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Recursive Formulas for the Generalized *LM*-Inverse of a Matrix

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Abstract. In this paper, we present recursive formulas for the sequential determination of the generalized *LM*-inverse of a general matrix. The formulas are developed for a matrix augmented by a column. These formulas are particularized to obtain also recursive relations for the generalized *L*-inverse of a general matrix augmented by a column.

Key Words. Generalized inverse, generalized *LM*-inverse, recursive formulas, least squares problem.

1. Introduction

Consider a set of linear equations

$$Bx = b, \tag{1}$$

where B is an m by n matrix, b is an m-vector, and x is an n-vector.

The generalized *LM*-inverse of the matrix B, which we denote as B^+_{LM}, is the matrix such that the solution x, uniquely given by

$$x = B^+_{LM} b, \tag{2}$$

minimizes both

$$G = \|L^{1/2}(Bx - b)\|^2 = \|Bx - b\|^2_L \tag{3}$$

and

$$H = \|M^{1/2}x\|^2 = \|x\|^2_M, \tag{4}$$

where L is an m by m symmetric positive-definite matrix and M is an n by n symmetric positive-definite matrix.

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The following relations are the four properties of the matrix B^+_{LM} (Ref. 1):

$$\text{(i)}\ \ BB^+_{LM}B = B, \tag{5}$$

$$\text{(ii)}\ \ B^+_{LM}BB^+_{LM} = B^+_{LM}, \tag{6}$$

$$\text{(iii)}\ \ \left(BB^+_{LM}\right)^T = LBB^+_{LM}L^{-1}, \tag{7}$$

$$\text{(iv)}\ \ \left(B^+_{LM}B\right)^T = MB^+_{LM}BM^{-1}. \tag{8}$$

It should be noted that the generalized *LM*-inverse is a more general inverse than the Moore-Penrose (MP) inverse of a matrix. The concept of MP inverses was first defined by Moore (Ref. 2) in 1920 and independently by Penrose (Ref. 3) in 1955. Greville (Ref. 4) provided the first recursive algorithm in 1960 for determining the Moore-Penrose inverse of a matrix. His algorithm updates the MP inverse of a matrix whenever new information is added in the form of an augmented row or an augmented column. Because of its ability to perform such sequential updating, the recursive determination of MP inverses has found extensive use in various areas of application such as statistical inference (Ref. 5), filtering theory, estimation theory (Ref. 6), system identification (Ref. 7), optimization and control, and recently analytical dynamics (Ref. 8). In 1997, Udwadia and Kalaba (Ref. 9) gave an alternative and simple constructive proof of Greville's formulas and later (Refs. 10–11) developed recursive relations for different types of generalized inverse of a matrix including the least-squares generalized inverse, the minimum-norm generalized inverse, and the Moore-Penrose (MP) inverse of a matrix. In 2005, Udwadia and Phohomsiri (Ref. 12) obtained recursive formulas for the generalized *M*-inverse of a matrix, which is a subset of the generalized *LM*-inverse and which obtains when $L = \alpha I_m$, where α is a positive scalar.

In this paper, we develop recursive formulas for determining the generalized *LM*-inverse B^+_{LM} of any given matrix B. The results that we provide here are applicable to the successive addition of a column vector to any (in general, rectangular) matrix A. Even more general than the *MP*-inverse of a matrix, the generalized *LM*-inverse finds applications in a vast variety of areas ranging from statistics, filtering, control theory, and optimization to signal processing and mechanics. This is because, as stated before, it allows the explicit determination of the unique least squares solution of the matrix equation

$$Bx = b,$$

given by

$$x = B^+_{LM}b,$$

where the solution vector x is such that both $(b - Bx)^T L(b - Bx)$ and $x^T Mx$ are minimized for any given, appropriately dimensioned, positive-definite weighting matrices L and M. Then, the recursive determination of B^+_{LM} becomes a matter of great importance when one wants to update the vector x in the presence of new or additional information.

We then show that, for some special cases of the weighting matrices L and M, our recursive formulas for the generalized LM-inverse reduce to those for the generalized M-inverse, the generalized L-inverse, and the standard Moore-Penrose inverse. In addition to the formulas for the recursive determination of B^+_{LM}, those obtained herein for the recursive determination of the generalized L-inverse of a matrix appear also, to the best of our knowledge, to have been unknown to date.

