge-Kutta method and those obtained from the Iterative Method. Again, the first iteration result is seen to be substantially different from that of the sixth; convergence to the Runge-Kutta response is observed within six iterations. Since $h(t) = 0$, the components of the original forcing function are zero. Figures 5(a)-5(c) show the pseudo-forces that are generated at each iteration corresponding to the uncoupled system of equations (7), clearly indicating that they are not negligible and therefore cannot be ignored if the response of the coupled system is to be approximated by the response of the uncoupled system. Furthermore, these pseudo-forces depend on the forcing function $h(t)$ of the coupled system. Figure 6 again shows the nature of the convergence of the scheme with iteration number.

Example 3: Here we consider a symmetric matrix F with the elements given by

$$F = \begin{bmatrix} 0.40 & 0.30 & 0.10 \\ 0.30 & 0.60 & 0.34 \\ 0.10 & 0.34 & 0.50 \end{bmatrix} \text{ and } \Lambda = \text{Diag}\{20.0,\ 20.0,\ 20.0\}. \tag{35}$$

along with the initial conditions

$$z_i(0) = 1.0,\ i = 1,\ 2,\ 3;\ \text{and,}\ \dot{z}_i(0) = 0,\ i = 1,\ 2,\ 3. \tag{36}$$

The forcing vector $h(t)$ is the same as that in equation (33).

The matrix F though symmetric is no longer diagonally dominant. The eigenvalues of F are 0.12, 0.347, and 1.033. Yet the eigenvalues of $D^{-1}F$($\mu_1 = 1.9909$, $\mu_2 = 0.7764$, and $\mu_3 = 0.2327$) are all less than 2 and by Theorem 3 of Section III the iterative scheme of Section I is ensured of convergence. Again, the elements of the matrix Λ are identical and the forcing function has a frequency identical to the undamped uncoupled fundamental frequency (i.e., $\sqrt{20}$ units). These sets of conditions promise a large interaction in the modes through the matrix F (Hasselman, 1976; Warburton and Soni, 1977). Figures 7(a)-7(c) show the displacement responses of the

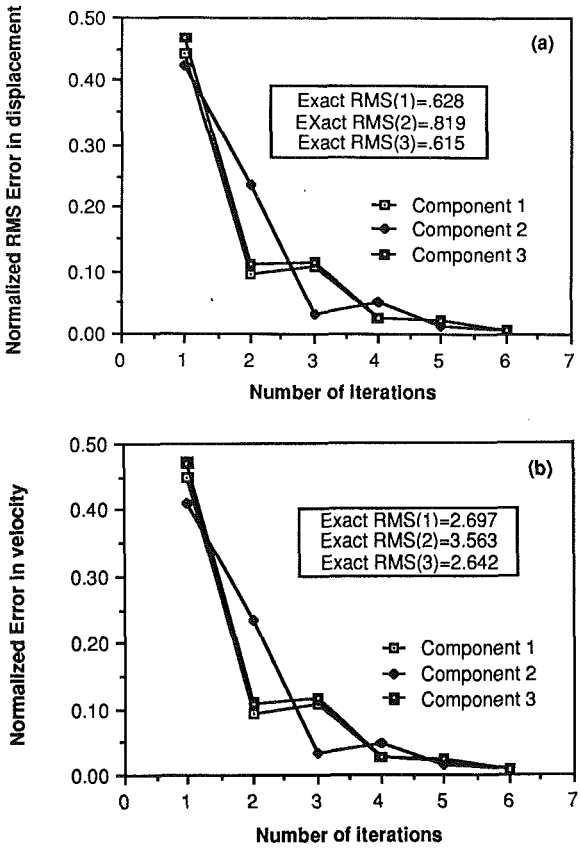

Fig. 9 Normalized RMS error of the displacement (*a*) and velocity (*b*) components (example 3) versus the number of iterations

