Methodology for Optimum Sensor Locations for Parameter Identification in Dynamic Systems

By Firdaus E. Udwadia[1]

Abstract: This paper provides a methodology for optimally locating sensors in a dynamic system so that data acquired from those locations will yield the best identification of the parameters to be identified. It addresses the following questions: (1) Given m sensors, where should they be placed in a spatially distributed dynamic system so that data from those locations will yield best estimates of the parameters that need to be identified?; and (2) given that we have already installed p sensors in a dynamic system, where should the next s be located? The methodology is rigorously founded on the Fisher information matrix and is applicable to both linear and nonlinear systems. A rapid algorithm is provided for use in large multi-degree-of-freedom systems. After developing the general methodology, the paper goes on to develop the method in detail for a linear N-degree-of-freedom, classically damped, system. Numerical examples are provided and it is verified that the optimal placement of sensors, as dictated by the methodology that is developed, could provide significantly improved estimates of the parameters to be identified.

INTRODUCTION

Reliable predictions of structural responses are closely dependent on the validity of the models chosen to represent the systems involved. When parametric models are used, a proper knowledge of the various parameter values becomes crucial in establishing the usefulness of such models. However, to actually come up with these parameter values, one often needs to collect response data from instruments located at various positions within the structure. The usefulness of such data, in turn, depends primarily on the instrument characteristics and on the chosen positions where the instruments are located. Consequently, for given types of instruments, which are to be used, one often wants to locate them such that data collected from those locations yield the best estimates of the modeled structural parameters.

Although various methods have been developed to identify the parameters that characterize flexible structures (e.g. Rodriguez 1985; Hart 1976; Mehra and Lainiotis 1976; Udwadia and Shah 1976; Dale and Cohen 1971; Ljung 1987), from records obtained in them under various loading conditions, few investigators, if any, have looked at the question of where to locate sensors in a large, spatially extended structure to acquire data for best parametric identification (Udwadia and Shah 1978; Rodriguez 1985). The problem of optimally locating sensors in a dynamic vibrating system mainly arises from considerations of: (1) Minimizing the cost of instrumentation, data processing, and data handling through the use of a smaller number of sensors, data channels, etc.; (2) obtaining better (more accurate) estimates of model parameters from noisy measurement data; (3) improving structural control through the use of superior structural models; (4) efficiently determining structural properties and their changes with a view to

[1]Prof. of Mech. Engrg., Civ. Engrg., and Decision Systems, Univ. of Southern California, Los Angeles, CA 90089-1453.

Note. Discussion open until July 1, 1994. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on January 11, 1991. This paper is part of the *Journal of Engineering Mechanics*, Vol. 120, No. 2, February, 1994. ©ASCE, ISSN 0733-9399/94/0002-0368/$1.00 + $.15 per page. Paper No. 1298.

acquiring improved assessments of structural integrity; and (5) improving the early fault-detection capability for large, flexible structural systems.

The problem addressed in the present paper can sufficiently be stated as follows: Given m sensors, where should they be located in a spatially distributed dynamic system so that records obtained from those locations yield the best estimates of those relatively unknown parameters that need to be identified? In the past, the optimum sensor location problem (OSLP) was solved by positioning the given sensors in the system, using the records obtained at those locations with a specific estimator, and repeating the procedure for different sensor locations. The set of locations which yield the best parameter estimates would then be selected as optimal. The estimates obtained, of course, depend on the type of estimator used. Thus the optimal locations are estimator-dependent, and an exhaustive search needs to be performed for each specific estimator. Such a procedure, besides being highly computationally intensive, suffers from the major drawback of not yielding any physical insight into why certain locations are preferable to others.

Work on the solution of the OSLP was perhaps first done by Shah and Udwadia (1978) and later by Rodriguez (1985). In brief, in the former paper they used a linear relationship between small perturbations in a finite dimensional representation of the system parameters and a finite sample of observations of the system time response. The error in the parameter estimates was minimized yielding the optimal locations. In the present paper, we developed a more-direct approach to the problem, which is both computationally superior and throws considerable light on the rationale behind the optimal selection process. The methodology is applicable to all spatially extended dynamic systems. In this report, special attention has been paid to large, flexible structural systems such as large space structures.

We uncoupled the optimization problem from the identification problem using the concept of an efficient estimator [e.g. the maximum likelihood estimator as the time history of data becomes very large (see Cramer; 1957)]. For such an estimator, the covariance of the parameter estimates is a minimum. Using this technique and motivated by heuristic arguments, a rigorous formulation and solution of the OSLP is presented.

MODEL FORMULATION

Most large, complex dynamic systems are spatially continuous in nature. Suitable discrete models are usually formulated for engineering applications through the use of finite-element or finite-difference techniques. Through the development of the optimum sensor location (OSL), criterion will be shown to be unrelated to the nature (linear, nonlinear, time-variant, time-invariant) of the system S under consideration; let us for the moment consider a linear dynamic vibrating system so that we have a vehicle for developing the methodology.

The governing differential equation of motion for a linear dynamic system may be considered as

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = F(t); \ \dot{\mathbf{X}}(0) = \dot{\mathbf{X}}_0, \ \mathbf{X}(0) = \mathbf{X}_0 \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

where $\dot{\mathbf{X}}_0$ and $\mathbf{X}_0$ = given initial conditions for the sytsem. The constant coefficient matrices **M**, **C**, and **K** are each of dimension ($N \times N$). The **M** matrix may be considered as the lumped or the consistent mass matrix (as is often the case in structural analysis), **C** as the damping matrix, and **K** as the stiffness matrix. **X** is an N-vector whose components x_j may be considered to be the displacement response (at the nodes of the finite-element or finite-difference mesh) of the system to the input N-vector $[F(t)]_N$. One or more elements of the coefficient matrices, in (1), may constitute the unknown parameters. To best estimate these parameters, one would locate sensors in the system in such a way that the measurements obtained thereat are most informative about the estimated parameters. To accomplish this task, let us collect all the possible unknown parameters that need to be estimated in a vector $\mathbf{\Theta}$ of dimension L. Hence

$$\mathbf{\Theta} = (\mathbf{\Theta}_M^T \ \mathbf{\Theta}_C^T \ \mathbf{\Theta}_K^T)_L^T \quad \text{.......... (2)}$$

where the subvectors Θ_M, Θ_C, and Θ_K have dimensions a, b, and c, respectively, and are related to parameters in the **M**, **C**, and **K** matrices.

We next describe the measurement model. The response of the dynamic system is assumed to be measured using m, $m < N$, available sensors. Solution of the OSLP is equivalent to the selection of the m locations out of N possible such locations so that the m time histories of response obtained at those locations yield the maximum amount of information about the system parameters.

To formulate the measurement model, let us first assume there are exactly N sensors available, so that each component of X is measured (i.e., $m = N$). These measured responses can be mathematically represented by the N-vector **Z** as follows:

$$Z_j(t) = g_j[X(\theta, t)] + N_j(t)], j = 1, 2, \ldots, N \quad \text{.......... (3)}$$

where Z_j = jth component of $Z(t)$; functional g_j = measurement process; and the dependence of the response **X** on the parameter θ is explicitly noted. We shall assume that g is a memoriless transformation of the system output which yields the measurements. The measurement noise $N_j(t)$ is taken as nonstationary Gaussian white noise with a variance of $\psi^2(t)$. Therefore

$$E[N_j(t_1)N_j(t_2)] = \psi^2(t_1)\delta_K(i - j)\delta_D(t_1 - t_2) \quad \text{.......... (4)}$$

where δ_K and δ_D = Kroneker- and dirac-delta functions, respectively. Having measured each element of the response vector **X**, a total of m out of N responses need to be selected so that they contain the most information about the system parameters and are maximally sensitive to any change in the parameter values. This selection process can be represented by an m-dimensional vector **Y** such that

$$\mathbf{Y}(t) = \mathbf{S}\mathbf{Z}(t) \quad \text{.......... (5)}$$

where $\mathbf{S}$ = $(m \times N)$ upper triangular selection matrix with each row containing null elements except for one, which is unity. The m different components of **Z** selected to be measured are so ordered in vector **Y**, that if the element in the ith row and kth column of **S** is unity, the $(i + 1)$th row can have unity in its sth column only if $s > k$. The matrix **S** then has the property that, $\mathbf{P} = \mathbf{S}^T\mathbf{S}$, is an $(N \times N)$ diagonal matrix with unity in its ith row if, and only if, $\mathbf{Z}_i$ is selected to be measured. The elements of P are otherwise zero. Hence, one can write

$$\mathbf{Y}(t) = \mathbf{S}\mathbf{g}[\mathbf{X}(\theta, t)] + \mathbf{S}\mathbf{N}(t) \quad \text{.......... (6)}$$

or

$$\mathbf{Y}(t) = H[\mathbf{X}(\theta, t)] + \mathbf{V}(t) \quad (7)$$

If g_i is linearly related to the response x_j, in general, then

$$\mathbf{H}[\mathbf{X}(\theta, t)] = \mathbf{SRX} \quad (8)$$

where $\mathbf{R}(t)$ in (8) can be thought of as a dynamic gain matrix. In the case that g_i is related to the response x_i only, matrix $\mathbf{R}$ will reduce to a diagonal matrix, $\mathrm{Diag}(\rho_1, \rho_2, \ldots, \rho_N)$.