2. A Property on Extremum of Least Squares Problems

In this section, we provide a useful property on the extremum of least squares problems (Ref. 13) that we shall use for the derivation of our main result.

Lemma 2.1. Let D be any given m by n matrix, not necessarily of full rank, and let c be any given m-vector. The extrema of $K(x) = \|Dx - c\|^2$ are all minima.

Proof. Let us start with differentiating $K(x)$ with respect to x to have

$$\partial K(x)/\partial x = 2D^T Dx - 2D^T c = 0. \tag{9}$$

Solving for x in Eq. (9) by using the definition of the generalized Moore-Penrose inverse, all the solutions to Eq. (9) are given by

$$x = (D^T D)^+ D^T c + \left[I - (D^T D)^+(D^T D)\right] w, \tag{10}$$

where the n by 1 vector w is arbitrary. We note that all the vectors x given by Eq. (10) extremize the function $K(x)$. Since $D^+ = (D^T D)^+ D^T$ (Ref. 1), the solutions can be written as

$$x = D^+ c + (I - D^+ D) w. \tag{11}$$

If we add any arbitrary nonzero n-vector e to the right-hand side of Eq. (11), we get

$$x = D^+ c + (I - D^+ D) w + e. \tag{12}$$

A substitution of Eq. (12) in $K = \|Dx - c\|^2$ gives

$$K = \|De - (1 - DD^+)c\|^2. \tag{13}$$

Since De and $(I - DD^+)c$ are orthogonal, Eq. (13) can be expressed as

$$K = \|(I - DD^+)c\|^2 + \|De\|^2. \tag{14}$$

From Eq. (14), we see that the additional vector e can only increase the value of K. Hence, the extrema that we obtain from Eq. (11) are all minima. □

3. Generalized *LM*-Inverse of a Columnwise Partitioned Matrix

In this section, we develop the recursive formulas for the generalized *LM*-inverse of an m by n matrix B being partitioned as $B = [A \mid a]$, where A is an m by $n-1$ matrix and a is a column vector of m components. Our general approach to obtain a constructive proof for the recursive relations is inspired by dynamic programming. We begin with stating our main result.

Result 3.1. Given the columnwise partitioned m by n matrix $B = [A \mid a]$, its generalized *LM*-inverse is given by

$$B^+_{LM} = [A \mid a]^+_{LM} = \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-} a d^+_L - p d^+_L \\ d^+_L \end{bmatrix}, \quad \text{for } d \neq 0, \tag{15}$$

$$= \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-} a h - p h \\ h \end{bmatrix}, \quad \text{for } d = 0, \tag{16}$$

where

$$d = (I - AA^+_{LM_-})a, \;\; p = (I - A^+_{LM_-}A)M_-^{-1}\widetilde{m}, h = \left(q^T/(q^T M q)\right) MU,$$
$$U = \begin{bmatrix} A^+_{LM_-} \\ 0_{1\times m} \end{bmatrix}, \; q = \begin{bmatrix} A^+_{LM_-} a + p \\ -1 \end{bmatrix}.$$

Here,

$$M = \begin{bmatrix} M_- & \widetilde{m} \\ \widetilde{m}^T & \overline{m} \end{bmatrix}$$

is a given n by n positive-definite matrix, where M_- is a symmetric positive-definite $(n-1)$ by $(n-1)$ matrix, $\widetilde{m}$ is a column vector of $n-1$ components, $\overline{m}$ is a scalar, and L is a given m by m positive-definite matrix.

Proof. Consider the system of linear equations

$$Bx = [A \mid a]x = b, \tag{17}$$

where $B = [A \mid a]$, A is an m by $n-1$ matrix, a is an m-vector, b is an m-vector, and x is an n-vector.

It should be noted again that L is an m by m symmetric positive-definite matrix and M is an n by n symmetric positive-definite matrix. The matrix M can be written as

$$M = \begin{bmatrix} M_- & \widetilde{m} \\ \widetilde{m}^T & \overline{m} \end{bmatrix}, \tag{18}$$

where M is a symmetric positive-definite $(n-1)$ by $(n-1)$ matrix, $\widetilde{m}$ is the column vector of $(n-1)$ components, and $\overline{m}$ is a scalar.