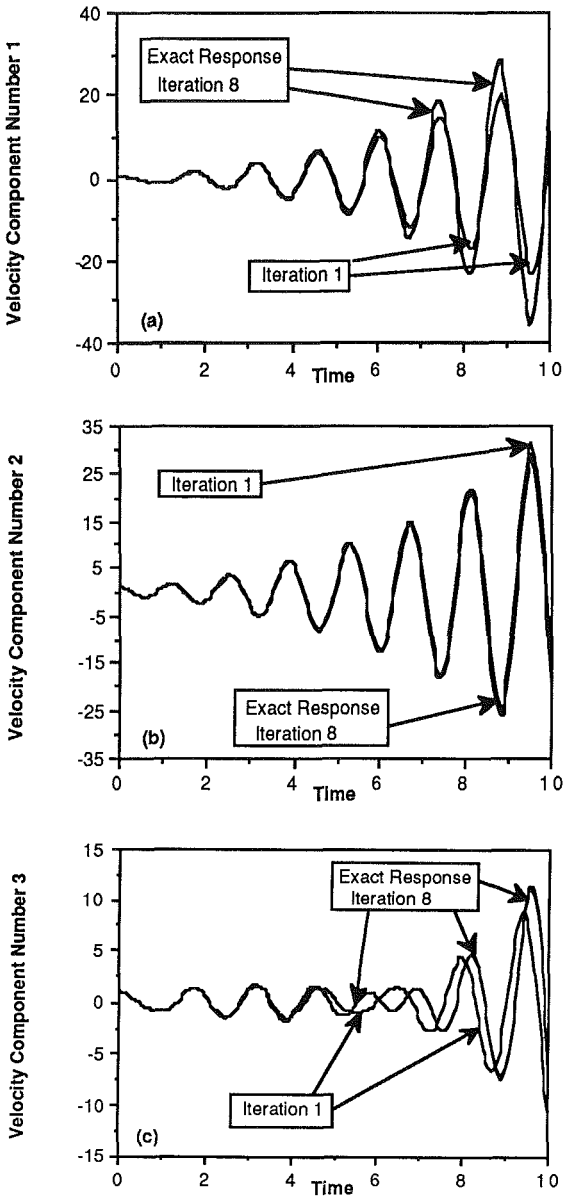

Fig. 10 Variations of the velocity components corresponding to the exact response, first, and eighth iterations (example 4) versus time

various components. Again, convergence of the response at the sixth iteration to the response computed by direct integration is shown. In Figs. 8(*a*)-8(*c*) the pseudo-force components at the fourth and sixth iterations are shown along with the components of the original forcing vector $h(t)$. The manner in which the convergence occurs is indicated in Figs. 9(*a*) and 9(*b*).

Example 4: To amplify the nature of the results obtained in Section III, we next consider a three degree-of-freedom system defined by the following matrices:

$$F = \begin{bmatrix} 0.2 & 0.7 & 0.6 \\ 0.5 & 0.1 & 0.3 \\ 0.4 & 0.4 & 0.3 \end{bmatrix}, \text{ and, } \Lambda = \text{Diag}\{20.0,\ 20.0,\ 20.0\} \tag{37}$$

along with the initial conditions

$$z_i(0) = 0,\ i = 1,\ 2,\ 3;\ \text{and,}\ \dot{z}_i(0) = 1.0,\ i = 1,\ 2,\ 3. \tag{32}$$

The forcing vector $h(t)$ is taken to be the same as that in relation (33).

Here the matrix F is nonsymmetric and nonpositive definite. In spite of this, Figs. 10(*a*)-10(*c*) show the convergence of the various components of the response of the system to results obtained from the Runge-Kutta computations. Figures 11(*a*)-11(*c*) show the pseudo-forces that need to be used to compensate for ignoring the off-diagonal terms of the matrix F. Figures 12(*a*) and 12(*b*) show the convergence of displacement and velocity components, respectively.

It should be noted that the results presented in Section III provide only the sufficient conditions for convergence of the iterative scheme. This means that convergence is possible and often does occur, in such schemes, even when these sufficient conditions are not satisfied. This is exemplified in this example where we find that neither Theorem 2 nor 3 of Section III is applicable. A brief summary of all four examples is provided in Table 1.