The problem of locating sensors in an optimal manner then reduces to determining the selection matrix $\mathbf{S}$ defined before. Alternately put, one needs to determine the m locations along the diagonal of the matrix $\mathbf{P}$ that should be unity. These locations must be so chosen as to obtain the best parameter estimates. We next go on to explain what we mean by best.

EFFICIENT ESTIMATORS AND FISHER INFORMATION MATRIX

In the present paper we shall assume that we have a reasonably good estimate or idea of the unknown parameter vector that we are attempting to identify through the proper placement of sensors. Thus our identification schemes, provided our measurements are not too noisy, would not be likely to converge to parameters that are different from those of the actual system. As opposed to locating sensors to assure the best global convergence (i.e. starting from any parameter value estimate) we shall aim to develop a methodology to locate sensors so that, when starting from a close nearby guess, the unknown parameters are best identified. [For further discussion of global versus local optimization, see Udwadia (1988).] This results in the conditional estimation problem for which we know that the covariance of the vector parameter estimates satisfy the relations (Nahi and Wallis 1968; Sage and Melsa 1971)

$$E[(\theta - \theta)(\theta - \hat{\theta})^T] \geq Q^{-1}(T) \quad (9)$$

where

$$Q(T) = \int_0^T \frac{\left(\frac{\partial \mathbf{H}}{\partial \theta}\right)^T \left(\frac{\partial \mathbf{H}}{\partial \theta}\right)}{\psi^2(t)\, dt} \quad (10)$$

and $\hat{\theta}$ denotes the estimate θ. The right-hand side of (9) is called the Cramer-Rao lower bound (CRLB).

An unbiased estimator that achieves the CRLB is called efficient. Such an unbiased, efficient estimator is also a minimum-variance estimator. One commonly useful class of estimators that is asymptotically unbiased and efficient is the one that comprises the maximum likelihood estimators (Gart 1959; Goodwin and Payne 1977). Hence for efficient estimators (minimum covariance) the inequality (9) (Nahi 1969) become an equality. Therefore, one can write

$$E[(\theta - \hat{\theta})(\theta - \hat{\theta})^T] = \mathrm{CRLB} = Q^{-1} \quad (11)$$

In this sequel such as estimator shall be assumed to exist.

The expression in (10) is known at the Fisher information (FI) matrix (Udwadia and Shah 1976). Therefore maximization of this Fisher information matrix (maximizing a certain norm of the matrix, such as the trace norm, etc.) would yield the minimum possible value of the covariance of the estimation error (Jazwinski 1970; Mehra 1974).

OPTIMUM SENSOR LOCATION FOR CONDITIONAL ESTIMATION OF VECTOR-VALUED PARAMETERS

Consider first a case in which one tries to estimate a single parameter, θ_1, which is to be identified in a dynamic system model with only one sensor provided. One would want to ideally choose a location i (out of N possible such locations) such that the measurement $y_1(t)$, $i \in [1, N]$, $t \in (0, T)$ at location i yields the best estimate of the parameter θ_1. Heuristically, one should place the sensor at such a location that the time history of the measurements obtained at that location is most sensitive to any changes in the parameter θ_1. Hence, it is really the slope of $H[X(\theta_1, t)]$ with respect to θ_1 that needs to be maximized. However, since only the absolute magnitude of this slope is of interest, it is logical to want to find i (or equivalently determine the selection matrix **S** described previously) such as to maximize $(\partial H/\partial\theta_1)^2$. Since this quantity is a function of time, one would want to locate a sensor which maximizes the average value of $(\partial H/\partial\theta_1)^2$ over the time interval $(0, T)$ during which the response is to be measured. This leads to maximizing the following integral:

$$q_i(T) = \int_0^T \left(\frac{\partial H}{\partial \theta_1}\right)^2 dt \quad \ldots\ldots (12)$$

When there is more than one parameter to be estimated, and the number of sensors is greater than unity, this intuitive approach needs to be extended in a more rigorous manner, and recourse to the Fisher information matrix is necessary.

Thus, to reduce the error in the estimates, one would want to maximize a suitable norm (e.g., Trace, etc.) (Goodwin and Payne 1977) of the Fisher information matrix $\mathbf{Q}(T)$. Therefore, introducing (8) into (10), (this constitutes an extension of the (12) which we heuristically derived for the scalar case, to the vector situation), one obtains

$$Q(T) = \int_0^T \frac{\mathbf{X}_\theta^T \mathbf{R}^T \mathbf{P} \mathbf{R} \mathbf{X}_\theta}{\psi^2(t)}\, dt \quad \ldots\ldots (13)$$

where the ij element of $\mathbf{X}_\theta$ can be written as

$$(\mathbf{X}_\theta)_{ij} = \frac{\partial x_i}{\partial \theta_j}, \; i \in (1, N), \; j \in (1, L) \quad \ldots\ldots (14)$$

where $\mathbf{X} = (x_i)_N$, $\theta = (\theta_i)_L$, and $\mathbf{P} = \mathbf{S}\mathbf{S}^T$. We note that the Fisher matrix is symmetric and is dependent on the length of the record available, as well as the locations of the sensors as determined by the matrix **P**.

If the m locations where the sensors are to be placed are denoted by s_k, $k = 1, 2, \ldots, m$, then

$$P = \sum_{k=1}^{m} \mathbf{I}_{s_k} \quad \ldots\ldots (15)$$

where the $(N \times N)$ diagonal matrix $\mathbf{I}_{s_k}$ has all its elements equal to zero except the element of the s_k row, which is unity. Noting that **P** is a diagonal matrix, (13) can be simplified to yield

$$Q[T; s_1, s_2, \ldots, s_m; S, \theta; I(t)] = \sum_{k=1}^{m} \int_0^T \frac{\mathbf{X}_\theta^T r_{s_k}^T r_{s_k} \mathbf{X}_\theta}{\psi^2(t)} dt \quad (16)$$

where r_{s_k} is the s_k row of the matrix **R**. Also in (16) explicit mention is made of the dependence of the Fisher matrix on the time length T of the available data, the system **S**, the parameter vector $\boldsymbol{\theta}$, and the time-variant input $\mathbf{I}(t)$. If the matrix **R** is diagonal, with diagonal elements ρ_1, ρ_1, ρ_N, then the ij element of the matrix **Q**, after some manipulation, reduces to

$$\mathbf{Q}_{ij}[T; s_1, s_2, \ldots, s_m; S, \theta; I] = \sum_{k=1}^{m} \int_0^T \left\{ \frac{\partial x_{s_k}}{\partial q_i} \frac{\partial x_{s_k}}{\partial \theta_j} \left[\frac{\rho(t)_{s_k}}{y(t)} \right] \right\} dt \quad (17)$$

One notes that each element of $\mathbf{Q}_{ij}$ represents the cross-sensitivity of measurement with respect to the response x_{s_k} of node s_k.

The optimal sensor locations are then obtained by picking m locations s_k, $k = 1, 2, \ldots, m$, out of a possible N, so that a suitable norm of the matrix **Q** is maximized (e.g. the trace norm) (Nahi and Wallis 1968). This may be specified by the condition

$$\max_{s_k \in (1,N)} \|Q\{T; s_1, s_2, \ldots, s_m; \mathbf{S}, \theta; \mathbf{I}(t)\}\| \quad (18)$$

It should also be noted that the criterion developed by (17) and (18) do not hinge upon the linearity of the system. The only equations involved are the measurement (8) and the relation (10). The methodology introduced herein may be applied to the systems governed by nonlinear differential equations [for details on nonlinear systems refer to Udwadia (1988)]. The nature of the system S enters in the determination of $\mathbf{X}_\theta$.

CHOICE OF MATRIX NORMS AND ALGORITHM

Since **Q** is a matrix, it is necessary to use a suitable scalar norm of it, $\|Q\|$, to obtain an idea of the information content, about a parameter vector $\boldsymbol{\theta}$, available from sensors at one or more locations, given the input, $\mathbf{I}(t)$. Various norms may be used as scalar measures of performance. Some commonly used norms are (Mehra 1974) as follows:

1. D-optimality: minimize the determinant of Q^{-1} or equivalently maximize the determinant of Q
2. A-optimality: minimize the trace of Q^{-1}
3. T-optimality: maximize the trace of Q.

