

New Approach to the Modeling of Complex Multibody Dynamical Systems

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In this paper, a general method for modeling complex multibody systems is presented. The method utilizes recent results in analytical dynamics adapted to general complex multibody systems. The term complex is employed to denote those multibody systems whose equations of motion are highly nonlinear, nonautonomous, and possibly yield motions at multiple time and distance scales. These types of problems can easily become difficult to analyze because of the complexity of the equations of motion, which may grow rapidly as the number of component bodies in the multibody system increases. The approach considered herein simplifies the effort required in modeling general multibody systems by explicitly developing closed form expressions in terms of any desirable number of generalized coordinates that may appropriately describe the configuration of the multibody system. Furthermore, the approach is simple in implementation because it poses no restrictions on the total number and nature of modeling constraints used to construct the equations of motion of the multibody system. Conceptually, the method relies on a simple three-step procedure. It utilizes the Udwadia–Phohomsiri equation, which describes the explicit equations of motion for constrained mechanical systems with singular mass matrices. The simplicity of the method and its accuracy is illustrated by modeling a multibody spacecraft system. [DOI: 10.1115/1.4002329]

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1 Introduction

The motion of a multibody system is generally highly nonlinear and often complex. Formulating equations of motion for general multibody systems can be a nontrivial task, and in many formulations the equations of motion are restricted to case specific multibody problems. Recently, a fundamental result in analytical dynamics obtained by Udwadia and Kalaba [1–4] led to a new view in the theory of constrained motion, which is intimately tied to multibody dynamics. They obtain explicitly a general set of equations of motion for holonomically and nonholonomically constrained mechanical systems in terms of the generalized coordinates that describe their configuration. In a further advance, Udwadia and Phohomsiri [5] obtained an explicit equation of motion (Udwadia–Phohomsiri (UP) equation) for constrained mechanical systems with singular mass matrices. While it may not be obvious how singular mass matrices might arise when describing the unconstrained motion of a mechanical system, we note that they do arise when modeling multibody systems. The approach proposed herein permits the deft handling of systems with singular mass matrices, thereby permitting the description of complex multibody systems through the use of a larger number of coordinates than the minimum required.

The value of a method that is used to obtain the equations of motion for a complex multibody system should be measured by its simplicity and the amount of effort required when deriving the governing equations of the correctly modeled multibody system as reasoned in surveys by Kane [6] and Schiehlen [7]. The methodology proposed herein aims at this goal. It significantly simplifies the effort in obtaining the equations of motion of such systems by allowing the modeler to apply the following: (1) use of more than

the minimum number of coordinates to describe the motion of each subsystem of a complex multibody system, (2) inclusion of any, and all, constraints that can be discerned in the description of the various connections between the subsystems used to model the complex system without worrying about which constraints may be functionally dependent, and (3) use of a formulation that seamlessly includes positive semidefinite and/or positive definite mass matrices.

In many popular multibody modeling methods available to date, the equations of motion are generated using recursive methods. Most recursive methods are employed by casting a multibody system into a tree topology wherein each individual body in the multibody system is attached to one or more of the other bodies that comprise the whole system. The bodies are attached to one another at joints located at arbitrarily given fixed points on each individual body. The coordinates in these systems are taken relative to the so-called joint coordinates (relative displacements and rotations between component bodies) so that the equations of motion are obtained in terms of the independent degrees of freedom. These approaches attempt to provide a reduction in the total number of coordinates with the primary goal of increasing computational performance, a topic on which considerable research interest seems to have been focused (e.g., Refs. [8–10]). While it is true that a reduction in the number of coordinates can ideally increase computational performance, it often does so at the cost of increased difficulty in arriving at the equations of motion. In addition, these approaches force the modeling to be conceptualized within a predefined modeling structure, one that requires the construction of the multibody system to take place along lineal lines of thought that more or less reflect the underlying paradigmatic tree-structure. This aspect could be a source of considerable inconvenience, especially when constraints need to be altered, because this might at times require a complete remodeling of the multibody system. Also, as pointed out by several researchers, applying general constraint equations or forcing functions can become problematic in some situations for recursive methods [11].

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Other approaches for generating the governing equations rely on the computation of Lagrange multipliers [12,13], or by the elimination of Lagrange multipliers via null space methods [14,15]. The so-called null space method relies on the computation of an orthogonal complement of the constraint matrix in which the constraints are required to be linear in the velocities and nonredundant. These methods are difficult to apply to general complex multibody systems and they fail in situations where the constraints are functionally dependent. While holonomic constraints are often easy to spot when they are not independent, nonholonomic constraints take the form of differential equations and in highly complex systems, which may have many such constraints, ensuring that they are all independent could become a nontrivial job. This is because two differential equations, though equivalent, can take on very different-looking forms when multiplied by various multiplying factors.

In the following, a new and simple approach of systematically modeling a complex system of N rigid bodies is developed wherein we posit no predefined modeling structure on the development of the equations of motion, and in implementation the desired computational performance may not be realized due to the computational overhead of the recursion. The term complex is used in this paper to denote those systems that are highly nonlinear, possibly nonautonomous, and that may yield motions at multiple time and distance scales. One of the salient advantages of our approach is its conceptual clarity. The modeling methodology and its formulation are simple, and the effort required to obtain the equations of motion is minimal, thereby allowing a uniform, improved, and widely applicable route for modeling complex multibody systems. The method exploits the appearance of singular mass matrices in Lagrangian mechanics by utilizing the UP equation, thereby overcoming a difficulty that is not easily handled with current methods [5]. In an example, we show its use for the modeling of a realistic multiscale, multibody spacecraft system to demonstrate the simplicity of the approach, its ease of implementation, and its numerical accuracy.

2 Modeling Complex Multibody Systems

A general formulation is developed in this section to describe the dynamics of multiple interconnected rigid bodies. Beginning with the concepts of generalized coordinates and kinetic energy, we describe how to obtain the explicit equations of motion for complex multibody systems using a simple straightforward three-step procedure. Conceptually, these three steps are the following:

- (i) description of the so-called unconstrained system of equations
- (ii) description of the constraints required to model the given multibody system
- (iii) description of the constrained multibody system using the previous two descriptions

In what follows, we develop each of these steps pointing out the ease and efficacy with which the equations of motion for general complex multibody dynamical systems can be obtained. In order to provide an explicit framework for our modeling methodology and establish our notation, we begin by considering the motion of a rigid body wherein we permit the use of an *arbitrary number of coordinates* to describe its motion.

2.1 Generalized Coordinates and Kinetic Energy of a Component Body in a Multibody System Using an Arbitrary Number of Coordinates to Describe Its Motion. Consider first a single rigid body that is a component of the multibody system that we desire to model. Let its mass be m and let the position to its center of mass be given by the vector \mathbf{R} whose components $R = [R_1, R_2, R_3]^T$ are given relative to an inertial frame of reference as shown in Fig. 1. The rectangular coordinate frame with axes $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ is fixed to the body and its origin is located at its center of mass. Without loss of generality, we assume that this coordinate

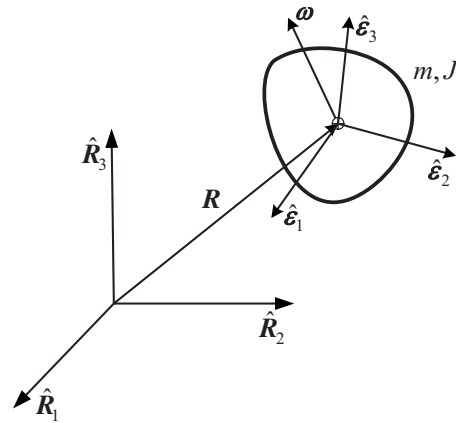


Fig. 1 A rigid body in an inertial frame of reference

frame is aligned along the principal axes of inertia. The principal moments of inertia about these axes are J_1 , J_2 , and J_3 , respectively, so that the 3 by 3 inertia matrix of the body is then given by $J = \text{diag}(J_1, J_2, J_3)$.