V Conclusions and Discussion

A simple, new, computationally efficient and heuristically-motivated iterative method is developed for the numerical solution of general, linear vibratory systems modeled by coupled differential equations. It is shown to be a better alternative to the approximate methods that have so far been used by previous investigators. Most, if not all, of the previous investigations have focused on the modification of the left-hand side of the system represented by equation (2) in such a way that the transformed damping matrix is diagonal. This generates unacceptable computational errors for certain cases. The iterative scheme presented here shows that by appropriately adjusting the right-hand side of the equation in each iteration, the exact solutions can be rapidly obtained. Furthermore, to simplify the use of the proposed scheme to

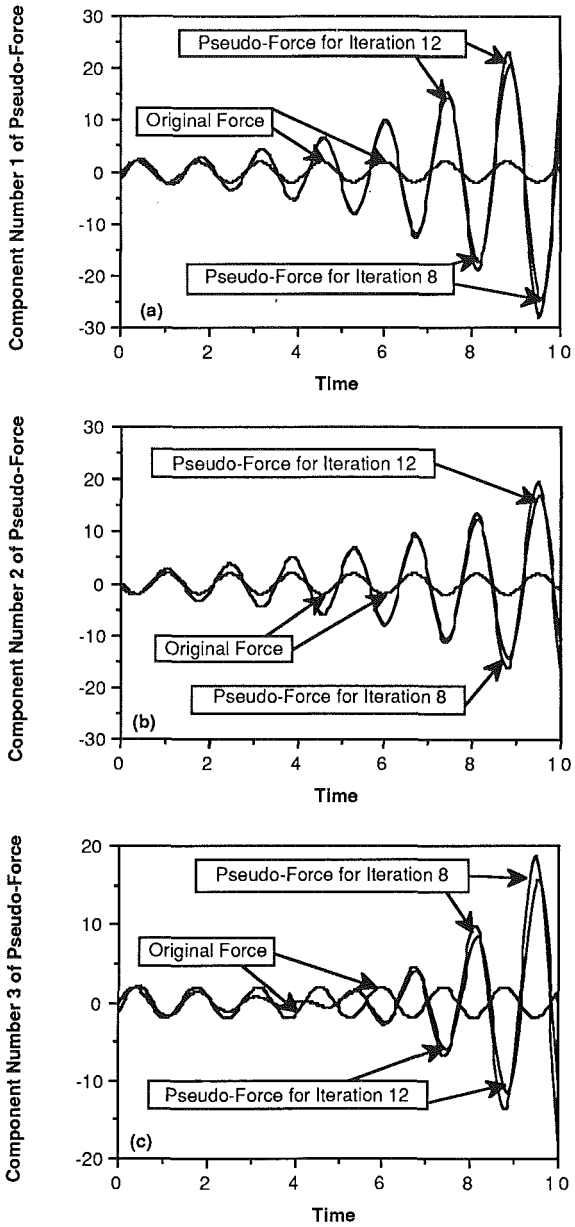

Fig. 11 Variations of the three components of the pseudo-force generated at the first, eighth, and twelveth iterations (example 4) with response to time

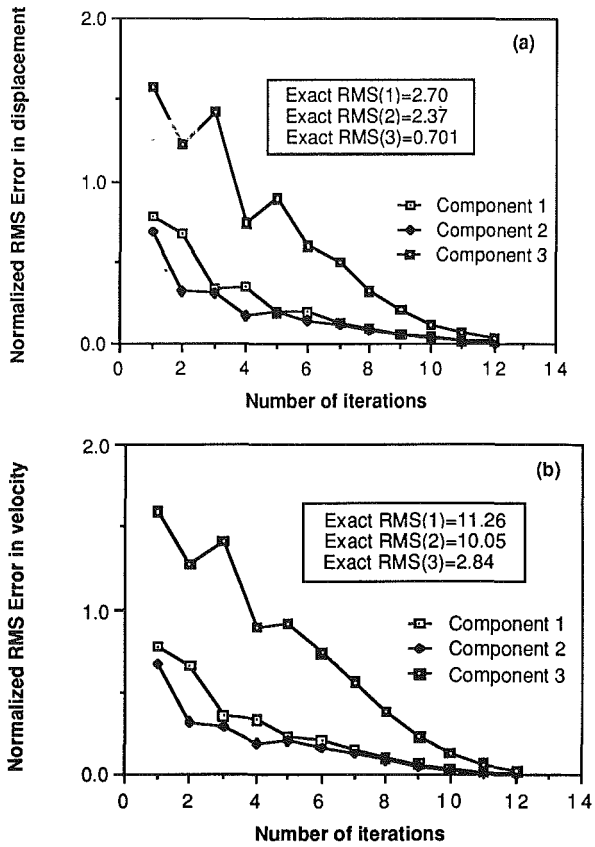

Fig. 12 Normalized RMS error for the displacement (*a*) and velocity (*b*) components (example 4) with respect to the number of iterations