An important advantage of D-optimality is its invariance under scale changes in the parameters and linear transformations of the output. However T-optimality has the advantage that the trace operator is linear and therefore Trace(Q) can be expressed as

$$\text{Trace}[Q(T)] = \sum_{k=1}^{k=m} q_{s_k}(T) \quad (19)$$

where

$$\bar{q}_{s_k}(T) = \text{Trace}\left\{ \int_0^T \left[\frac{\partial x_{s_k}}{\partial \theta_i} \frac{\partial x_{s_k}}{\partial \theta_j} \left[\frac{\rho(t)_{s_k}}{\psi(t)} \right]^2 dt \right] \right\} \quad (20)$$

Where m is large, this relationship allows the optimal sensor locations, got by maximizing Trace(**Q**) [as given by (18)], to be obtained in a simple sequential manner. The algorithm to be used can be described in the following three steps:

1. For each s_k, $k = 1, 2, \ldots, N$, determine $\bar{q}_{s_k}$
2. Sort the N numbers $\bar{q}_{s_k}$, $k = 1, 2, \ldots, N$, in an array of descending order, starting with the largest
3. The s_k, $k = 1, 2, \ldots, m$ locations that correspond to the largest m values of $\bar{q}_{s_k}$ are the m optimal sensor locations.

Should r sensors be already fixed in place at locations s_k, $k = 1, 2, \ldots, r$, the best locations for an additional m sensors can be found by including a further step in the preceding algorithm, as follows:

1. Perform step 1 as before
2. Perform step 2 as described previously, and from the sorted array obtained in step 2, delete $\bar{q}_{s_k}$, $k = 1$
3. The $\bar{q}_{s_k}$, $k = 1, 2, \ldots, m$ locations corresponding to the largest values in the remaining sorted array yield the optimal sensor locations for the next m sensors.

Due to computational ease and efficiency with which the trace criterion can be used, and the simplicity with which the maximization defined in (18) can be carried out, in this sequel we shall exclusively use it. (For an analytical explanation of the trace norm in terms of estimation error minimization, see Appendix I). A rigorous comparison of the results between A-, T- and D-optimality will be left for a future study; it suffices to say that in almost all the examples studied by the writer they provide that same ordering of sensor locations.

OPTIMAL SENSOR LOCATION FOR N–DEGREE-OF-FREEDOM LINEAR SYSTEMS

We use our previous results now to obtain solutions of the OSL problem for multi–degree-of-freedom linear systems. Consider the N–degree-of-freedom classically damped dynamic system whose governing differential equation

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F}(t), \mathbf{X}(t_0) = \mathbf{X}_0, \dot{\mathbf{X}}(t_0) = \dot{\mathbf{X}}_0, \quad (21)$$

where X_0 and $\dot{X}_0$ are the given initial conditions for the system. Using normal modes, the response vector $\mathbf{X}(t)$ can be determined. Introducing

$$\mathbf{X}(t) = \mathbf{\Phi}\eta(t) \quad (22)$$

where $\Phi = (N \times N)$ weighted model matrix (transformation matrix); and $\eta(t)$ may be referred to as the N-vector of generalized coordinates (reponse coordinates). Then

$$\ddot{\eta} + 2\xi\Lambda\dot{\eta} + \Lambda\eta = \Phi^T F(t), \eta(t_0) = \Phi^T M X_0, \dot{\eta}(t_0) = \Phi^T M \dot{X}_0 \quad (23)$$

where the $(N \times N)$ diagonal matrices $\mathbf{\Lambda}$ and $\boldsymbol{\xi}$ are

$$\mathbf{\Lambda} = \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \text{Diag}(\omega_1, \omega_2, \omega_3, \ldots, \omega_N) \quad (24a)$$

and

$$\xi = \text{Diag}(\xi_1, \xi_2, \ldots, \xi_N) \qquad (24b)$$

The solution of (23) is given as

$$\eta_i(t) = \eta_{0_i} u_i(t - t_0) + \dot{\eta}_{0_i} v_i(t - t_0) + \int_{t_0}^{t} h_i(t - \tau) p_i(\tau)\, d\tau \qquad (25)$$

where η_{0_i} and $\dot{\eta}_{0_i}$ = initial conditions and

$$u_i(t) = \frac{1}{\omega_{d_1}} \exp(-\xi_i \omega_i t) \left(\cos \omega_{d_1} t + \frac{\xi \omega_i}{\omega_{d_i}} \sin \omega_{d_i} t \right) \qquad (26)$$

$$v_i(t) = \frac{1}{\omega_{d_1}} \exp(-\xi_i \omega_i t) \sin \omega_{d_i} t \qquad (27)$$

$$h_i(t) = v_i(t) \qquad (28)$$

$$\omega_{d_i} = \omega_i (1 - \xi_i^2)^{1/2} \qquad (29)$$

and

$$p_i(t) = \boldsymbol{\Phi}^T \mathbf{F}(t) \qquad i = 1, 2, \ldots, N \qquad (30)$$

If the OSLP is to be solved for estimation of $\boldsymbol{\Theta}$ where $\boldsymbol{\Theta}$ was previously defined in (2), then one can differentiate (21) with respect to $\boldsymbol{\Theta}$. This yields

$$\mathbf{M}\dot{\mathbf{X}}_\Theta + \mathbf{C}\dot{\mathbf{X}}_\Theta + \mathbf{K}\mathbf{X}_\Theta = \mathbf{F}_\Theta(t) - (\mathbf{M}_\Theta \ddot{\mathbf{X}} + \mathbf{C}_\Theta \dot{\mathbf{X}} + \mathbf{K}_\Theta \mathbf{X}); \dot{\mathbf{X}}_\Theta(t_0) = 0, \mathbf{X}_\Theta(t_0) = 0 \qquad (31)$$

where

$$(\mathbf{X}_\Theta)_{ij} = \frac{\partial x_i}{\partial \theta_j} \qquad (32)$$

$$[\overline{\mathbf{M}_\Theta \ddot{\mathbf{X}}}] = (\mathbf{M}_{\theta_1}\ddot{\mathbf{X}} \quad \mathbf{M}_{\theta_2}\ddot{\mathbf{X}} \ldots \mathbf{M}_{\theta_L}\ddot{\mathbf{X}}) \qquad (33)$$

$$(\overline{\mathbf{C}_\Theta \dot{\mathbf{X}}}) = (\mathbf{C}_{\theta_1}\dot{\mathbf{X}} \quad \mathbf{C}_{\theta_2}\dot{\mathbf{X}} \ldots \mathbf{C}_{\theta_L}\dot{\mathbf{X}}) \qquad (34)$$

$$(\overline{\mathbf{K}_\Theta \mathbf{X}}) = \{\mathbf{K}_{\theta_1}\mathbf{X} \quad \mathbf{K}_{\theta_2}\mathbf{X} \ldots \mathbf{K}_{\Theta_L}\mathbf{X}) \qquad (35)$$

$$[\mathbf{F}_\Theta(t)]_{ij} = \frac{\partial f_i}{\partial \theta_j} \qquad (36)$$

with

$$\boldsymbol{\Theta} = (\boldsymbol{\Theta}_M^T \, \boldsymbol{\Theta}_C^T \, \boldsymbol{\Theta}_K^T)_L^T = (\theta_j)_L^T \qquad (37)$$

for $i = 1, \ldots, N$, and $j = 1, \ldots, L$

$$\mathbf{X}_\Theta = \boldsymbol{\Phi}\mathbf{Z}(t) \qquad (38)$$

where $\mathbf{Z}$ is an $N \times L$ matrix, yields

$$\ddot{\mathbf{Z}} + 2\boldsymbol{\xi}\boldsymbol{\Lambda}\dot{\mathbf{Z}} + \boldsymbol{\Lambda}\mathbf{Z} = \mathbf{G}(t); \dot{\mathbf{Z}}(t_0) = 0; \mathbf{Z}(t_0) = 0 \qquad (39)$$

where

$$\mathbf{G}(t) = \mathbf{\Phi}^T[\mathbf{F}_\Theta - (\overline{\mathbf{M}_\Theta\ddot{\mathbf{X}}} + \overline{\mathbf{C}_\Theta\dot{\mathbf{X}}} + \overline{\mathbf{K}_\Theta\mathbf{X}})] \quad \ldots\ldots (40)$$

Eq. (40) can further be simplified using (22) to

$$\mathbf{G}(t) = \mathbf{\Phi}^T[\mathbf{F}_\Theta - (\mathbf{M}_\Theta\mathbf{\Phi}\ddot{\eta} + \mathbf{C}_\Theta\mathbf{\Phi}\dot{\eta} + \mathbf{K}_\Theta\mathbf{\Phi}\eta)] \quad \ldots\ldots (41)$$

where $\dot{\eta}$ and $\ddot{\eta}$ can be obtained by proper differentiation of (25). Hence

$$\dot{\eta}_i(t) = \eta_{0_i}W_i(t - t_0) + \dot{\eta}_{0_i}Y_i(t - t_0) + \int_{t_0}^{t} \overline{h}_i(t - \tau)p_i(\tau)\, d\tau \quad \ldots\ldots (42)$$

where

$$W_i(t) = -\exp(-\xi_i\omega_i t)\left[\omega_{d_i} + \frac{(\xi_i\omega_i)^2}{\omega_{d_i}}\right] \sin \omega_{d_i}t \quad \ldots\ldots (43)$$

$$Y_i(t) = \exp(-\xi_i\omega_i t)\left[\cos \omega_{d_i}t - \left(\frac{\xi_i\omega_i}{w_{d_i}}\right) \sin \omega_{d_i}t\right] \quad \ldots\ldots (44)$$

$$h_i(t) = \mathbf{Y}_i(t) \quad \ldots\ldots (45)$$

and

$$p_i(t) = \mathbf{\Phi}^T\mathbf{F}(t),\ i = 1, 2, \ldots, N \quad \ldots\ldots (46)$$