The two sets of unit vectors $\{\hat{\mathbf{R}}_1, \hat{\mathbf{R}}_2, \hat{\mathbf{R}}_3\}$ and $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ are related through the transformation

$$\hat{\mathbf{R}} = S\hat{\mathbf{e}} = [S_1, S_2, S_3]\hat{\mathbf{e}} \quad (1)$$

where S is the so-called (orthogonal) active rotation matrix [3] whose columns we denote by S_1 , S_2 , and S_3 . We define the three-vector (3 by 1 column vector) $\hat{\mathbf{R}} = [\hat{\mathbf{R}}_1, \hat{\mathbf{R}}_2, \hat{\mathbf{R}}_3]^T$ and likewise the three-vector $\hat{\mathbf{e}} = [\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3]^T$. The absolute angular velocity of the body $\boldsymbol{\omega}$ has components in the body-fixed coordinate frame given by the three-vector $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]^T$. These components are related to the active rotation matrix S in Eq. (1) by the relation

$$[\tilde{\boldsymbol{\omega}}]_{i,j} := -\epsilon_{ijk}\omega_k = S^T\dot{S} \quad (2)$$

where ϵ_{ijk} is the usual permutation symbol and the skew-symmetric matrix

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad (3)$$

Let us now describe the configuration of the body at any instant of time t by the n -vector

$$\mathbf{q} := [r^T, u^T]^T \quad (4)$$

where we assume that the position to the center of mass of the body is described by a v -vector (v by 1 column vector) r and the orientation of the body is described by a w -vector u . At each instant of time, the position and orientation of the body is, therefore, described by a total of $n = v + w$ parameters (or generalized coordinates).

In general, we know that the minimum number of independent coordinates required to describe the configuration of the body is six since a rigid body has six degrees of freedom. However, we make *no assumptions on the number (and nature) of the n generalized coordinates used in Eq. (4)*, and they may, for ease of modeling, be allowed to exceed this minimum number. This forms one of the key ideas in the method proposed herein.

Given a set of $n > 6$ generalized coordinates with which we may choose to describe the configuration of the body, we must have $n - 6$ relations (or constraints) between the generalized coordinates because only six independent coordinates are really required. The position three-vector R to the center of mass of the rigid body is then given by

$$R = R(r_1, r_2, \dots, r_v) := R(r) \quad (5)$$

so that \dot{R} is then simply

$$\dot{R} = \frac{\partial R}{\partial r} \dot{r} := G(r) \dot{r} \quad (6)$$

where G is a 3 by v matrix. Similarly, each element of the rotation matrix S is a function of the elements of the w -vector u so that¹

$$S = S(u_1, u_2, \dots, u_w) := S(u) \quad (7)$$

Using Eq. (2), we can then express the components of the angular velocity vector ω in the body-fixed coordinate frame so that the three-vector ω can be expressed as

$$\omega = H(u) \dot{u} \quad (8)$$

where $H(u)$ is a 3 by w matrix.

By Eqs. (6) and (8), the total kinetic energy of the body is

$$T = \frac{1}{2} m \dot{R} \cdot \dot{R} + \frac{1}{2} \omega^T J \omega = \frac{1}{2} m \dot{r}^T G^T G \dot{r} + \frac{1}{2} \dot{u}^T H^T J H \dot{u} = \frac{1}{2} \dot{q}^T M(q) \dot{q} \quad (9)$$

where the n by n block-diagonal matrix

$$M(q) = \text{diag}(mG^T G, H^T J H) \quad (10)$$

and the matrices G and H are functions of q . It should be noted that the matrix M may not be positive definite, in general, but only *positive semidefinite*. This comes about because we have modeled the rigid body with an arbitrary number ($n > 6$) of generalized coordinates.

For example, suppose we choose the generalized coordinate u as the four-vector of (unit norm) quaternions, a coordinate well suited for the parameterization of rigid body rotations [16]. The resultant matrix H in Eq. (8) would then become a 3 by 4 matrix of rank three. This causes the matrix $H^T J H$ in Eq. (10) to become positive semidefinite. Consequently, the matrix M will also become positive semidefinite (singular).

2.2 Description of the Unconstrained System and the Unconstrained Equations of Motion. The next key idea in our approach is to assume that the n generalized coordinates are all *independent* of each other. We, thus, apply Lagrange's equation under the assumption, that *all* the components of the generalized coordinate n -vector q are independent. Thus, for any one body that comprises the multibody system, we have

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = \hat{Q}_k, \quad k = 1, 2, \dots, n \quad (11)$$

where \hat{Q} is an n -vector that contains the generalized "given" forces and torques (part of which may be derived from a potential) acting on the body. The n -vector $\hat{Q} = [\Lambda^T, \Gamma^T]^T$ is composed of the generalized force v -vector Λ and the generalized torque w -vector Γ . Equation (11) will yield, in general, a set of n second-order nonlinear, nonautonomous differential equations, which are simply expressed in the form

$$M \ddot{q} = Q \quad (12)$$

The n -vector Q on the right hand side of Eq. (12) contains, as usual, the generalized "impressed" force-torque vector \hat{Q} and other additional terms generated by applying Lagrange's equation. We note that because of the flexibility provided in the choice of the number (and nature) of the coordinates in the n -vector q and because of our assumption that all the components of q are independent, the equations of motion in Eq. (12) are obtained with

¹In general, we could take $R = R(r_1, r_2, \dots, r_v, t) := R(r, t)$ in Eq. (5) and $S = S(u_1, u_2, \dots, u_w, t) := S(u, t)$ in Eq. (7), but for the sake of simplicity and clarity of exposition we do not explicitly include the time t in these equations.

considerable ease. Also, the generalized force-torque vector \hat{Q} is near-trivial to obtain since the virtual displacements of all the components of q are assumed independent of each other. However, since the generalized coordinates q may not in actuality be independent of one another as we have assumed, the matrix M given by Eq. (10) is, in general, only positive semidefinite.

Now that we have described the equations of motion for a single component body of a multibody system in terms of the generalized coordinates used to describe its position and orientation, we can proceed in a similar manner to obtain the equations of motion for each of the N bodies that comprise the entire multibody system. Let body " i " of the multibody system have mass m^i and an inertia matrix J^i . We shall denote quantities relevant to body i by the superscript i . The position and orientation of body i is then described by the n^i -component column vector of generalized coordinates q^i as described in Eq. (4). Thus, we have a total of

$$K = \sum_{i=1}^N n^i \quad (13)$$

generalized coordinates that specify the configuration of the multibody system, and we assemble these generalized coordinates into the K -vector $q = [(q^1)^T, (q^2)^T, \dots, (q^N)^T]^T$. The n^i -vectors Q^i (and \hat{Q}^i) are now, in general, functions of the K -vectors q and \dot{q} that describe the configuration of the multibody system and its generalized velocity.

Under the assumption that the generalized coordinates describing the configuration of each rigid body (the components of the vector q^i) are all independent of one another and that the coordinates q^i and q^j for all $i \neq j$, $i, j \in (1, N)$ are independent of one another, we next assemble the Lagrange equations for the entire multibody system as

$$M \ddot{q} := \begin{bmatrix} M^1 & 0 & \dots & 0 \\ 0 & M^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M^N \end{bmatrix} \begin{bmatrix} \ddot{q}^1 \\ \ddot{q}^2 \\ \vdots \\ \ddot{q}^N \end{bmatrix} = \begin{bmatrix} Q^1 \\ Q^2 \\ \vdots \\ Q^N \end{bmatrix} := Q \quad (14)$$

where the K by K block-diagonal matrix M is, in general, positive semidefinite.

We refer to these equations as "unconstrained" since they have been arrived at under the assumption that all the components of the K -vector q are independent of one another. In other words, when we write the Lagrange equations (Eq. (14)), we assume that the virtual displacements in each of the coordinates (components of the K -vector q) are independent of the virtual displacements in any of the other coordinates. This forms another key feature in our approach. We note that from a numerical implementation standpoint, assembling the so-called unconstrained equations of motion of the system (as done in Eq. (14) by considering each individual body that makes up the complex multibody system) is highly amenable to parallel processing. This makes the present approach an excellent candidate for parallelization since the equations of motion of each body (or a subgroup of them) can be independently assembled.

2.3 Description and Specification of the Modeling Constraints. In the second step of our three-step procedure, we impose all the necessary modeling constraints on the N bodies so that we appropriately model the complex multibody system. Conceptually, we can think of these constraints as forming two categories. The first category of constraints deals with the fact that we may have chosen more than the minimum number (i.e., $n > 6$) of generalized coordinates q^i that describe the configuration of body i . The second category deals with the fact that since the individual bodies comprising the multibody system are interconnected, the components of the vectors q^i and \dot{q}^i may be affected by

those of q^j and \dot{q}^j at each instant of time. The second category, thus, includes all those physical interactions described by means of constraints between bodies i and j that form the multibody system. We use the following notation to describe them.

