NEW DIRECTIONS IN THE CONTROL OF NONLINEAR MECHANICAL SYSTEMS

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ABSTRACT

The newly developed equations of motion for constrained mechanical systems are used as a means for the explicit determination of the control forces needed to cause mechanical systems to satisfy given constraints, such as motion along a given curve in phase space. Two simple examples are described utilizing this new approach to the control of nonlinear mechanical systems.

INTRODUCTION

Consider a nonlinear mechanical system, consisting of n particles, described in a rectangular inertial frame of reference by the equation

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\dot{\mathbf{x}},t). \tag{1}$$

Here the 3n-vector of position is denoted by the vector $\mathbf{x}$, and M is a constant, positive definite, diagonal 3n by 3n matrix. The dots refer to differentiation with respect to time. The "given force" $\mathbf{F}$ and the matrix M are assumed known.

We now desire to control this mechanical system so that, in addition, it satisfies the m consistent equations given by

$$\varphi_i(\mathbf{x},\dot{\mathbf{x}},t) = 0,\ i = 1, 2, \ldots, m, \tag{2}$$

which need not be functionally independent. This set of equations (2) may then be thought of as an additional set of *constraint* equations which the mechanical system defined by equation (1) is further required to satisfy.

On differentiating equations (2) we obtain the equation

$$A(\mathbf{x},\dot{\mathbf{x}},t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x},\dot{\mathbf{x}},t), \tag{3}$$

where the elements of the m by 3n matrix A as well as those of the m-vector $\mathbf{b}$ are known functions of $\mathbf{x}$, $\dot{\mathbf{x}}$ and t.

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Given the set of initial conditions $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$ which satisfy the constraint equations (2), the equation of motion of the constrained mechanical system (described jointly by equations (1) and (3)) can now be explicitly expressed as (see [1])

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\dot{\mathbf{x}},t) + M^{1/2}C^{+}(\mathbf{b} - A\mathbf{a}) \tag{4}$$

where the 3n-vector $\mathbf{a} = M^{-1}F(\mathbf{x},\dot{\mathbf{x}},t)$ and the matrix $C = A(\mathbf{x},\dot{\mathbf{x}},t)M^{-1/2}$. The superscript "+" denotes the Moore-Penrose inverse of the matrix C. We note that the vector $\mathbf{a}$ is simply the acceleration corresponding to the unconstrained system as obtained by using equation (1).

EXPLICIT DETERMINATION OF THE CONTROL FORCE

Noting that the presence of the constraints brings into play a set of constraint forces, we can express the equation of motion of the constrained system as

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x},\dot{\mathbf{x}},t) + \mathbf{F}^{c}(\mathbf{x},\dot{\mathbf{x}},t)\ , \tag{5}$$

where $\mathbf{F}^{c}(\mathbf{x},\dot{\mathbf{x}},t)$ is the force of constraint brought into play by Nature so that the constraints (2) are exactly satisfied. Comparing equations (4) and (5) we find that the force of constraint can be expressed explicitly as

$$\mathbf{F}^{c}(\mathbf{x},\dot{\mathbf{x}},t) = M^{1/2}C^{+}(\mathbf{b} - A\mathbf{a}). \tag{6}$$

We may alternately think of this force as the "control force" exerted by Nature so that the unconstrained (uncontrolled) system given by equation (1) is forced to satisfy the constraint set (2) in the presence of the 'given' forces. Thus we have obtained an explicit expression for a control force which would cause the system to satisfy the constraints. We next give two examples where this control philosophy is illustrated.

EXAMPLES

(1) Let a particle of unit mass be located in a Cartesian coordinate frame of reference at x=0, y= L at time t =0. The particle is subjected to the downward force of gravity acting along the positive y direction. We want this particle to follow the given trajectory

$$x^2 + y^2 = L^2, \tag{7}$$

where L is a given constant. We are desirous of finding a control force that will do this for us, given that the particle's initial velocity at time t =0 is $\dot{x}(0) = \dot{x}_0$ and $\dot{y}(0) = 0$. (Note that our initial conditions satisfy equation (7)).