Also

$$\ddot{\eta}_i(t) = \eta_{0_i}\overline{W}_i(t - t_0) + \dot{\eta}_{0_i}\overline{Y}_i(t - t_0) + \int_{t_0}^{t} \overline{\overline{h}}_i(t - \tau)p_i(\tau)\, d\tau + p_i(t) \quad \ldots (47)$$

where

$$\overline{W}_i(t) = \exp(-\xi_i\omega_i t)\left\{\left[\frac{(\xi_i\omega_i)^3}{\omega_{d_i}} + \omega_{d_i}(\xi_i\omega_i)\right] \sin \omega_{d_i}t - [\omega_{d_i}^2 + (\xi_i\omega_i)^2]\cos \omega_{d_i}t\right\} \quad \ldots\ldots (48)$$

$$\overline{Y}_i(t) = \exp(-\xi_i\omega_i t)\left\{\left[\frac{(\xi_i\omega_i)^2}{\omega_{d_i}} + \omega_{d_i}\right] \sin \omega_{d_i}t - 2\xi_i\omega_i \cos \omega_{d_i}t\right\} \quad \ldots (49)$$

$$\overline{h}_i(t) = Y_i(t) \quad \ldots\ldots (50)$$

Therefore, substituting (42) and (47) into (31) gives $\mathbf{G}(t)$. Consequently the solution of (31) can be written as

$$[Z(t)]_{ij} = \int_{t_0}^{t} h_i(t - \tau)G_{ij}(\tau)\, d\tau \quad \ldots\ldots (51)$$

where $h_i(t)$ is the same as in (28).

If in (21) we assume that $\mathbf{C}$ is expressed as a linear combination of $\mathbf{K}$ and $\mathbf{M}$, i.e., $2\alpha\mathbf{K} + 2\beta\mathbf{M}$, where α and β are known constants, then the percentage of damping, ξ_i, can be expressed as

$$\xi_i = \alpha\omega_i + \frac{\beta}{\omega_i}, \; i = 1, 2, \ldots, N \quad (52)$$

and yields

$$G(t) = \Phi^T\{F_\Theta - [\overline{M_\Theta\Phi(\ddot{\eta} + 2\beta\dot{\eta})}] + \overline{K_\Theta\Phi(\eta + 2\alpha\dot{\eta})}$$
$$+ 2\overline{(\beta_\Theta M + \alpha_\Theta K)\Phi\dot{\eta}}\} \quad (53)$$

In several practical applications one often needs to identify only the mass parameters or only the stiffness parameters or only the damping parameters. We then have the following three cases:

1. If vector $\boldsymbol{\Theta}$ of (2) contains only the subvector $\boldsymbol{\Theta}_M$ (estimating masses), then

$$G(t) = \boldsymbol{\Phi}^T[F_\Theta - \overline{M_\Theta\Phi(\ddot{\eta} + 2\beta\dot{\eta})}] \quad (54a)$$

2. If the vector $\boldsymbol{\Theta}$ of (2) only contains the subvector $\boldsymbol{\Theta}_K$ (estimating stiffness), then

$$G(t) = \boldsymbol{\Phi}^T[F_\Theta - K_\Theta\overline{\Phi(\eta + 2\beta\dot{\eta})}] \quad (54b)$$

3. Finally, if vector $\boldsymbol{\Theta}$ is reduced to subvector $\boldsymbol{\Theta}_C$, $\boldsymbol{\Theta}_c = [\alpha \quad \beta]^T$, then

$$G(t) = (\boldsymbol{\Phi}^T F_\alpha - 2\boldsymbol{\Lambda}\dot{\eta} \quad \boldsymbol{\Phi}^T F_\beta - 2\mathbf{I}\dot{\eta}) \quad (54c)$$

Once the solution of (31) is obtained using (38), the Fisher matrices may be evaluated using (16). Hence we get

$$Q = \sum_{k=1}^{m} \int_{t_\Omega}^{T} \frac{\mathbf{Z}^T\boldsymbol{\Phi}^T r_{s_k}^T r_{s_k} Z\Phi}{\psi^2(t)}\, dt \quad (55)$$

To further expose the structure of (55) let us assume that for the given N–degree-of-freedom dynamic system described by (21) one seeks the solution of the OSLP for identifying all the system model parameters $\boldsymbol{\theta}_L$. Let $\mathbf{R} = \mathbf{I}$ and $\Psi(t) = \Psi_0$. We note that

$$\mathbf{X}_\Theta = \Phi_{N\times N}(z_1\, z_2 \ldots z_L)_{N\times L} \quad (56)$$

Denoting

$$I_{s_K} = \begin{bmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 1 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix} \leftarrow s_k\text{th row} \quad (57)$$

and

$$H_{s_k} \overset{\Delta}{=} \Phi^T I_{s_k}\Phi \quad (58)$$

Eq. (55) can be written as

$$\mathbf{Q} = \sum_{k=1}^{m} \int_{t_0}^{T} \begin{pmatrix} z_1^T \\ z_2^T \\ \vdots \\ z_L^T \end{pmatrix} \mathbf{H}_{s_k}(z_1\ z_2\ .\ .\ .\ z_L)\ dt \qquad (59)$$

where m = number of sensors to be used. [Since $\Psi(t)$ is a constant it will not affect the OSL and can be factored out.] The preceding equation can further be expanded to

$$Q = \sum_{k=1}^{m} \int_{t_0}^{T} \begin{bmatrix} z_1^T H_{s_k} z_1 & \cdots & z_1^T H_{s_k} z_L \\ z_2^T H_{s_k} z_1 & \cdots & \vdots \\ \vdots & & \\ z_L^T H_{s_k} z_1 & \cdots & z_L^T H_{s_k} z_L \end{bmatrix} dt \qquad (60)$$

Eq. (60) is the Fisher information matrix for the given N–degree-of-freedom dynamic system. If a particular parameter, say, θ_i is not to be estimated, then the ith row and the ith column of the matrix of (60) would be absent. Therefore, if only parameters θ_1 and θ_3 of $\boldsymbol{\Theta}$ are to be estimated, then the first and the third rows and columns in the matrix (60) would only be present and (60) would reduce to

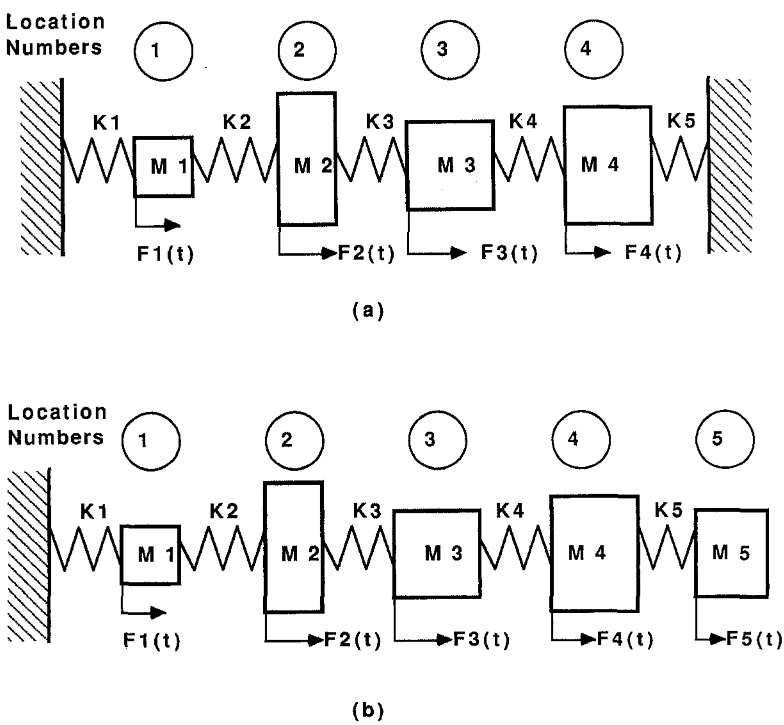

FIG. 1. A Multi–Degree-of-Freedom System for Numerical Study with Rayleigh Damping