Table 1 Summary of numerical results

Example Number	F Matrix and Its Characteristics	Diagonal Λ Matrix	Forcing Vector h(t)	Initial Conditions	*Normalized* RMS Error (Displacement)
1	$\begin{bmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0.3 \\ 0.1 & 0.3 & 0.5 \end{bmatrix}$ Nonsymmetric; Positive Definite; Diagonally Dominant; THEOREM 2 OF SECTION III APPLIES	$\lambda_1 = 20.0$ $\lambda_2 = 20.0$ $\lambda_3 = 20.0$	$h_1(t) = 2\sin\sqrt{20}\,t$ $h_2(t) = -2\sin\sqrt{20}\,t$ $h_3(t) = 2\sin\sqrt{20}\,t$	$z_1(0) = 0$ $z_2(0) = 0$ $z_3(0) = 0$ $\dot{z}_1(0) = 1$ $\dot{z}_2(0) = 1$ $\dot{z}_3(0) = 1$	$e_1 = 0.47E\text{-}03$ $e_2 = 0.51E\text{-}03$ $e_3 = 0.71E\text{-}03$ (Iteration 6)
2	Same as in Example Number 1	Same as in Example Number 1	$h_i(t) = 0,\ i = 1, 2, 3$	Same as in Example Number 1	$e_1 = 0.28E\text{-}02$ $e_2 = 0.41E\text{-}02$ $e_3 = 0.39E\text{-}02$ (Iteration 6)
3	$\begin{bmatrix} 0.4 & 0.3 & 0.1 \\ 0.3 & 0.6 & 0.34 \\ 0.1 & 0.34 & 0.5 \end{bmatrix}$ Symmetric; Positive Definite: Non–Diagonally Dominant; THEOREM 3 OF SECTION III APPLIES	Same as in Example Number 1	Same as in Example Number 1	$z_1(0) = 1$ $z_2(0) = 1$ $z_3(0) = 1$ $\dot{z}_1(0) = 0$ $\dot{z}_2(0) = 0$ $\dot{z}_3(0) = 0$	$e_1 = 0.58E\text{-}02$ $e_2 = 0.77E\text{-}02$ $e_3 = 0.60E\text{-}02$ (Iteration 6)
4	$\begin{bmatrix} 0.2 & 0.7 & 0.6 \\ 0.5 & 0.1 & 0.3 \\ 0.8 & 0.4 & 0.3 \end{bmatrix}$ Nonsymmetric; Nonpositive; Non Diagonally Dominant; RESULTS OF SECTION III DO NOT APPLY	Same as in Example Number 1	Same as in Example Number 1	Same as in Example Number 1	$e_1 = 0.96E\text{-}02$ $e_2 = 0.63E\text{-}02$ $e_3 = 0.28E\text{-}01$ (Iteration 12)

practical problems in structural and mechanical engineering, we have provided the algorithm in the form of a pseudo-code. We list the important features of the work presented here as follows:

(*i*) An important aspect of the presently introduced method is the conceptual understanding that many coupled systems of differential equations which are met with in structural dynamics can be interpreted in terms of uncoupled systems, *provided* that they are subjected to an appropriately calculated *pseudo-forcing vector*. This yields considerable physical insight into the dynamic behavior of large-scale, nonclassically damping structural and mechanical systems.

(*ii*) Unlike several approaches (Singh and Ghafory-Ashtiany, 1986; Veletsos and Ventura, 1986) requiring calculation of complex frequencies and complex mode shapes, thus making the physical interpretation of the elements of the solution difficult, the method presented here is straightforward, computationally efficient, and does not require computations involving complex variables.