The first category of constraints is dealt with by imposing the coordinate constraints (requirements)

$$\phi_k^i(q^i, t) = 0, \quad i = 1, 2, \dots, N, \quad k = 1, 2, \dots, n^i - 6 \quad (15)$$

on Eq. (14). These $g = K - 6N$ constraints reflect the fact that for each body i , the components of the generalized coordinate vector q^i may not be independent of one another. To “construct” the desired complex multibody system from its component bodies, we define the physical interactions, or additional modeling constraints, between each of the bodies that make up the multibody system. We include interactions that are governed by the constraint equations

$$\Phi_k(q, t) = 0, \quad k = 1, 2, \dots, h \quad (16)$$

$$\Psi_k(q, \dot{q}, t) = 0, \quad k = 1, 2, \dots, s \quad (17)$$

which include holonomic and nonholonomic constraints, respectively. Equations (15)–(17) represent all the so-called modeling constraints, which comprise a total of $m = g + h + s$ relations involving the components of the K -vectors q and \dot{q} , and the time, t . When the consistent set of these m modeling constraint equations are sufficiently smooth, we can differentiate them with respect to time and obtain the constraint equation

$$A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t) \quad (18)$$

where A is an m by K matrix and b is an m -vector. Each row of Eq. (18) corresponds to one of the m modeling constraints in Eqs. (15)–(17). The general set of modeling constraints in Eq. (18) that may be imposed on the “unconstrained” multibody system (Eq. (14)) include constraints that are (1) nonlinear functions of q and \dot{q} , (2) explicitly dependent on time, and (3) functionally dependent. The first of these permits nonlinear constraints to be used and not just those in the so-called Pfaffian form; the second permits the constraints to yield nonautonomous dynamical systems; and the third provides one of the key features of our approach with the following purpose in mind.

In a complex multibody system, it can be difficult to determine which of the constraints are functionally dependent. This is especially so for systems with nonholonomic constraints since these constraints take the form of differential equations as mentioned before. This often makes it difficult when modeling a complex multibody system to decipher whether a given set (or subset) of constraints implies another. In fact, were we to repeat a constraint (possibly in a different form) as part of our set of constraints, standard methods like the Lagrange multiplier methods will fail (see Sec 2.4 below). This difficulty is directly averted in our approach, thereby greatly easing the modeler’s effort. Thus, another key feature of the approach is the facility provided to the modeler in placing as many constraints that (s)he can uncover in the multibody system without worrying about (a) whether a constraint has been repeated (possibly in some other form) and/or (b) whether a particular subset of constraints, in certain regions (or points in time) in the system’s phase space, might imply another constraint that has also been included in the set.

We note that the methodology proposed herein follows along logical lines, lines that would be used to mentally construct the multibody system. First, one assembles each body that constitutes the multibody system, as was done in Eq. (14), by using the most convenient set of generalized coordinates (with no restriction on the total number and nature) needed to describe it. Then, one assembles the necessary constraints engendered, as shown in Eq. (18) (or alternatively, in Eqs. (15)–(17)). The method does not require the modeler to undertake the task of identifying all the independent constraints and weeding out the dependent ones. *Any and every constraint* that can be identified by the modeler can be

included in this set of m constraints.

Finally, we present the last step in our three-step modeling procedure. Using the description of the unconstrained system given by Eq. (14) along with the description of the modeling constraints given by Eq. (18), we obtain the explicit equations of motion of the complex multibody system.

2.4 Explicit Equations of Motion for Complex Multibody Systems. The presence of the modeling constraints cause generalized forces to be exerted on the unconstrained system described by Eq. (14). Thus, to accommodate these modeling constraints on the multibody system, an additional generalized force K -vector \mathcal{Q}^m must be applied so that the required motion satisfies the constraints in Eq. (18). The equation of motion for the complex multibody system is then simply expressed as

$$\mathcal{M}\ddot{q} = \mathcal{Q} + \mathcal{Q}^m(q, \dot{q}, t) \quad (19)$$

The explicit acceleration for constrained mechanical systems when the K by K matrix \mathcal{M} is positive semidefinite is obtained by using the UP equation. Udwardia and Phohomsiri [5] showed that a necessary and sufficient condition for the equation of motion of such a constrained system to yield a unique generalized acceleration at each instant of time—a requirement demanded by physical observation of mechanical systems—is that the matrix

$$\hat{\mathcal{M}} = [\mathcal{M} \quad A^T] \quad (20)$$

has rank K . The satisfaction of this requirement (the UP rank condition) can also be viewed as a check to the modeler on the validity of the modeling process that has been carried out. This is useful because it provides in addition to the facility given to the modeler (1) in choosing an arbitrary number of ($n^i > 6$) generalized coordinates to model each rigid body and (2) in placing as many constraints as can be deciphered in the multibody system, a check that the modeling has been done in a manner that respects the fact that the accelerations in a physical system must be uniquely ascertainable.

Thus, when $\mathcal{M} \geq 0$ and the aforementioned rank condition is satisfied, the explicit acceleration of the constrained system is given by [5]

$$\ddot{q} = \begin{bmatrix} (I - A^+A)\mathcal{M} \\ A \end{bmatrix}^+ \begin{bmatrix} \mathcal{Q} \\ b \end{bmatrix} := \bar{\mathcal{M}}^+ \begin{bmatrix} \mathcal{Q} \\ b \end{bmatrix} \quad (21)$$

where the $(\cdot)^+$ notation denotes the Moore-Penrose matrix inverse. If needed, the modeling constraint force, which arises as a result of the presence of the modeling constraints, can be explicitly determined by substituting the expression for \ddot{q} from Eq. (21) into Eq. (19) so that

$$\mathcal{Q}^m = \mathcal{M}\bar{\mathcal{M}}^+ \begin{bmatrix} \mathcal{Q} \\ b \end{bmatrix} - \mathcal{Q} \quad (22)$$

We note that nowhere in this discussion is the notion of a Lagrange multiplier invoked. There are several advantages to doing this.

- (1) The statement of the problem of constrained motion does not include the notion of a Lagrange multiplier and, therefore, quite naturally, nowhere in the final solution of the problem does it appear.
- (2) A Lagrange multiplier is an intermediary notion that was developed by Lagrange to handle constrained motion; it needs to be eliminated when one seeks the final solution.
- (3) As is clear from the discussion above, recent developments in analytical dynamics point to the fact that the equations of constrained motion can be obtained explicitly and directly without the use of this intermediary notion (and calculation of the Lagrange multiplier) so that parsimony (Occam’s razor) and simplicity would dictate that it need not be invoked.

(4) Finally, the standard Lagrange multiplier formulation of the problem of constrained motion requires the solution of the augmented $(K+m)$ by $(K+m)$ matrix equation

$$\mathcal{M}_L \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} := \begin{bmatrix} \mathcal{M} & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} Q \\ b \end{bmatrix} \quad (23)$$

where λ is the Lagrange multiplier m -vector. As shown in the Appendix, for the matrix \mathcal{M}_L to be nonsingular and, hence, for the formulation in Eq. (23) to be usable, the following two requirements must be met: (1) The constraints *must* be functionally independent and (2) the rank of $\hat{\mathcal{M}}$ in Eq. (20) must be K . We observe that only the second requirement is needed for Eq. (21) to be valid. Thus, Eq. (21) is more general than Eq. (23), making it useable when Eq. (23) fails to give the correct equations of motion of the system.

In fact, as any two (or more) constraints “approach” functional dependence, the numerical accuracy of the solution given by Eq. (23) deteriorates until finally, the equation becomes unusable when the two constraints become dependent. In order to ensure that \mathcal{M}_L is nonsingular when using the Lagrange multiplier method, the modeler is, therefore, required to ferret through all the constraints and ensure that they are all independent at each instant of time. Since the first requirement stated above is not needed when Eq. (21) is used, we find that it contains considerable practicality. By using Eq. (21), it is not necessary to consider which constraint subset may be functionally dependent. As mentioned earlier, this is a key feature of our approach. Even in the relatively low dimensional example that is used to illustrate our methodology in Sec. 3, we see that this feature becomes important in facilitating the modeling process and that use of the standard Lagrange multiplier method (Eq. (23)) will fail.