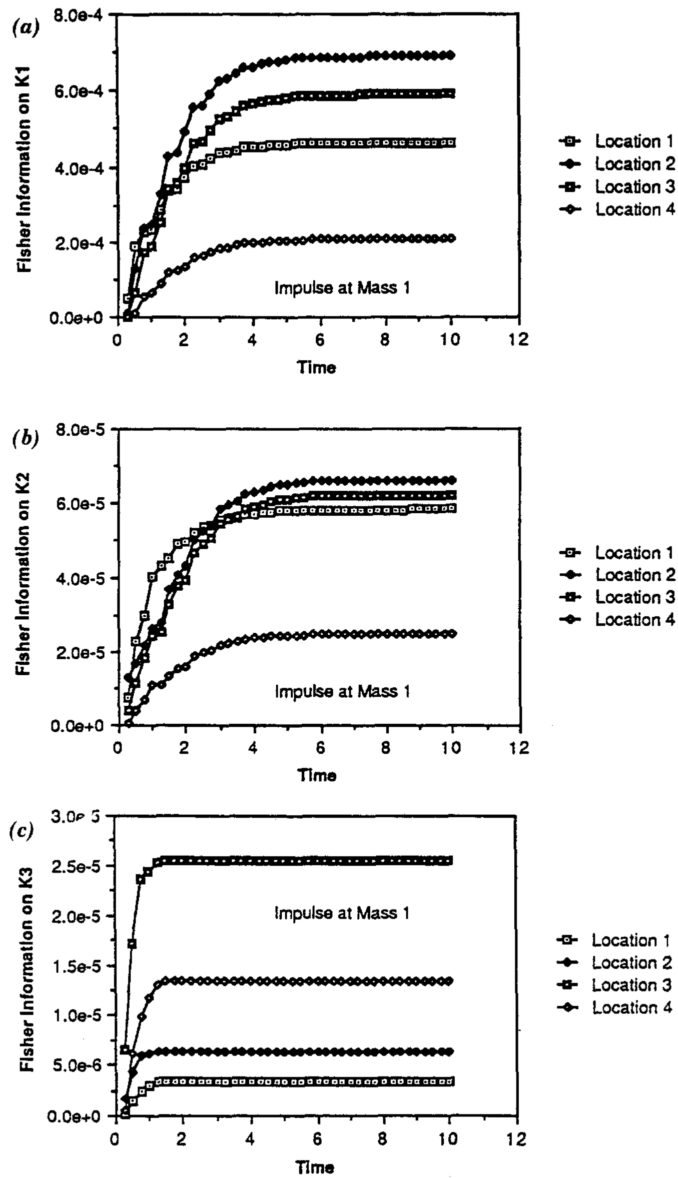

FIG. 2. Optimal Sensor Locations for Impulsive Force Applied at Mass m_1

$$Q = \sum_{k=1}^{m} \int_{t_0}^{T} \begin{bmatrix} z_1^T H_{s_k} z_1 & z_1^T H_{s_k} z_3 \\ z_3^T H^{s_k} z_1 & z_3^T H_{s_k} z_3 \end{bmatrix} dt \quad \text{.......................} \quad (61)$$

Result, (60) is particularly useful for computational purposes. We note that as long as the measurement noise has a constant power spectrum, the actual value of the power in the noise will not affect the OSL. This is an important

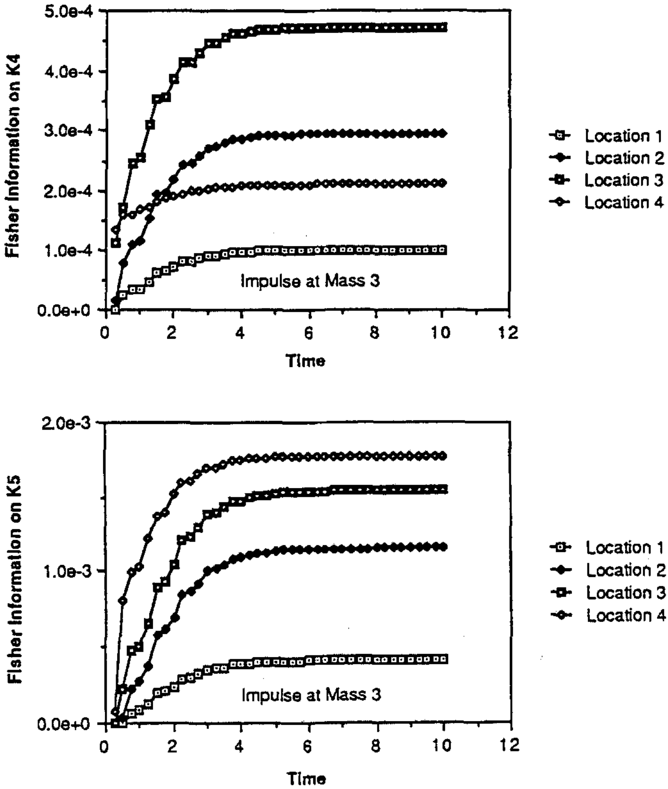

FIG. 2. *(Continued)*

result from a practical standpoint and is true for both linear and nonlinear systems.

APPLICATIONS TO MULTI-DEGREES-OF-FREEDOM SYSTEMS

Fig. 1 shows two different multi–degree-of-freedom systems that will be used to illustrate the OSLP methodology developed. The numerical results obtained will indicate the nature of the solutions for the OSLP and oftentimes their nonintuitive character. To illustrate the dependence of the OSL on the nature of the location(s) and types of inputs, two types of excitations have been used—transient and impulsive.

Fig. 1(a) shows a four degree-of-freedom system. The system parameters are: $m_1 = m_2 = 2$; $m_3 = m_4 = 1$; $k_1 = k_2 = 100$; $k_3 = 75$; $k_4 = 50$; $k_5 = 50$. The damping is taken to be of Rayleigh form (i.e. $\mathbf{C} = 2\alpha\mathbf{M} + 2\beta\mathbf{K}$) with $\alpha = 0.001$ and $\beta = 0.04$. The measurement noise $\Psi(t)$ is taken to be ψ_0.

Further, it is assumed that we have the ability to apply an impulsive force $f(t)$ (to any one of the masses m_i, $i = 1, 2, 3, 4$) whose impulse

$$I = \int_0^\infty f(t)\,dt = 10 \quad \text{(62)}$$

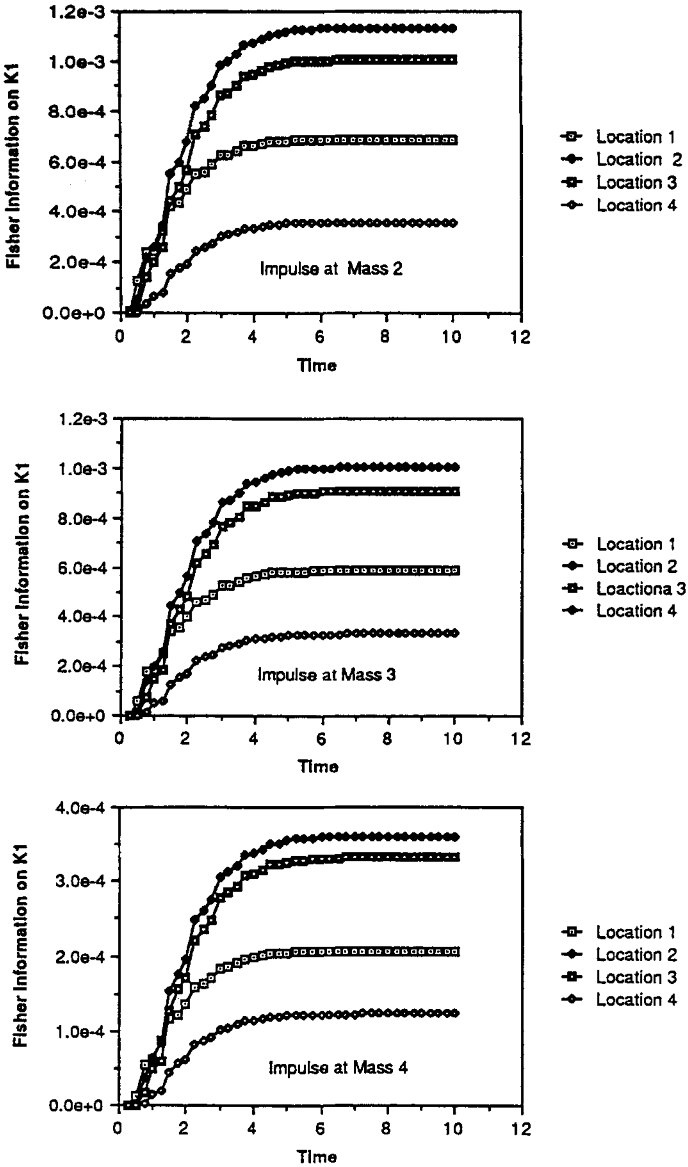

FIG. 3. Fisher Information for Stiffness k_1 When Using Impulses at Masses m_2, m_3, and m_4

The parameter values provided are assumed to be inappropriate and consistent units. We shall investigate the following:

1. If the impulsive force described precedingly is applied to one of the masses, say mass m_j, $j \in (1, 4)$, then where should we locate a sensor to best identify one of the stiffnesses k_i, $i \in (1, 5)$?