Since the equations of motion of the unconstrained (uncontrolled) system can be written as

$$\begin{aligned} \ddot{x} &= 0 \\ \ddot{y} &= g, \end{aligned} \tag{8}$$

the matrix $M = I$, and the acceleration $\mathbf{a}$ of the uncontrolled system is simply the vector $[0 \quad g]^T$. Differentiating equation (7) twice we obtain the equation

$$[x \quad y]\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = -(\dot{x}^2 + \dot{y}^2) \tag{9}$$

so that the matrix $C = A = [x \quad y]$, and the vector **b** is simply the scalar $-(\dot{x}^2 + \dot{y}^2)$. The Moore-Penrose inverse of C is given by

$$C^+ = \frac{1}{x^2 + y^2}\begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{L^2}\begin{bmatrix}x\\ y\end{bmatrix} \tag{10}$$

so that the "control force" that Nature would use to control this system would be simply given by equation (6) as

$$\mathbf{F}^c(\mathbf{x},\dot{\mathbf{x}},t) = -\frac{1}{L^2}\begin{bmatrix}x\\ y\end{bmatrix}(\dot{x}^2 + \dot{y}^2 + gy). \tag{11}$$

It must be remembered that there are a large number of "control forces" which will cause the particle to follow the trajectory specified by equation (7). *One* such set of control forces, based on our observation of the way Nature operates, is given by equation (11). These control forces, in the presence of the external influence of gravity, will ensure that the particle will follow the desired trajectory. Hence the equation of motion of the "controlled" system is given by

$$\begin{bmatrix}\ddot{x}\\ \ddot{y}\end{bmatrix} = \begin{bmatrix}0\\ g\end{bmatrix} - \frac{1}{L^2}\begin{bmatrix}x\\ y\end{bmatrix}(\dot{x}^2 + \dot{y}^2 + gy); \quad x(0) = 0,\ y(0) = L,\ \dot{x}(0) = \dot{x}_0,\ \dot{y}(0) = 0. \tag{12}$$

This is nothing other than the equation of motion of a pendulum of length L, as can be easily verified!

(2) Consider a particle of mass m subjected to the forces F_x, F_y, and F_z, in the x, y, and z Cartesian coordinate directions respectively. These forces are given functions of $x,\dot{x},y,\dot{y},z,\dot{z}$ and t. Let us require that in the presence of these forces, the particle be controlled so that it follows a trajectory along which

$$\dot{y}(t) = z(t)\dot{x}(t) + \sin(\alpha t) \tag{13}$$

where α is a given constant. The initial conditions of the particle at time t=0 are given by $x(0) = a,\ y(0) = b,\ z(0) = c,\ \dot{x}(0) = d,\ \dot{y}(0) = cd,\ \dot{z}(0) = e$.

A control force which would be capable of doing this can be explicitly found as follows. The unconstrained (uncontrolled) motion of the particle is given by

$$m[I]\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}F_x(x,\dot{x},y,\dot{y},z,\dot{z}\ ,t)\\ F_y(x,\dot{x},y,\dot{y},z,\dot{z}\ ,t)\\ F_z(x,\dot{x},y,\dot{y},z,\dot{z}\ ,t)\end{bmatrix}. \tag{14}$$

Differentiating equation (13) once we get

$$[-z \quad 1 \quad 0]\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \dot{z}\dot{x} + \alpha\cos(\alpha t) \tag{15}$$

so that the matrix $A = [-z \quad 1 \quad 0]$, and the vector $\mathbf{b}$ is the scalar $\dot{z}\dot{x} + \alpha\cos(\alpha t)$. Using equation (6) we get the control force to be

$$\mathbf{F}^c = \frac{1}{1+z^2}\begin{bmatrix}-z\\ 1\\ 0\end{bmatrix}\left\{m\dot{z}\dot{x} + m\alpha\cos\alpha t + zF_x - F_y\right\}. \tag{16}$$

This control force, in the presence of the given forces F_x, F_y, and F_z will cause the particle to follow a trajectory that satisfies equation (13), given that it has the prescribed initial conditions. The equation of motion of the controlled dynamic system will then be

$$m[I]\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}F_x(x,\dot{x},y,\dot{y},z,\dot{z},t)\\ F_y(x,\dot{x},y,\dot{y},z,\dot{z},t)\\ F_z(x,\dot{x},y,\dot{y},z,\dot{z},t)\end{bmatrix} + \frac{1}{1+z^2}\begin{bmatrix}-z\\ 1\\ 0\end{bmatrix}\left\{m\dot{z}\dot{x} + m\alpha\cos\alpha t + zF_x - F_y\right\}. \tag{17}$$

CONCLUSIONS

In this paper we have presented a new approach to the control of nonlinear mechanical systems through the use of the newly developed equation of motion for constrained systems. These ideas are only preliminary; as such, a great deal more research needs to be done. In particular, issues such as the uniqueness, the existence and the stability of equation (4) need to be attended to before we can go much further. We leave these issues for future communications.

REFERENCE

1. Udwadia F. E. and R. E. Kalaba, "A New Perspective on Constrained Motion, " *Proceedings of the Royal Society of London*, vol. 439, pp. 407-410, 1992.