We note that Eq. (21) can be computed in real time. The modeler does not need to worry about whether the matrix \mathcal{M} is positive definite or positive semidefinite; the equation is applicable in both cases. Furthermore, one or more of the modeling constraints may be easily removed, inserted, or altered (these changes being reflected in equation set (18)), thereby allowing one to assess the effect of imposing a certain set of constraints as opposed to some other set of constraints. Thus, the sensitivity of the ensuing dynamics of a multibody system to the use of one constraint as opposed to another can be handily studied. Similarly, the effect on the motion due to alterations in the parameters in one or more of the constraints can also be easily studied. This provides a robust modeling procedure since it is not required to remodel the entire multibody system with the removal, addition, or alteration of one or more of the modeling constraints.

Finally, we note that we do not need to use an inertial frame to represent the motion of the so-called unconstrained system. We can write the proper equations of motion for each of the component bodies expressed in *any* suitable set of coordinates, express the constraints in terms of the appropriate coordinates, and use the UP equation to get the equation of motion of the multibody system. In the following, we use the general methodology developed in this section and show its applicability by formulating the model of a multibody spacecraft system and investigating the numerical results that we obtained.

3 Example: Multibody Spacecraft System

In this example, we carry out the model development of a realistic multibody spacecraft system. In order to illustrate the central ideas in the approach proposed herein, it will suffice to use just two nonlinearly interacting rigid bodies; more rigid bodies (and more interconnections among them) would no doubt add to the complexity of the system, though possibly at the expense of obfuscating the main ideas underlying the methodology proffered herein. Our aim, therefore, in choosing this example, besides its

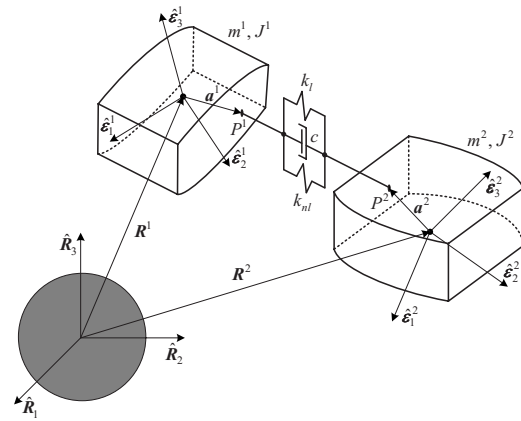


Fig. 2 A multibody spacecraft system consisting of two interconnected rigid bodies ($N=2$) in a uniform gravitational field. The connection between the two bodies at points P^1 and P^2 is modeled by two springs and a damper. The spring constants k_l and k_{nl} refer to the linear and cubically nonlinear restoring forces exerted by the springs, and the linear damping coefficient is denoted by c . The two bodies are free to rotate about and move along the line P^1P^2 , which is fixed relative to each body.

realistic nature, is to highlight the key features of the methodology. The problem considered, however, is complex in that the system is highly nonlinear and it is subjected to a nonlinear set of external forces, causing the dynamics to evolve at multiple distance and time scales as it simultaneously translates, tumbles, and vibrates. We also provide numerical results showing such a system’s response as it orbits in a circular low-Earth orbit, thereby illustrating the numerical accuracy of the approach.

3.1 Model Development. Consider a multibody spacecraft system in orbit around a central body with a uniform gravity field as illustrated by Fig. 2. The system is modeled using two rigid bodies ($N=2$) with masses m^1 and m^2 and principal inertia tensors J^1 and J^2 . As stated before for the sake of illustrating the underlying methodology, we will only take two rigid bodies. The two bodies are connected at arbitrary locations P^1 and P^2 by a linear spring with stiffness coefficient k_l and a nonlinear spring with stiffness coefficient k_{nl} . It is also assumed that a linear viscous damper with a damping coefficient c is present along the line P^1P^2 (see Fig. 2). The connection points P^1 and P^2 are located by the position vectors $R^1 + a^1$ and $R^2 + a^2$ where the vectors a^1 and a^2 are fixed relative to the respective body-fixed coordinate frames in the two bodies. The two rigid bodies are free to (1) move along the line P^1P^2 , which is fixed in a direction relative to the two coordinate frames $\{\hat{e}_1^i, \hat{e}_2^i, \hat{e}_3^i\}$, $i=1,2$, and (2) rotate independently about this line.

The system, therefore, has eight degrees of freedom since relative motion occurs in translation along the line P^1P^2 and in rotation about the line P^1P^2 . In what follows, we will use the notation established earlier to denote quantities relevant to each of the two bodies.

Step 1: Unconstrained equations of motion. To carry out the first step in our three-step procedure for obtaining the equations of motion of this multibody system, we determine the unconstrained equations of motion for the two rigid bodies that comprise the multibody spacecraft. Let the generalized coordinate vector for each body be taken as

$$q^i = [R_1^i, R_2^i, R_3^i, u_1^i, u_2^i, u_3^i, u_4^i]^T, \quad i=1,2 \quad (24)$$

so that $n^i=7$, $i=1,2$. We shall assume that the orbital position R^i of the center of mass of body i is represented by the inertial coordinate three-vector $R^i=[R_1^i, R_2^i, R_3^i]^T$ and the orientation of

body i is represented by the unit quaternion four-vector $u^i = [u_1^i, u_2^i, u_3^i, u_4^i]^T$. By a unit quaternion, we mean

$$(u^i)^T u^i = 1, \quad i = 1, 2 \quad (25)$$

We note that the number of coordinates chosen to describe the configuration of each of the two bodies that constitute the multi-body system exceeds six, the minimum number needed and, therefore, the seven coordinates chosen in Eq. (24) cannot all be independent of one another.

The rotation matrix S^i (Eq. (1)) associated with body i is parameterized by the quaternion u^i as

$$S^i(u^i) = \begin{bmatrix} u_2^i & u_1^i & -u_4^i & u_3^i \\ u_3^i & u_4^i & u_1^i & -u_2^i \\ u_4^i & -u_3^i & u_2^i & u_1^i \end{bmatrix} \begin{bmatrix} u_2^i & u_3^i & u_4^i \\ u_1^i & -u_4^i & u_3^i \\ u_4^i & u_1^i & -u_2^i \\ -u_3^i & u_2^i & u_1^i \end{bmatrix} \quad (26)$$

Therefore, the chosen generalized coordinates R^i and u^i (see Eqs. (5) and (7)) yield the corresponding matrices (see Eqs. (6) and (8))

$$G^i = I_3^i, \quad i = 1, 2 \quad (27)$$

$$H^i = 2E^i, \quad i = 1, 2 \quad (28)$$

where the matrix E^i is the 3 by 4 matrix given by

$$E^i = \begin{bmatrix} -u_2^i & u_1^i & u_4^i & -u_3^i \\ -u_3^i & -u_4^i & u_1^i & u_2^i \\ -u_4^i & u_3^i & -u_2^i & u_1^i \end{bmatrix} \quad (29)$$

The angular velocity of body i in its body-fixed reference frame is given by Eq. (8) so that

$$\omega^i = 2E^i \dot{u}^i \quad (30)$$

The generalized force-torque vectors $\hat{Q}^i = [(\Lambda^i)^T, (\Gamma^i)^T]^T$, $i = 1, 2$, exerted on the individual bodies are generated by the presence of gravitational, elastic, and damping forces. The uniform gravitational potential of body i is

$$U_g^i(R^i) = -\frac{\mu_g m^i}{\|R^i\|}, \quad i = 1, 2 \quad (31)$$

where $\|\cdot\|$ denotes the two-norm operation and the gravitational parameter μ_g is the product of the gravitational constant and the mass of the Earth. The elastic potential of the linear and nonlinear spring is given by

$$U_e(q) = \frac{1}{2}k_r(\|D\| - \ell_e)^2 + \frac{1}{4}k_{nl}(\|D\| - \ell_e)^4 \quad (32)$$