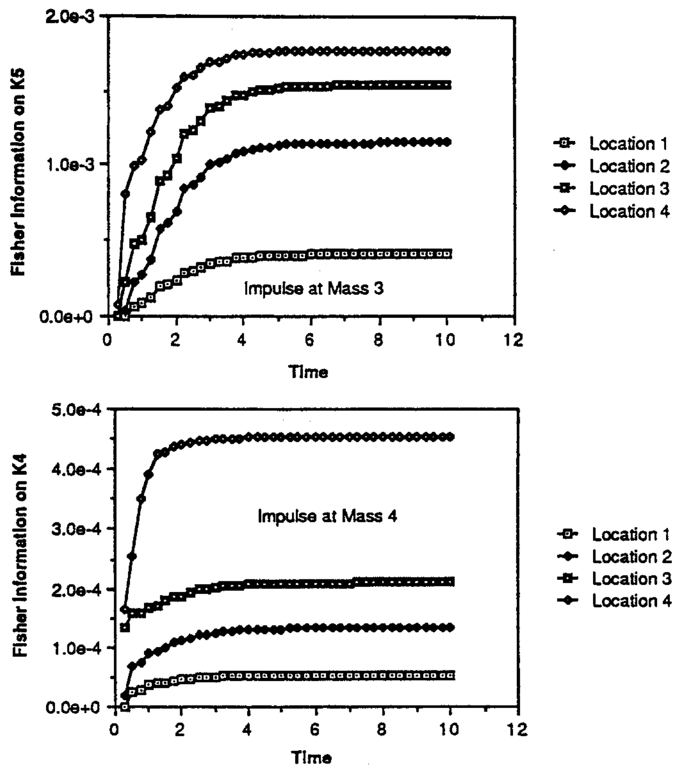

FIG. 4. Fisher Information: (*a*) for Stiffness k_5 at Various Sensor Locations When Applying Impulsive Force at Mass m_3; (*b*) for Stiffness k_4 at Various Sensor Locations When Applying Impulsive Force at Mass m_4

2. Were we required to place more than one sensor to identify k_i how would we find the optimal locations? Could we rank order the locations 1, 2, 3, 4 shown in Fig. 9(*a*) indicating the order in which they should be populated by sensors so as to best identify k_i?

3. Can we get an idea regarding the information gained (or reduced) by placing a sensor at location r as opposed to location $k[r, k \in (1, 4)]$?

4. What are the answers to the first three questions if we want to identify not just one stiffness k_i but a group of them, say k_1 and k_5?

Fig. 2(*a*) shows how the information on the stiffness parameter k_1 changes with time for records obtained at various locations when an impulsive force is applied at mass m_1 with $\mathbf{I} = 10$ units. As seen from Fig. 1, the information obtained from location 2 [show later in Fig. 9(*a*)] is the maximum and this says that the sensor (for identification of k_1), if only one such sensor be available, should be placed at location 2. We note that our intuitive idea of using a sensor at location 1 would have provided about 36% less information that the optimal sensor location obtained. The graph also gives the rank ordering for optimal sensor locations as: location 2, location 3, location 1, location 4; location 2 being the best, location 4 being the worst.

Fig. 2(*b*), (*c*), (*d*), and (*e*) indicates the optimal sensor locations (OSL)

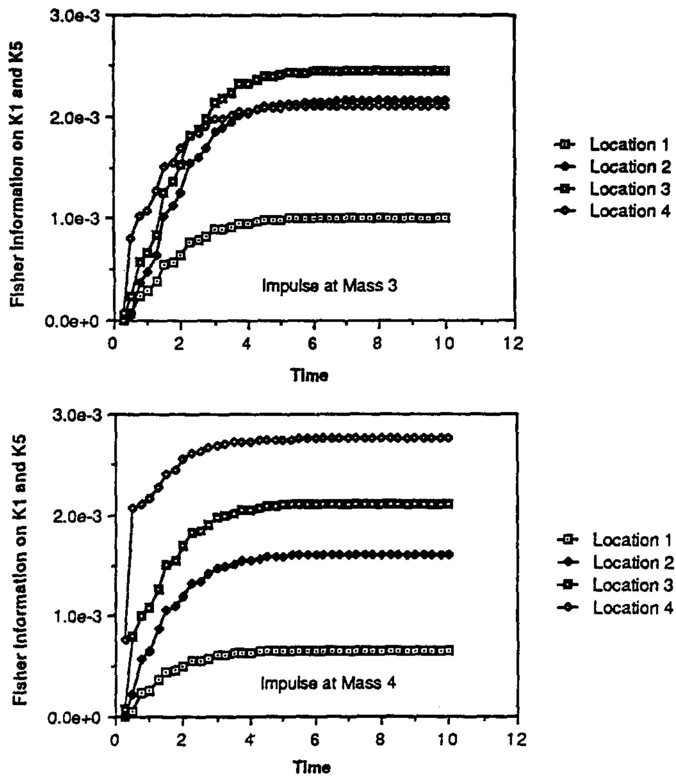

FIG. 5. Fisher Information on k_1 and k_5 When Applying Impulsive Force: (*a*) at Mass m_3; (*b*) at Mass m_4

for identification of the parameters k_2, k_3, k_4, k_5, respectively using an impulsive force applied at location 1. We note that the OSLs depend on the parameter that is required to be identified. Also of interest is the fact that the Fisher information, $Q(t)$, at each location is a function of time. Thus Fig. 2(*b*) shows that were k_2 to be identified using simply a two second length of record beginning at zero time, then location 1 would be the optimal location. However, the use of a longer duration of record for identification of k_2 would yield location 2 as the optimal location as seen in Fig. 2(*b*).

Fig. 3(*a*), (*b*), and (*c*) shows similar results for identification of the stiffness parameter k_1 using an impulsive force (with $\mathbf{I} = 10$) applied at masses m_2, m_3, and m_4, respectively. We see that the extent of information obtained about a parameter for the purposes of identifying it and therefore, in general, the optimal sensor locations, depend on the location where the force is applied. Thus Figs. 2(*a*) and 3(*a*) show that the information about the parameter k_1 from measurements taken at location 2 is about 1.7 times greater if the impulse is applied at mass m_2 rather than at mass m_1. In fact, Figs. 2(*a*), 3(*a*), 3(*b*), and 3(*c*) show that to identify k_1 using one sensor, the best location for both applying the impulsive force and for obtaining a measurement record, is location 2.

Fig. 4(*a*) shows the Fisher information for identification of k_5 when ap-

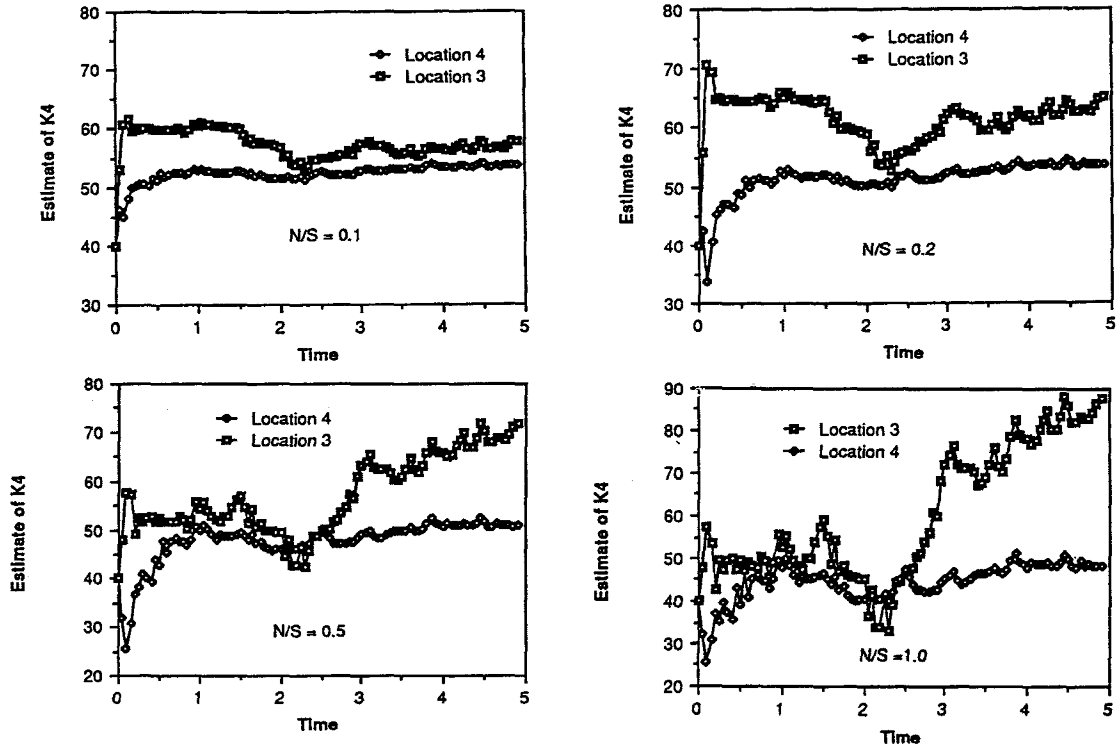

FIG. 6. Verification of Optimal Sensor Location Methodology for Different Noise-to-Signal Ratios