We assume that the generalized LM_--inverse of A, that we denote by the $(n-1)$ by m matrix $A^+_{LM_-}$, is known. Our aim is to obtain B^+_{LM} based on the known matrices A, L, M, $A^+_{LM_-}$, and the column vector a.

Let us denote

$$x = \begin{bmatrix} z \\ r \end{bmatrix}, \tag{19}$$

where z is an $(n-1)$-vector and r is a scalar. Next, we shall find the solution x such that

$$\begin{aligned} G(z,r) &= \|Bx - b\|_L^2 \\ &= \left\| [A \mid a] \begin{bmatrix} z \\ r \end{bmatrix} - b \right\|_L^2 \\ &= \|Az + ar - b\|_L^2 \end{aligned} \tag{20}$$

and

$$H = \|M^{1/2} x\|^2 \tag{21}$$

are both minimized.

For a fixed value of $r = r_0$, the value of $z(r_0)$ that minimizes $G(z(r_0), r_0)$ is given by (see Property 5.1, Appendix)

$$z(r_0) = A^+_{LM_-}(b - ar_0) + (I - A^+_{LM_-}A)t_1, \tag{22}$$

where t_1 is an arbitrary $(n-1)$-vector. Substituting Eq. (22) in Eq. (20) and using Eq. (5), we obtain

$$\begin{aligned} G(z(r_0), r_0) &= \|AA^+_{LM}(b - ar_0) + ar_0 - b\|_L^2 \\ &= \|dr_0 - (I - AA^+_{LM_-})b\|_L^2, \end{aligned} \tag{23}$$

where the m-vector d is given by

$$d = (I - AA^+_{LM_-})a.$$

We next find r_0 such that $G(z(r_0), r_0)$ is minimized. As we can notice, when $d = 0$, $G(z(r_0), r_0)$ is not a function of r_0. So, we shall consider two separate cases:

$d \neq 0$ and $d = 0$. It should be noted that, when $d = 0$, the column vector a is a linear combination of the columns of the matrix A (see Property 5.2, Appendix). When $d \neq 0$, the column vector a is not a linear combination of the columns of the matrix A.

(a) Case 1: $d \neq 0$. Let us first find r_0 that minimizes G and then let us find $(I - A^+_{LM_-}A)t_1$ that minimizes H. Using the definition of the generalized L-inverse, the scalar r_0 that minimizes G is given by

$$r_0 = d^+_L(I - AA^+_{LM_-})b + (1 - d^+_L d)t_2, \tag{24}$$

where t_2 is an arbitrary scalar.

Since $d^+_L A = 0$ and $d^+_L d = 1$ (see Properties 5.4 and 5.5, Appendix), we have

$$r_0 = d^+_L b. \tag{25}$$

Substituting Eqs. (22) and (25) in Eq. (19), we get

$$\begin{aligned} x &= \begin{bmatrix} z(r_0) \\ r_0 \end{bmatrix} \\ &= \begin{bmatrix} A^+_{LM_-}(b - ad^+_L b) + (I - A^+_{LM_-}A)t_1 \\ d^+_L b \end{bmatrix}, \end{aligned} \tag{26}$$

which can be written as

$$x = f + Et_1, \tag{27}$$

where

$$f = \begin{bmatrix} A^+_{LM_-}(I - ad^+_L) \\ d^+_L \end{bmatrix} b, \quad E = \begin{bmatrix} (I - A^+_{LM_-}A) \\ 0_{1\times(n-1)} \end{bmatrix},$$

and $0_{1\times(n-1)}$ is the zero row vector with $n - 1$ components.

Let us now find t_1 that minimizes H. Substituting Eq. (27) in Eq. (21) and then taking the partial derivative of $H(t_1) = \|M^{1/2}(f + Et_1)\|^2$ with respect to t_1, we get for the minimum

$$\partial H/\partial t_1 = \partial\|M^{1/2}(f + Et_1)\|^2/\partial e = 2E^T M(f + Et_1) = 0, \tag{28}$$

which gives

$$E^T MEt_1 = -E^T Mf. \tag{29}$$

We note that, by Lemma 2.1, the extrema of $H(t_1)$ are all minima.