(*iii*) Much of the previous effort to uncouple the equations of nonclassically damped systems has been based on replacing the damping matrix by a diagonal matrix. For certain systems and certain types of excitations, this leads to unacceptable computational errors (Hasselman, 1976). Or,

approximate solutions have been obtained by incorporating the contributions from only a finite number of modes. This may result in acceptable approximations for classically damped systems, but not necessarily for nonclassical systems (Duncan and Taylor 1979). The approach introduced here explains physically why such behavior is to be commonly expected from nonclassically damped systems.

(*iv*) Analytical results are provided giving sufficient conditions for the convergence of the iterative method. Such conditions are indeed necessary to provide, since convergence is of crucial importance in obtaining numerically accurate dynamic responses. It is shown that for diagonally-dominant general damping matrices, F, the method converges. Further results are given for passive systems in which the damping matrix is positive-definite and symmetric. These conditions cover a wide range of commonly occurring structural and mechanical systems.

(*v*) Several numerical examples are presented and it is shown that even under the least ideal conditions the iterative method works well and converges rapidly to the exact solution. The examples illustrate the important role played by the pseudo-force for such convergence. The figures show the extent of error created if the pseudo-force is not taken into account and point out why previous approaches have failed to obtain higher numerical accuracies. It is shown that the iterative approach presented here holds high promise for obtaining very accurate responses to nonclassically damped linear systems, within a few iterations.

References

Caughey, T. K., 1960, "Classical Normal Modes in Damped Linear Structures," ASME Journal of Applied Mechanics, Vol. 27, pp. 269–271.

Caughey, T. K., and O'Kelly, M. J., 1963, "General Theory of Vibrations of Damped Linear Dynamic Systems," Report of the Dynamics Laboratory, California Institute of Technology, Pasadena, Calif.

Clough, R. W., and Mojtahedi, S., 1976, "Earthquake Response Analysis Considering Non-Proportional Damping," *Earthquake Engineering and Structural Dynamics*, Vol. 4, pp. 489–496.

Cronin, D. L., 1976, "Approximation for Determining Harmonically Excited Response of Non-Classically Damped Systems," *ASME Journal of Engineering for Industry*, pp. 43–47.

Duncan, P. E., and Taylor, R. Eatock, 1979, "A Note on the Dynamic Analysis of Non-Proportioanlly Damped Systems," *Earthquake Engineering and Structural Dynamics*, Vol. 7, pp. 99–105.

Foss, K. A., 1958, "Co-Ordinates that Uncouple the Equations of Motion of Damped Linear Dynamic Systems," ASME Journal of Applied Mechanics, Vol. 25, pp. 361–364.

Hasselman, T. K., 1976, "Modal Coupling in Lightly Damped Structures," *AIAA Journal*, Vol. 14, pp.

Nigam, N., and Jennings, P. C., 1968, "Digital Calculation of Response Spectra From Strong Motion Earthquake Records," EERL Report No. 700-00, California Institute of Technology, Pasadena, Calif.

Noble, B., 1969, *Applied Linear Algebra*, Prentice-Hall, Englewood Cliffs, N. J.

Press, W. H., Flannery, B. P., Teukolsky, S. A., and Vetterling, W. T., 1986, *Numerical Recipes*, Cambridge University Press, Cambridge, U.K.

Singh, M. P., and Ghafory-Ashtiany, M., 1986, "Modal Time History of Non-Classically Damped Structures for Seismic Motions," *Earthquake Engineering and Structural Dynamics*, Vol. 13, pp. 133–146.

Spanos, P. D., Cao, T., Jacobson, C., Nelson, D., and Hamilton, D., 1988, "Decoupled Dynamic Analysis of Combined Systems by Iterative Determination of Interface Accelerations," *Earthquake Engineering and Structural Dynamics*, Vol. 16, pp. 491–500.

Thomson, W. T., Calkins, T., and Caravani, P., 1974, "A Numerical Study of Damping," *Earthquake Engineering and Structural Dynamics*, Vol. 3, pp. 97–103.

Veletsos, A. S., and Ventura, C. E., 1986, "Modal Analysis of Non-Classically Damped Linear Systems," *Earthquake Engineering and Structural Dynamics*, Vol. 14, pp. 217–243.

Warburton, G. B., and Soni, S. R., 1977, "Errors in Response Calculations for Non-Classically Damped Structures," *Earthquake Engineering and Strucutral Dynamics*, Vol. 5, pp. 365–376.