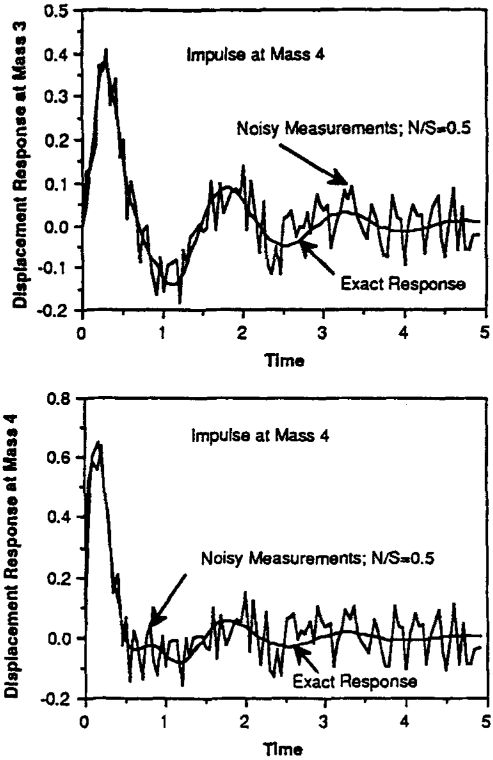

FIG. 7. Noisy Response Measurements Used for Identification of k_4

plying an impulsive force at mass m_3. Fig. 4(*b*) shows the information gained when identifying stiffness k_1 using an impulsive force applied to mass m_4.

Fig. 5(*a*) indicates results for the situation where both k_1 and k_5 are to be simultaneously identified using noisy measurements at one or more locations with an impulsive force (**I** = 10) applied at mass m_3. While the identification of k_1 alone would show the OSL to be at location 2 [see Fig. 3(*b*)], and that of k_5 alone to be location 5 [see Fig. 4(*a*)], the OSL for simultaneous identification of both these parameters is location 3. The locations can be rank ordered as location 3, location 2, location 4, and location 1; location 3 being the best. Should more sensors be available, they would then successively populate the mass locations as per this ordering so that identification of these two parameters can be best carried out. Fig. 5(*b*) shows a similar result except that the impulse (**I** = 10) is applied now at mass m_4. We observe that the rank ordering of locations, as per our trace criterion, has now significantly changed to: location 4, location 3, location 2, location 1; location 4 being the best. Having answered the four questions that were posed previously, we next go on to verify some of our results.

Having determined the optimal sensor locations, we show now that the results of system identification of the system parameters when using data from the optimal sensor locations just obtained are indeed superior to those obtained from data gathered from other sensor locations.

Consider the results depicted in Fig. 4(*b*), in which the OSL is obtained

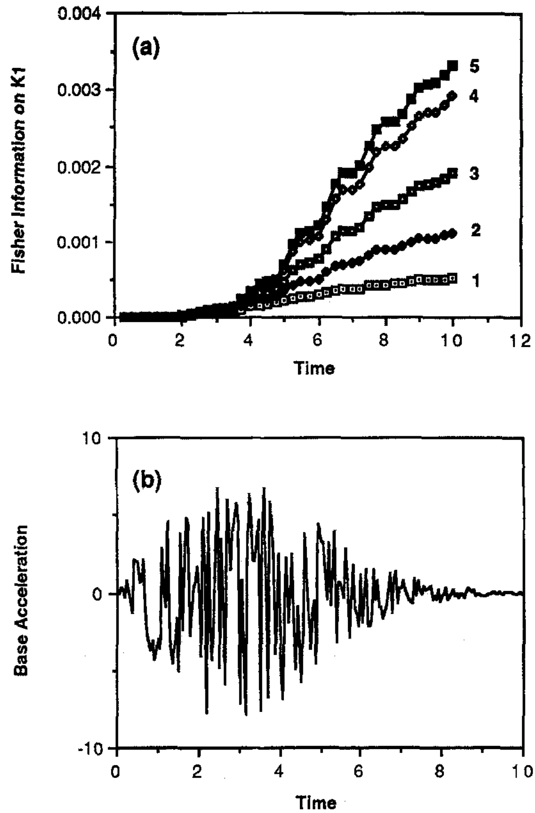

FIG. 8. Optimum Sensor Locations for Transient-Base Excitation of System Shown in Fig. 1(*b*)

for the situation where an impulsive force (I = 10) is applied to mass m_4 and identification k_4 is intended. The correct value for k_4 is 50 units. The results show [Fig. 4(*b*)] that location 4 is far superior to location 3. Real-time, on-line parameter identification using the recursive prediction error method (RPEM) was carried out using records obtained from locations 3 and 4 in response to the impulsive force at m_4. For a description of the RPEM method see, for example, Ljung (1987). For comparison purposes, the same identification scheme (and computer program) was used for records obtained from both locations. The identification scheme was started off with a close-by initial guess, namely $k_4 = 40$. The results of this identification are shown in Figs. 6(*a–d*) for four different levels of noise-to-signal ratios (indicated in Fig. 6 by *N/S*). To provide a feel for the extent of noise prevalent in the records we show in Fig. 7(*a* and *b*) the noisy displacement records used for identification for the noise level: $N/S = 0.5$. We note that while locations 3 and 4 provide about the same accuracy of identification for *N/S* values less than 0.1, as the *N/S* ratio increases, location 4 as predicted by Fig. 4(*b*) is indeed superior. In fact identification of k_4 can be carried out with reasonable accuracy even when $N/S = 1.0$ as seen in Fig. 6(*d*) with measurements from location 4. On the other hand measurements from

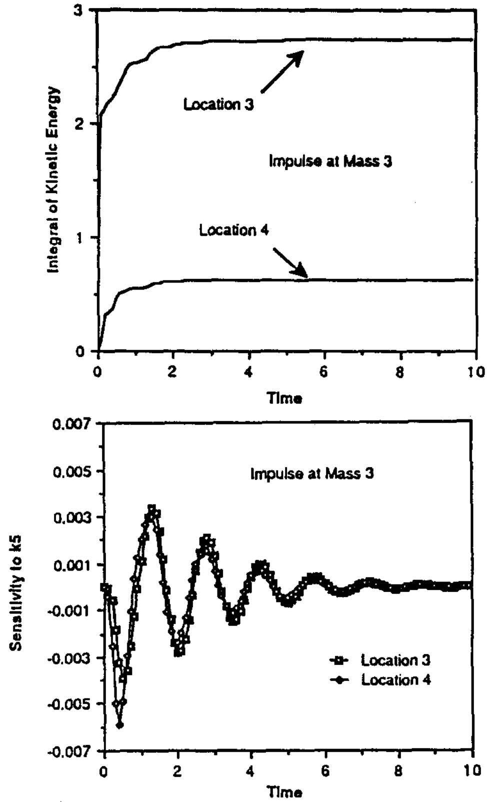

FIG. 9. Comparison of Kinetic Energy and Sensitivity of Measurements

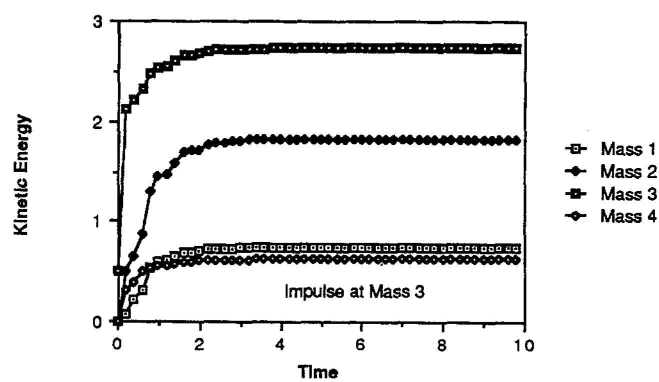

FIG. 10. Erroneous Results for Optimal Sensor Locations Using Kinetic Energy Criterion

location 3 cause the same identification scheme to diverge for $N/S = 0.5$ and $N/S = 1.0$.

Thus, with the improved information at the optimal sensor location(s) about the parameter(s) to be identified, one could not only improve the accuracy of the identification but could allow estimates to be obtained that would otherwise be unobtainable due to divergence of the parameter estimates in noisy measurement environments.