In Eq. (32), ℓ_e is the unstretched length of the two springs between the points P^1 and P^2 , and the relative distance vector D between the points P^1 and P^2 is given by

$$D = R^1 + a^1 - R^2 - a^2 \quad (33)$$

The viscous damping is described by the Rayleigh dissipation function

$$U_d(q, \dot{q}) = \frac{1}{2}c\dot{D} \cdot \dot{D} \quad (34)$$

where

$$\dot{D} = \dot{R}^1 + \omega^1 \times a^1 - \dot{R}^2 - \omega^2 \times a^2 \quad (35)$$

Here, we note that the components of the vectors R^i , ω^i , and a^i in Eqs. (33) and (35) are all to be resolved in a consistent frame of reference. Using Eqs. (31), (32), and (34), the generalized forces on body i , assuming no “impressed forces” are applied to it, are, thus,

$$\Lambda^i = -\frac{\partial U_g^i}{\partial R^i} - \frac{\partial U_e}{\partial R^i} - \frac{\partial U_d}{\partial \dot{R}^i}, \quad i = 1, 2 \quad (36)$$

Similarly, the generalized torques on body i , assuming no “impressed torques” are applied to it, are

$$\Gamma^i = -\frac{\partial U_e}{\partial u^i} - \frac{\partial U_d}{\partial \dot{u}^i}, \quad i = 1, 2 \quad (37)$$

The first step in our approach is accomplished by using Lagrange’s equation under the key assumption that each of the components of the generalized coordinate vectors q^i , $i = 1, 2$, are independent of the others, (see Eq. (11)). This gives

$$M^i(q)\ddot{q}^i = Q^i(q, \dot{q}), \quad i = 1, 2 \quad (38)$$

where

$$M^i = \begin{bmatrix} m^i j_3^i & 0 \\ 0 & 4(E^i)^T J^i E^i \end{bmatrix}, \quad Q^i = \begin{bmatrix} \Lambda^i \\ -8\dot{E}^T J^i E \dot{u} + \Gamma^i \end{bmatrix}, \quad i = 1, 2 \quad (39)$$

The expressions for some of the individual terms of Λ^i and Γ^i are inordinately long and have not, for brevity, been given here. The unconstrained equations of motion for the two bodies are, thus, obtained as (see Eq. (14))

$$\mathcal{M}\ddot{q} := \begin{bmatrix} M^1 & 0 \\ 0 & M^2 \end{bmatrix} \begin{bmatrix} \dot{q}^1 \\ \dot{q}^2 \end{bmatrix} = \begin{bmatrix} Q^1 \\ Q^2 \end{bmatrix} := Q \quad (40)$$

where \mathcal{M} is a 14 by 14 matrix and the vector Q is a 14-vector since $K = 14$.

This first step illustrates the following three important features of the method. (1) The ease with which these equations can be written; this is because we have used far more coordinates (a total of 14) to describe the configuration of the system than the minimum number required, which is eight. (2) Lagrange’s equations are determined under the assumption that all the coordinates are independent of one another. (3) The matrix \mathcal{M} in Eq. (40) is singular because $4(E^i)^T J^i E^i$ is singular, which is a consequence of the prior two features.

Step 2: *Description of constraints.* We next impose the necessary constraints on the two rigid bodies that are a consequence of

- (i) not having used the minimum number of coordinates to describe the configuration of each body
- (ii) not yet having expressed the proper interconnections between the two bodies in our unconstrained equations of motion

The method does not impose any restrictions on the number (or nature) of constraints to be used; the modeler can include whatever constraints are decipherable, irrespective of whether they are independent or not.

Thus, we have the two required unit quaternion constraints (see Eq. (15))

$$\phi_1^1 = (u^1)^T u^1 - 1 = 0 \quad (41)$$

and

$$\phi_1^2 = (u^2)^T u^2 - 1 = 0 \quad (42)$$

which are required so that the quaternions u^1 and u^2 represent real physical rotations.

In addition, we have the modeling constraints due to the physical interaction of the two bodies. This interaction is governed by the requirement that the direction of the relative distance vector D is fixed relative to the two body-fixed coordinate frames. This requirement is modeled by the modeling constraint equations

$$\hat{n}^1 \times D = 0 \quad \text{and} \quad \hat{n}^2 \times D = 0 \quad (43)$$

The unit vectors \hat{n}^1 and \hat{n}^2 in Eq. (43) are determined by the locations of the spring and damper connections at the points P^1 and P^2 (see the numerical example in the next subsection for their description). Equation (43) leads, in general, to a total of six constraints, one for each component of the cross product, so that we have the set of constraints (see Eq. (16))

$$\Phi_k(q) = 0, \quad k = 1, 2, \dots, 6 \quad (44)$$

It is important to point out that Eq. (43) is an intuitive modeling constraint, which is a modeling requirement that is easily derived when mentally constructing the multibody system from the two rigid bodies.

Now, by appropriately differentiating each of the $m=8$ modeling constraint equations with respect to time, we can form the modeling constraint matrix equation as in Eq. (18). This process generates an 8 by 14 matrix A whose rank is six, and an eight-vector b . For ease of implementation, this differentiation can be easily carried out symbolically using platforms like MAPLE, MATHEMATICA, or MATLAB.

Step 3: *Determination of the equations of motion of the multibody system.* At this point we have all the information needed to obtain both the explicit acceleration of the multibody spacecraft system and the explicit force of constraint needed to satisfy the imposed modeling constraint set. We simply use the UP equation (Eqs. (21) and (22)), which is valid because the UP rank condition (the matrix \hat{M} (Eq. (20)) has full rank) is satisfied with the rank of \hat{M} being 14. The fact that this matrix has full rank implies that the generalized acceleration can be uniquely found, which is necessary for obtaining the equations of motion of the multibody system and is also a useful check on the validity of our modeling. On the other hand, were the standard Lagrange multiplier method to be used (see Eq. (23)), it would fail since the matrix \mathcal{M}_L is singular. Finally, since the equations of motion of the multibody system are directly and explicitly found, it is simple to numerically implement them using a standard ode solver.

As we shall shortly see, the nonlinear spacecraft system described in this section has complex dynamical behavior: it includes translational, tumbling, and vibrational motion, as well as multiple characteristic time scales ranging from several thousand seconds to a few tens of seconds, and multiple characteristic distance scales ranging from several thousand kilometers to 10^{-4} m. Such nonlinear, multiscale systems are often difficult to accurately model and usually pose considerable challenges from a numerical standpoint.

3.2 Numerical Results. We now provide numerical results showing the response of the modeled multibody spacecraft system. The initial conditions of the two-body spacecraft system shown in Fig. 3 are specified so that the system is in a circular low-Earth orbit where the center of mass of body 1 is at a constant altitude of 300 km. The mass m^i and the principal inertia matrix J^i of each component body i are given by

$$m^1 = 2200 \text{ kg} \quad (45)$$

$$J^1 = \text{diag}(2300, 4500, 3600) \text{ kg m}^2 \quad (46)$$

$$m^2 = 1200 \text{ kg} \quad (47)$$

$$J^2 = \text{diag}(1700, 2000, 600) \text{ kg m}^2 \quad (48)$$

The linear and nonlinear elastic spring constants are $k_l = 25 \text{ N m}^{-1}$ and $k_{nl} = 1 \text{ N m}^{-3}$, respectively, and the damping coefficient is taken to be $c = 0.2 \text{ N m}^{-1} \text{ s}$. The equilibrium length of the spring is $\ell_e = 2 \text{ m}$. The spring connections are located relative to the two spacecraft by the position vectors a^1 and a^2 (see Fig. 3) whose components in the body-fixed frame of reference are given by the three-vectors