Since

$$E^T ME = M_-(I - A^+_{LM_-}A)$$

(see Property 5.6, Appendix) and since

$$E^T Mf = [(I - A^+_{LM_-}A)^T \ 0^T_{n-1}] \begin{bmatrix} M_- & \widetilde{m} \\ \widetilde{m}^T & \overline{m} \end{bmatrix} \begin{bmatrix} A^+_{LM_-}(I - ad^+_L) \\ d^+_L \end{bmatrix} b$$
$$= M_-(I - A^+_{LM_-}A)M_-^{-1}\widetilde{m}d^+_L b,$$

Eq. (29) becomes

$$\begin{aligned} M_-(I - A^+_{LM_-}A)t_1 &= -E^T Mf \\ &= -M_-(I - A^+_{LM_-}A)M_-^{-1}\widetilde{m}d^+_L b, \end{aligned} \tag{30}$$

which yields, since M_- is nonsingular,

$$\begin{aligned} (I - A^+_{LM_-}A)t_1 &= -(I - A^+_{LM_-}A)M_-^{-1}\widetilde{m}d^+_L b \\ &= -pd^+_L b, \end{aligned} \tag{31}$$

where p denotes the $(n-1)$-vector

$$p = (I - A^+_{LM_-}A)M_-^{-1}\widetilde{m}.$$

Substituting $(I - A^+_{LM_-}A)t_1 = -pd^+_L b$ in Eq. (26), we have

$$\begin{aligned} x &= B^+_{LM} b \\ &= \begin{bmatrix} A^+_{LM_-}(b - ad^+_L b) + (I - A^+_{LM_-}A)t_1 \\ d^+_L b \end{bmatrix} \\ &= \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-}ad^+_L - pd^+_L \\ d^+_L \end{bmatrix} b. \end{aligned} \tag{32}$$

Thus, Eq. (32) gives

$$B^+_{LM} = \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-}ad^+_L - pd^+_L \\ d^+_L \end{bmatrix},$$

when

$$d = (I - AA^+_{LM_-})a \neq 0.$$

□

(b) Case 2: $d = 0$. When $d = 0$, we have

$$G = \|(I - AA^+_{LM_-})b\|^2_L,$$

which is independent of our choice of r_0, which is fixed. Therefore, the vector x that minimizes G for any given fixed value of r_0 is given by

$$\begin{aligned} x &= \begin{bmatrix} z(r_0) \\ r_0 \end{bmatrix} \\ &= \begin{bmatrix} A^+_{LM_-}(b - ar_0) + (I - A^+_{LM_-}A)t_1 \\ r_0 \end{bmatrix}, \end{aligned} \tag{33}$$

which can be written as

$$x = \begin{bmatrix} z(r_0) \\ r_0 \end{bmatrix} = g + Et_1, \tag{34}$$

where

$$g = \begin{bmatrix} A^+_{LM_-}(b - ar_0) \\ r_0 \end{bmatrix}, \quad E = \begin{bmatrix} (I - A^+_{LM_-}A) \\ 0_{1\times(n-1)} \end{bmatrix}.$$

Substituting Eq. (34) in Eq. (21), we have

$$H(t_1) = \|M^{1/2}(g + Et_1)\|^2. \tag{35}$$

Minimizing Eq. (35) with respect to $t_1(r_0)$, we obtain

$$\begin{aligned} \partial H/\partial t_1 &= \partial \|M^{1/2}x\|^2/\partial t_1 \\ &= \partial \|M^{1/2}(g + Et_1)\|^2/\partial t_1, \\ &= 2E^T M(g + Et_1) = 0. \end{aligned} \tag{36}$$

As before, we note that, by Lemma 2.1, the extrema of $H(t_1)$ are all minima.

Since

$$E^T ME = M_-(I - A^+_{LM_-}A)$$

(see Property 5.6, Appendix), from the last equality of Eq. (36) we have

$$M_-(I - A^+_{LM_-}A)t_1 = -E^T Mg, \tag{37}$$

which can be simplified after premultiplication by M_-^{-1} as

$$(I - A^+_{LM_-}A)t_1 = -M_-^{-1}E^T Mg. \tag{38}$$

Since

$$\begin{aligned} E^T Mg &= \begin{bmatrix} (I - A^+_{LM_-}A)^T & 0^T_{1\times(n-1)} \end{bmatrix} \begin{bmatrix} M_- & \widetilde{m} \\ \widetilde{m}^T & \overline{m} \end{bmatrix} \begin{bmatrix} A^+_{LM_-}(b - ar_0) \\ r_0 \end{bmatrix} \\ &= M_-(I - A^+_{LM_-}A)M_-^{-1}\widetilde{m}r_0, \end{aligned}$$