We next present an example of OSL for a transient excitation provided at the base of the five–degree-of-freedom system depicted in Fig. 1(*b*). The system parameters are: $m_1 = m_2 = 2$; $m_3 = m_4 = 1$; $m_5 = 0.5$; $k_1 = k_2 = 100$; $k_3 = 75$; $k_4 = 50$; $k_5 = 50$. The damping is again taken to be of Rayleigh form (i.e., $\mathbf{C} = 2\alpha\mathbf{M} + 2\beta\mathbf{K}$) with $\alpha = 0.001$ and $\beta = 0.04$. Fig. 8 shows the base acceleration and the results for optimal sensor location for identification of parameter k_1. It is interesting to note that though considerations of uniqueness in identification would dictate location 1 to be optimal (Udwadia 1978), considerations of identifying the parameter k_1 starting from a close-by estimate shows that location 4 is optimal. This example therefore brings out the difference between global convergence and local convergence as discussed previously.

Kinetic Energy Criterion

Some investigators have proposed that, heuristically speaking, displacement sensors should be located where the kinetic energy of the system is a maximum. While this may be a somewhat intuitive approach to the problem, we have found that this kinetic energy criterion does not yield, in general, the optimal sensor locations. Firstly, such a criterion is not dependent on the parameters that are required to be identified as any such criterion should. Secondly, and perhaps more importantly, our concern in locating sensors for best identification of parameters hinges around the sensitivity of measurements to the parameters to be identified and not on the kinetic energy of the system. For the system considered in Fig. 1(*a*) the results shown in Fig. 4(*a*) indicate that the optimal sensor location is at location 4. Fig. 9(*a*) shows that the kinetic energy (KE) at location 4 is lower than that at location 3. Yet location 4 is obtained as the OSL from Fig. 4(*e*). This is explained by Fig. 9(*b*), which shows that though the KE at location 4 is lower than at location 3, the sensitivity (actually, its absolute value) of measurement to parameter k_5 is higher at location 4 than at location 3. The results of the KE criterion for system depicted for the fixed-fixed system considered earlier are shown in Fig. 10 for an impulsive force applied to the mass at location 3. The rank ordering of sensor locations using the KE criterion is quite different from say that given by Figs. 4(*a*) or 5(*a*). Again we see that the KE criterion would lead to a different and erroneous rank ordering of locations for, say, the identification of k_5, or of k_1 and k_5.

CONCLUSIONS

In this paper a methodology has been developed for optimally locating sensors for parameter identification using noisy measurement data. Such a methodology, based on rigorous statistical thinking, has hereto been unavailable. The methodology is predicated on starting any such identification process with a nearby, close initial estimate of the parameters to be identified. The optimal sensor location methodology proposed herein, decouples the optimization problem from the identification problem through the concept of an efficient estimator. It is applicable to both linear and nonlinear systems. The methodology also answers, in a rational manner, where to locate additional instruments, given that several are already in place in a dynamic system. A simple algorithm, which is computationally efficient, has been developed for obtaining the OSL for linear and nonlinear systems. The following are the main conclusions:

The optimal sensor locations are shown to depend on: (1) The nature of the system (the structure of the differential equations); (2) the specific parameter values of the different parameters in the system model; (3) the number of sensors to be used; (4) the duration of time over which the identification is to be carried out; (5) the specific parameters to be identified; and (6) the nature and location(s) of the input time functions (applied forces).

The methodology has been applied to, and detailed expressions obtained for, linear multi-degree-of-freedom systems. Numerical results have been also presented. The methodology provides a ranking of the locations from the best sensor location to the worst.

The methodology has been validated by actually using data from various locations and showing that those locations that are predicted to be optimal by the methodology do indeed provide the best identification of the parameters from noisy measurement data.

The results have shown that the heuristically obtained kinetic energy criterion, which is sometimes alluded to in the literature, has little to do with optimally locating sensors and, to that extent, is inappropriate for use in developing methodologies relevant to such problems.

APPENDIX I. INTERPRETATION OF TRACE NORM

We present here a more formal interpretation of the trace norm. Let us introduce an error criterion

$$J = E_{\theta,Y}[f(\theta, \hat{\theta})] \qquad (63)$$

then

$$J \cong E_\theta \left\{ E_{Y|\theta} \left[f(\theta, \theta) + \frac{\partial f}{\partial \theta} (\hat{\theta} - \theta) + \frac{1}{2} (\hat{\theta} - \theta)^T \frac{\partial^2 f}{\partial \theta^2} (\hat{\theta} - \theta) \right] \right\} \qquad (64)$$

where θ, $\hat{\theta}$, and Y = true value of the estimated parameters, the estimate, and the measurement yielding the estimates, respectively. Since $f(\theta, \hat{\theta})$ is a function of error between the θ and $\hat{\theta}$, then $f(\theta, \theta) = 0$.

However, if $\hat{\theta}$ is close to θ, then $\partial f/\partial \theta \approx \partial f/\partial \hat{\theta}$. Using this approximation one can write

$$J \cong E_\theta \left\{ \frac{\partial f}{\partial \theta} E_{Y|\theta}(\hat{\theta} - \theta) + E_{Y|\theta} \left[\frac{1}{2} (\hat{\theta} - \theta)^T \frac{\partial^2 f}{\partial \hat{\theta}^2} (\hat{\theta} - \theta) \right] \right\} \qquad (65)$$

Notice that if $\hat{\theta}$ is an efficient unbiased estimator, then $E(\hat{\theta}) = \theta$. Hence

$$E_{Y|\theta}(\hat{\theta} - \theta) = E_{Y|\theta}(\hat{\theta}) - E_{Y|\theta}(\theta) = 0 \qquad (66)$$

Therefore J simplifies to

$$J \cong E_\theta \left\{ E_{Y|\theta} \left[\frac{1}{2} (\hat{\theta} - \theta)^T \frac{\partial^2 f}{\partial \hat{\theta}^2} (\hat{\theta} - \theta) \right] \right\} \qquad (67)$$

which can be written as

$$J \cong E_\theta \frac{1}{2}\left[\text{trace}\left(\frac{\partial^2 f}{\partial \hat{\theta}^2}\text{cov }\hat{\theta}\right)\right] \quad \text{(68)}$$

To minimize the error between the estimate $\hat{\theta}$ and θ, one would want to minimize the right-hand side of (68). If f is quadratic in θ then, the second derivative of f with respect to θ in (68) is a constant matrix.

APPENDIX II. REFERENCES

Cramer, H. (1957). *Mathematical methods in statistics*. Princeton University Press, Princeton, N.J.

Dale, O. B., and Cohen, R. (1971). "Multiparameter identification in linear continuous vibratory systems." *J. Dynamic Systems, Measurement and Control*, 45–52.

Gart, J. J. (1959). "An extension of the Cramer-Rao Inequality." *Ann. of Mat. Statistics*, 32(2), 367–380.

Goodwin, G., and Payne, R. (1977). *Dynamic system identification*. Academic Press, New York, N.Y.

Hart, G. C., ed. (1976). "Dynamic response of structures: Instrumentation, testing and system identification." *Proc., ASCE Engrg. Mech. Specialty Conf.*, ASCE, New York, N.Y.

Jazwinski, A. H. (1970). *Stochastic processes and filtering*. Academic Press, New York, N.Y.

Ljung, L. (1987). *System identification: theory for the user*. Prentice-Hall, Englewood Cliffs, N.J.

Mehra, R. K., and Lainiotis, D. E. (1976). *System identification—advances and case studies*. Academic Press, New York, N.Y.

Nahi, N. E. (1969). *Estimation theory and applications*. John Wiley and Sons, New York, N.Y.

Nahi, N. E., and Wallis, D. (1968). "Optimal control for information maximization in least square parameter estimation." *U.S.C. Rep. USCEE 253*, Univ. of Southern California, Los Angeles, Calif.

Rodriguez, G., ed. (1985). "Proc., workshop on identification and control of flexible space structures." *JPL Publication 85-29*, Vols. I and II.

Sage, A. P., and Melsa, J. L. (1971). *Estimation theory with applications to communications and control*. McGraw Hill, New York, N.Y.

Shah, P., and Udwadia, F. E. (1978). "A methodology for optimal sensor locations for identification of dynamic systems." *J. Appl. Mech.*, 45, 188–196.

Udwadia, F. E., and Sharma, D. K. (1978). "Some uniqueness results related to building structural identification." *SIAM J. Appl. Math.*, 34(1), 104–118.

Udwadia, F. E., and Shah, P. (1976). "Identification of structures through records obtained during strong ground shaking." *ASME Trans., J. Engrg. and Industry*, 98(1), 1347–1362.

Udwadia, F. E., Shah, P., and Sharma, D. (1978). "Uniqueness of damping and stiffness distributions in the study of soil and structural systems." *J. Appl. Mech.*, 45, 181–187.

Udwadia, F. E. (1988). "Optimal sensor locations for system identification, report of the air force astronautics laboratory." *AFAL-TR-87-087*.