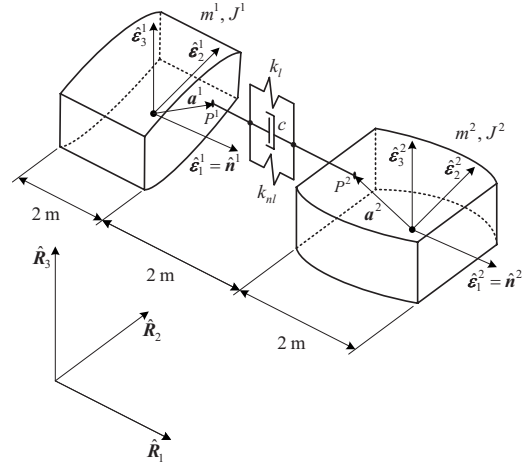


Fig. 3 A view illustrating the initial configuration of the system

$$a^1 = [1, 0, 1]^T \text{ m} \quad (49)$$

and

$$a^2 = [-1, 0, 1]^T \text{ m} \quad (50)$$

The components of the initial position and velocity vectors in the inertial frame are

$$R^1(0) = [\mathcal{R}_E + 300, 0, 0]^T \text{ km} \quad (51)$$

$$\dot{R}^1(0) = [0, \sqrt{\mu_g/R_1^1(0)}, 0]^T \text{ km s}^{-1} \quad (52)$$

$$R^2(0) = [\mathcal{R}_E + 300.004, 0, 0]^T \text{ km} \quad (53)$$

$$\dot{R}^2(0) = [0, \sqrt{\mu_g/R_1^1(0)}, 0]^T \text{ km s}^{-1} \quad (54)$$

where the gravitational parameter $\mu_g = 3.986 \times 10^5 \text{ km}^3 \text{ s}^{-2}$ and $\mathcal{R}_E = 6378.1 \text{ km}$ is the equatorial radius of Earth. Thus, initially, the distance between the two masses is 4 m; the springs are unstretched, and the distance between them is 2 m (see Fig. 3). The velocities in Eqs. (52) and (54) correspond to the required velocity for an equatorial circular orbit. The initial rotational state is chosen so that

$$u^1(0) = u^2(0) = [1, 0, 0, 0]^T \quad (55)$$

$$\dot{u}^1(0) = \dot{u}^2(0) = [0, 0, 0, 0]^T \quad (56)$$

Given the spring connection locations a^1 and a^2 in Eqs. (49) and (50) and the initial position and orientation of the two bodies, the required components of the unit vectors \hat{n}^1 and \hat{n}^2 expressed in the respective body-fixed reference frames of the two bodies are simply (see Fig. 3)

$$n^1 = [1, 0, 0]^T \quad (57)$$

and

$$n^2 = [1, 0, 0]^T \quad (58)$$

The numerical integration of the multibody spacecraft system found by Eq. (21) is carried out for a time duration $t \in (0, 16293) \text{ s}$ using a variable time step Runge-Kutta scheme with a relative error tolerance of 10^{-10} and an absolute error tolerance of 10^{-13} . The duration of integration corresponds to approximately three orbital periods.

In order to check the fidelity of our multibody model, whose description is provided by the equations of motion obtained in step three of our methodology, our attention must first be drawn to the extent to which all the modeling constraints are satisfied. The numerically integrated equations of motion must result in these constraints being satisfied, or else our model would be deficient.

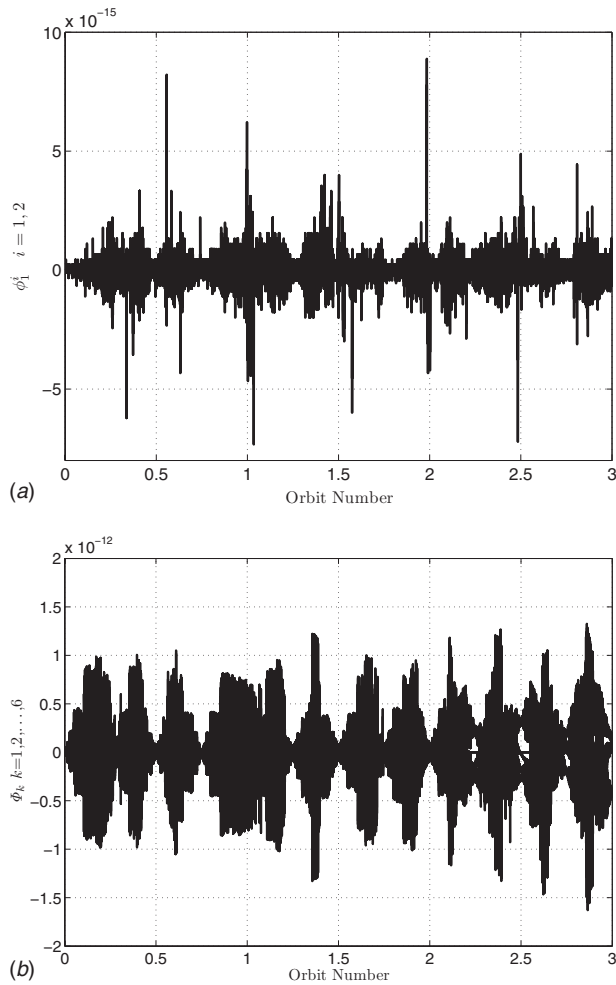


Fig. 4 (a) The coordinate modeling constraints ϕ_1^i , $i=1,2$, over three orbits and (b) the physical modeling constraints Φ_k , $k=1,2,\dots,6$, over three orbits; error in the eight modeling constraints showing their satisfaction throughout the integration

To investigate this, we show in Figs. 4(a) and 4(b) the superimposed plots of the errors in the modeling constraints ϕ_1^i , $i=1,2$, and Φ_k , $k=1,2,\dots,6$, respectively, over the duration of the integration. Noting the vertical scales, we find that all eight modeling constraints are satisfied better than the local error tolerances used when numerically integrating our modeled multibody spacecraft system.

The radii $\|R^1\|$ and $\|R^2\|$ throughout the integration are shown in Fig. 5. The quaternions u^i , $i=1,2$, for the two spacecraft are shown in Fig. 6 revealing large angle rotational motions as the system orbits. The body-fixed angular rates ω^i throughout the simulation are found by Eq. (29), and they are shown in Fig. 7. Finally, Fig. 8 shows the difference $\|\mathbf{D}\| - \ell_e$ between the magnitude of the relative distance $\|\mathbf{D}\|$ and the unstretched length ℓ_e of the spring over the three orbit simulation. The thick solid line in Fig. 7(a) contains higher frequency oscillations and for illustration, the oscillations are shown over a small fraction of an orbital period in Fig. 7(b). The resulting vibrational motion has an approximate period of 32 s and it decays throughout the orbital motion. The spring is stretched throughout most of the three orbit period while contractions only occur in the first orbit.

As a final indication of the numerical accuracy of the method proposed herein, we consider the two relations, Eqs. (33) and (35). These two relations must be analytically valid throughout the complex motion of this oscillating system as it tumbles and vibrates in its motion around the central body in the nonlinear gravi-

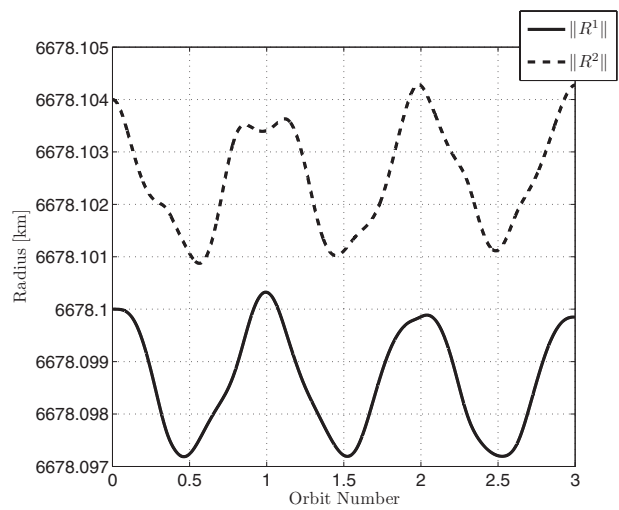


Fig. 5 The radial distance to bodies 1 and 2 over three orbits

tational field. Computationally, we can determine the extent to which our numerical results satisfy, at each instant of time, the relations