by Eq. (38) we have

$$\begin{aligned} (I - A^+_{LM_-}A)t_1 &= -(I - A^+_{LM_-}A)M_-^{-1}\widetilde{m}r_0 \\ &= -pr_0, \end{aligned} \tag{39}$$

where again p denotes

$$p = (I - A^+_{LM_-}A)M_-^{-1}\widetilde{m}.$$

We find next the value of r_0 that minimizes $H(t_1(r_0), r_0)$. Using Eq. (39) in Eq. (33), we obtain by Eq. (21)

$$\begin{aligned} H(r_0) &= \|M^{1/2}x\|^2 \\ &= \left\| M^{1/2} \begin{bmatrix} A^+_{LM\_}(b - ar_0) - pr_0 \\ r_0 \end{bmatrix} \right\|^2 \\ &= \|M^{1/2}(Ub - qr_0)\|^2, \end{aligned} \tag{40}$$

where

$$U = \begin{bmatrix} A^+_{LM\_} \\ 0_{1\times m} \end{bmatrix}, \quad q = \begin{bmatrix} \nu + p \\ -1 \end{bmatrix}, \quad \nu = A^+_{LM\_}a,$$

and $0_{1\times m}$ is the zero row vector with m components.

Minimizing the right-hand side of Eq. (40) with respect to r_0, we get

$$\partial H/\partial r_0 = 2q^T M(Ub - qr_0) = 0. \tag{41}$$

Since M is positive definite, the scalar $q^T Mq$ is greater than zero and therefore the value of r_0 that minimizes H is obtained from Eq. (41) as

$$\begin{aligned} r_0 &= (q^T/(q^T Mq))MUb \\ &= hb, \end{aligned} \tag{42}$$

where

$$h = (q^T/(q^T Mq))MU.$$

Using $(I - A^+_{LM\_}A)t_1 = -pr_0$ and $r_0 = hb$ in Eq. (33), we obtain

$$\begin{aligned} x &= [A\,|\,a]^+_{LM}b \\ &= \begin{bmatrix} A^+_{LM\_}b - A^+_{LM\_}ar_0 + (I - A^+_{LM\_}A)t_1 \\ r_0 \end{bmatrix} \\ &= \begin{bmatrix} A^+_{LM\_} - A^+_{LM\_}ah - ph \\ h \end{bmatrix} b. \end{aligned} \tag{43}$$

Thus, we have

$$\begin{aligned} B^+_{LM} &= [A\,|\,a]^+_{LM} \\ &= \begin{bmatrix} A^+_{LM\_} - A^+_{LM\_}ah - ph \\ h \end{bmatrix}, \end{aligned}$$

for $d = (I - AA^+_{LM\_})a = 0$,

where

$$p = (I - A^+_{LM\_}A)M_-^{-1}\widetilde{m}, \; h = (q^T/(q^T Mq))MU,$$

$$U = \begin{bmatrix} A^+_{LM\_} \\ 0_{1\times m} \end{bmatrix}, \; q = \begin{bmatrix} \nu + p \\ -1 \end{bmatrix}, \; \nu = A^+_{LM\_}a. \qquad \square$$

In the following three corollaries, we next particularized the relations (15) and (16) to obtain recursive formulas for the generalized M-inverse, the generalized L-inverse, and the standard Moore-Penrose inverse of a matrix. We note that, to the best of our knowledge, the explicit recursive relations for the generalized L-inverse are provided here for the first time.