$$R^1 + a^1 - D - R^2 - a^2 = 0 \quad (59)$$

and

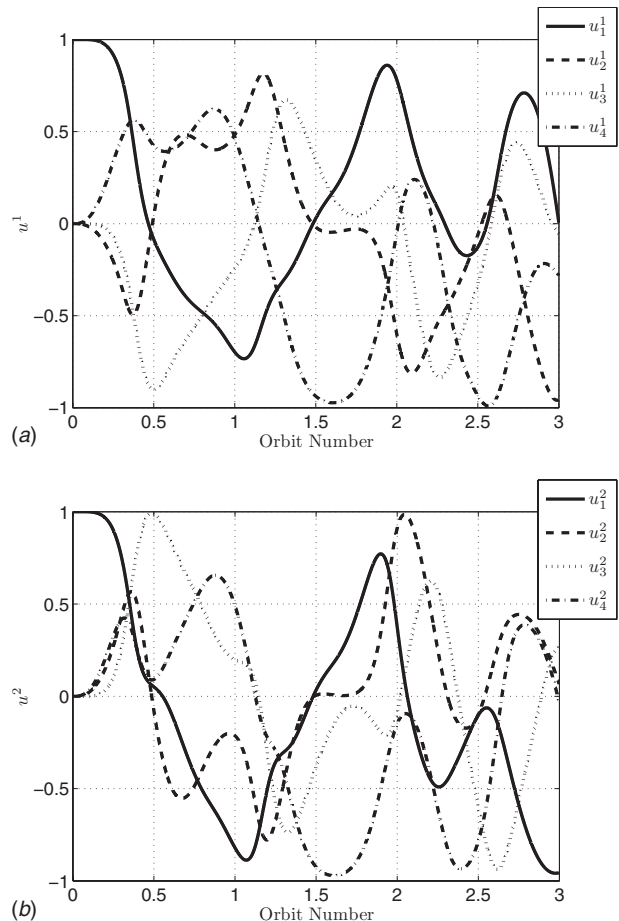


Fig. 6 (a) Unit quaternion of body 1 and (b) unit quaternion of body 2; unit quaternions u^i , $i=1,2$, over three orbits

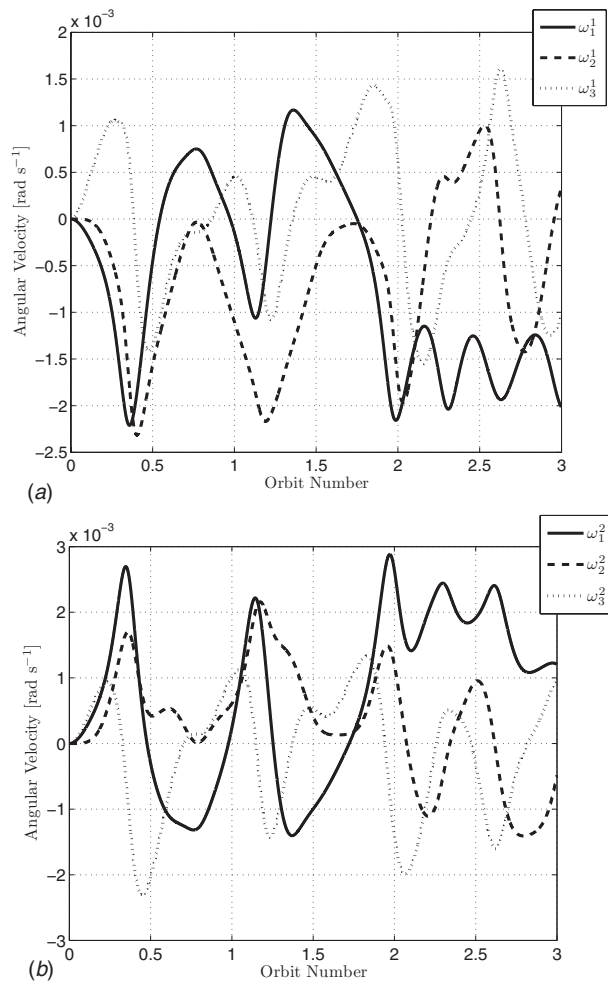


Fig. 7 (a) Angular velocity of body 1 found by Eq. (40) and (b) angular velocity of body 2 found by Eq. (40); body-fixed angular velocities ω^i , $i=1,2$, over three orbits

$$\dot{R}^1 + \omega^1 \times a^1 - \dot{D} - \dot{R}^2 - \omega^2 \times a^2 = 0 \quad (60)$$

Expressing each of the vectors on the left hand side of Eq. (59) in terms of their components measured in the inertial frame of reference, we have

$$e_1 = R^1 + S^1 a^1 - D - R^2 - S^2 a^2 \quad (61)$$

where the column vectors R^1 , R^2 , and D are 3 by 1 column vectors containing components in the inertial frame of R^1 , R^2 , and D , respectively. The 3 by 1 column vectors a^1 and a^2 contain the components of a^1 and a^2 in the body-fixed frame (as in Eqs. (49) and (50)). The rotation matrices S^1 and S^2 are given by Eq. (26). Similarly, the left hand side of Eq. (60) yields

$$e_2 = \dot{R}^1 + S^1 \dot{\omega}^1 a^1 - \dot{D} - \dot{R}^2 - S^2 \dot{\omega}^2 a^2 \quad (62)$$

By Eqs. (59) and (60), theoretically speaking, the three-vectors e_1 and e_2 must both be zero at each instant of time. Figure 9 plots the numerical values of the three components of e_1 and the first component of e_2 versus orbit number. As seen from Figs. 9(a)–9(d), Eqs. (59) and (60) are satisfied to orders of magnitude commensurate with the relative tolerance used in the numerical integration scheme. Though they are not shown in Fig. 9 for conciseness, the second and third components of the three-vector e_2 are also zero to the same order of accuracy displayed in Fig. 9(d). This points out, as do Figs. 1 and 2, the high numerical accuracy attained by the approach presented herein.

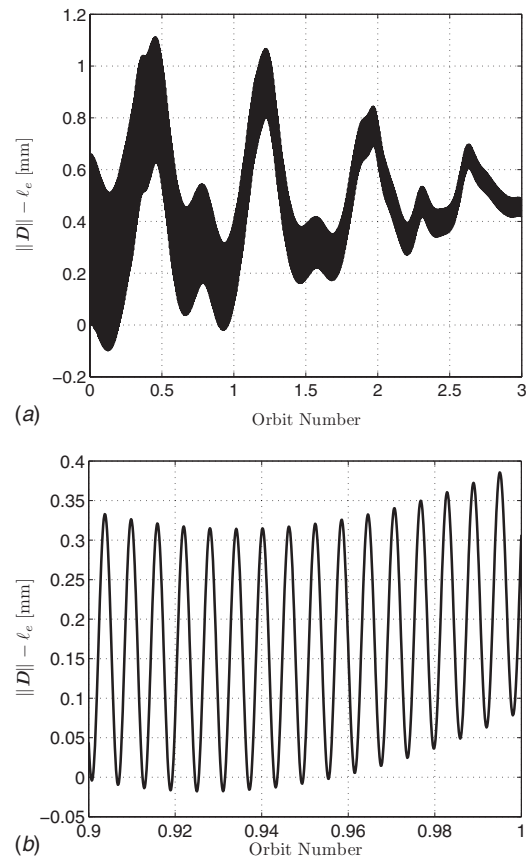


Fig. 8 (a) Vibrational motion between bodies 1 and 2 showing stretching and contracting and (b) vibrational motion over a fractional orbit period showing higher frequency oscillations; vibrational motion between the spring and damper connections P^1 and P^2 over three orbits

4 Conclusions

This paper provides a simple and general three-step approach for developing the equations of motion for complex multibody systems composed of interconnected rigid bodies. The three steps involve (1) description of the unconstrained multibody system in terms of each of its component bodies, (2) description of the constraints between the coordinates that describe the configuration of each component body and those describing the interconnections between these components, and (3) the description of the constrained system that yields the equation of motion for the complex multibody system. Throughout this treatment, our emphasis has been on the ease with which the equations of motion can be obtained, that is, the facility and flexibility afforded to the modeler in formulating the equations of motion, their ease of computational implementation, and their numerical accuracy.

Each step in the abovementioned modeling process proposed herein has the following certain key features that current methods lack. (1) In the first step, the modeler is given the convenience of choosing more coordinates than the minimum number required to describe the configuration of one or more of the bodies that constitute the system. (2) In the second step, all the modeling constraints, which the modeler can discern, describing the coordinate relations and the physical connections between the bodies are included in the modeling process. The modeler is relieved of the trouble of finding the minimum number of (functionally) independent constraints that are required to model the interconnections between the various components of the multibody system. The modeler can put down as many constraints as (s)he wants even if they are not independent, as long they all consistently describe the

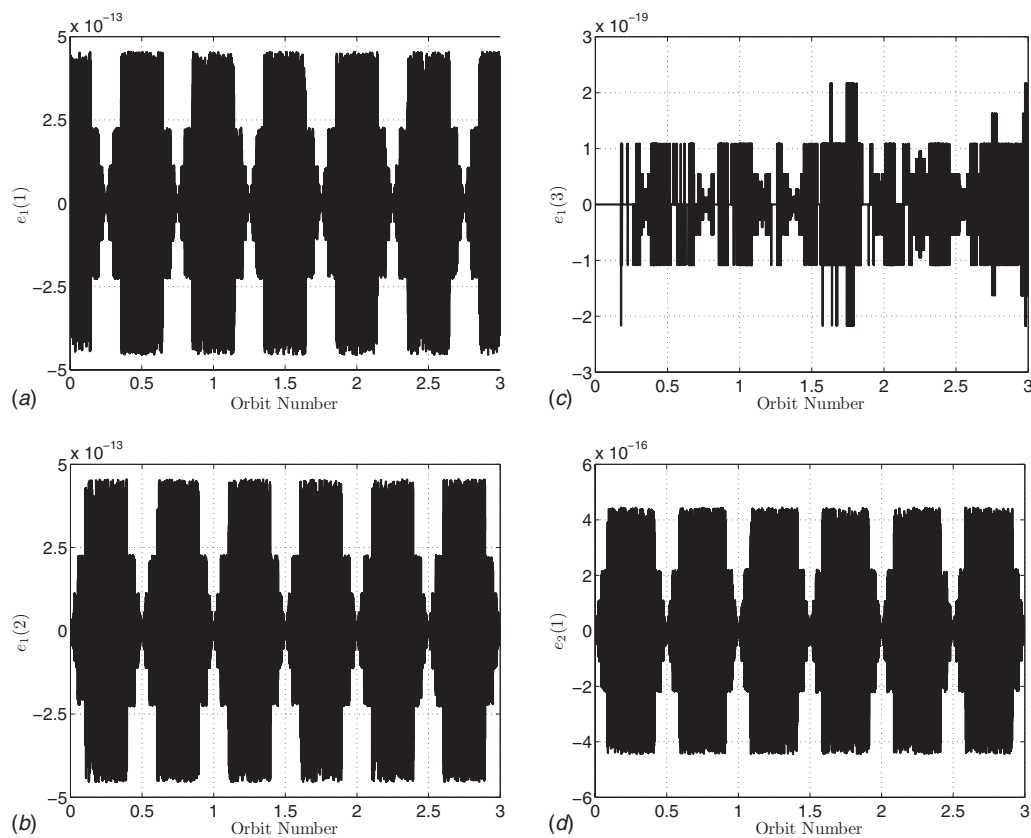


Fig. 9 (a) Error in the first component of e_1 , (b) error in the second component of e_1 , (c) error in the third component of e_1 , and (d) error in the first component of e_2 ; errors e_1 and e_2 versus orbit number

multibody system. (3) The last step simply uses the Udwadia–Phohomsiri equation to give the explicit equation of motion of the multibody dynamical system [5] and the explicit force of constraint required to satisfy the imposed modeling constraints. No attention needs to be paid to whether the mass matrix is positive definite or semipositive definite. A check on the validity of the modeling is also provided to the modeler through the UP rank condition. To the best of the authors' knowledge, no currently available general methodologies for modeling complex multibody systems are capable of achieving these useful features collectively in as uniform and simple a manner. These features, which in turn lead to several others that distinguish the approach from those available hereto, have been explained in detail in the paper.

Besides its simplicity and effectiveness, this three-step procedure has a certain intuitive feel to it since it conceptually flows along the same logical line of thinking where one would start with a set of N component rigid bodies and subsequently use them to mentally construct the desired multibody system through the addition of the appropriate interconnections (constraints) between them, while permitting oneself the luxury of using more coordinates (than the minimum needed) to describe the configuration of each component body.

The main contributions of this paper are the following.

1. The facility with which more coordinates than the minimum number can be used in the formulation of problems in multibody dynamics. Using more generalized coordinates than necessary to describe the configuration of one or more of the component bodies can often provide greater convenience and flexibility to the modeler, especially when dealing with complex systems. However, this leads, in general, to a mass matrix \mathcal{M} that may not be positive definite, but rather positive semidefinite. Our ability to directly deal with such matrices in the formulation of the equations of motion for

multibody systems ultimately rests on deeper results from analytical dynamics, specifically the recently developed UP equation for constrained systems [5].

2. The initial assumption that all the coordinates are independent greatly simplifies writing Lagrange's equations for the component bodies and especially the determination of the generalized force-torque vector.
3. Having chosen more than the minimum number of coordinates to facilitate her/his formulation, the modeler is provided information on the suitability of her/his choice of coordinates by the UP rank condition [5]. This condition requires that the matrix $\hat{\mathcal{M}} = [\mathcal{M} \ A^T]$ has full rank K (equal to the number of its rows) so that the final, resulting acceleration of the multibody system at each instant of time is unique, a requirement based on physical observations of the motion of mechanical systems.
4. The freedom to model a multibody system by using modeling constraints that may be functionally dependent or redundant is an aspect that can have considerable value in increasing the ease with which a complex multibody system is modeled. The modeler would indeed be required to identify the functionally dependent constraints if (s)he uses standard Lagrange multiplier methods. This is because these methods fail when functionally dependent constraints are used. The approach developed in this paper is seamlessly used in these situations.
5. Sensitivity of the ensuing dynamics to the removal/addition/alteration of constraints can be easily carried out because no reformulations of the entire system are required, as are often necessary in other approaches.
6. From an implementation point of view, the approach appears amenable to parallelization, and this may open up new com-

putational approaches when dealing with large-scale, complex multibody systems.

7. The example of a realistic interconnected two-body spacecraft in a low-Earth orbit illustrates all the key features of the modeling approach presented herein, including its ease of use and its accuracy. The dynamics of the nonlinear system is complex as it undergoes translation, tumbling, and vibration at multiple time and distance scales. The numerical results show that all the constraints are well satisfied. In fact, they are numerically satisfied (see Figs. 1 and 2) to orders of magnitude commensurate with the relative error tolerance used in the numerical integration of the equations of motion.
8. The modeling approach developed herein yields an explicitly generated set of equations of motion for multibody systems that are simple to construct, easy to computationally implement, and yield numerically accurate results.

Appendix

We prove here that the $(K+m)$ by $(K+m)$ matrix \mathcal{M}_L in Eq. (23) is nonsingular if and only if the following two conditions are satisfied.

- (1) the rank of the m by K matrix A is m
- (2) the rank of the matrix $\hat{\mathcal{M}}=[\mathcal{M} \ A^T]$ is K

where the matrix \mathcal{M} is positive definite or positive semidefinite.

Proof. (a) Let us assume that the two requirements stated above are satisfied. We shall show that the matrix \mathcal{M}_L is nonsingular. All we need to show then is that the equation

$$\mathcal{M}_L \begin{bmatrix} \alpha \\ \beta \end{bmatrix} := \begin{bmatrix} \mathcal{M} & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{A1})$$

has one and only one solution $\alpha=\beta=0$. Equation (A1) implies that

$$\begin{aligned} \mathcal{M}\alpha &= A^T\beta \\ A\alpha &= 0 \end{aligned} \quad (\text{A2})$$

Premultiplying the first of these by α^T and using the second, we get

$$\alpha^T \mathcal{M} \alpha = (A\alpha)^T \beta = 0 \quad (\text{A3})$$

from which it follows that $\mathcal{M}^{1/2}\alpha=0$ since \mathcal{M} is positive semidefinite (or positive definite). Hence, $\mathcal{M}\alpha=0$. Thus, by the first equation in relation (A2) we must have $A^T\beta=0$, and since the

rank of A^T is m , the unique solution of this equation is $\beta=0$. Equation (A1) then becomes

$$\hat{\mathcal{M}}^T \alpha = \begin{bmatrix} \mathcal{M} \\ A \end{bmatrix} \alpha = 0 \quad (\text{A4})$$

whose unique solution is $\alpha=0$, since the rank of $\hat{\mathcal{M}}^T$ is K .

(b) Let us now assume that the matrix \mathcal{M}_L is nonsingular. Then, the columns of the matrix \mathcal{M}_L must be linearly independent.

Thus, the submatrix $\hat{\mathcal{M}}^T$, which has K columns, must have rank K ; similarly, the submatrix $[-A \ 0]^T$, which has m columns, must have rank m . Hence, the rank of $\hat{\mathcal{M}}$ is K and the rank of A is m .

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