Corollary 3.1. When $L = \alpha I_m$ for $\alpha > 0$, the generalized LM-inverse of the matrix $B = [A \mid a]$ becomes the generalized M-inverse and the recursive relations are

$$[A \mid a]^+_{L=\alpha I_m, M} = [A \mid a]^+_M = \begin{bmatrix} A^+_{M_-} - A^+_{M_-} ad^+ - pd^+ \\ d^+ \end{bmatrix}, \quad \text{for } d \neq 0, \tag{44}$$

$$[A \mid a]^+_{L=\alpha I_m, M} = [A \mid a]^+_M = \begin{bmatrix} A^+_{M_-} - A^+_{M_-} ah - ph \\ h \end{bmatrix}, \quad \text{for } d = 0, \tag{45}$$

where

$$d = (I - AA^+_{M_-})a, \quad p = (I - A^+_{M_-}A)M_-^{-1}\widetilde{m}, \quad h = (q^T/(q^T Mq))MU,$$

$$U = \begin{bmatrix} A^+_{M_-} \\ 0_{1\times m} \end{bmatrix}, \quad q = \begin{bmatrix} \nu + p \\ -1 \end{bmatrix}, \quad \nu = A^+_{M_-} a.$$

Proof. When $L = \alpha I_m$ for $\alpha > 0$, we have $d_L^+ = d^+$ and $A^+_{LM_-} = A^+_{M_-}$. Thus, for $d \neq 0$, we have, by relation (15),

$$[A \mid a]^+_{L=\alpha I_m, M} = \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-} ad_L^+ - pd_L^+ \\ d_L^+ \end{bmatrix}_{L=\alpha I_m} = \begin{bmatrix} A^+_{M_-} - A^+_{M_-} ad^+ - pd^+ \\ d^+ \end{bmatrix}.$$

For $d = 0$, using relation (16) we get

$$[A \mid a]^+_{L=\alpha I_m, M} = \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-} ah - ph \\ h \end{bmatrix}_{L=\alpha I_m} = \begin{bmatrix} A^+_{M_-} - A^+_{M_-} ah - ph \\ h \end{bmatrix}.$$

In the above relations,

$$p = (I - A_{M_-}^{+} A)M_{-}^{-1}\widetilde{m}, \quad h = (q^T / q^T M q) M U,$$

$$U = \begin{bmatrix} A_{M_-}^{+} \\ 0_{1\times m} \end{bmatrix}, \quad q = \begin{bmatrix} \nu + p \\ -1 \end{bmatrix}, \quad \nu = A_{M_-}^{+} a.$$

These relations are identical to the recursive relations for B_M^+ given in Ref. 12. □

Corollary 3.2. When $M = \beta I_n$ for $\beta > 0$, the generalized LM-inverse of the matrix $B = [A\,|a]$ becomes the generalized L-inverse and the recursive relations are

$$[A\,|\,a]_{L,M=\beta I_n}^{+} = [A \mid a]_L^{+} = \begin{bmatrix} A_L^{+} - A_L^{+} a d_L^{+} \\ d_L^{+} \end{bmatrix}, \qquad \text{for } d \neq 0, \tag{46}$$

$$[A\,|\,a]_{L,M=\beta I_m}^{+} = [A\,|\,a]_L^{+} = \begin{bmatrix} A_L^{+} - A_L^{+} a h \\ h \end{bmatrix}, \qquad \text{for } d = 0, \tag{47}$$

where

$$d = (I - AA_L^{+})a, \quad h = \nu^T A_L^{+} / (1 + \nu^T \nu), \quad \nu = A_L^{+} a.$$

Proof. When $M = \beta I_n$ for $\beta > 0$, we have

$$A_{LM_-}^{+} = A_L^{+}, \quad \widetilde{m} = 0, \quad p = 0,\ U = \begin{bmatrix} A_L^{+} \\ 0_{1\times m} \end{bmatrix},\ q = \begin{bmatrix} \nu \\ -1 \end{bmatrix},\ \nu = A_L^{+} a.$$

Using relation (15), for $d \neq 0$ we have

$$[A\mid a]_{L,M=\beta I_n}^{+} = \begin{bmatrix} A_{LM_-}^{+} - A_{LM_-}^{+} a d_L^{+} - p d_L^{+} \\ d_L^{+} \end{bmatrix}_{M=\beta I_n} = \begin{bmatrix} A_L^{+} - A_L^{+} a d_L^{+} \\ d_L^{+} \end{bmatrix}.$$

For $d = 0$, using relation (16), we obtain

$$[A\,|\,a]_{L,M=\beta I_n}^{+} = \begin{bmatrix} A_{LM_-}^{+} - A_{LM_-}^{+} a h - p h \\ h \end{bmatrix}_{M=\beta I_n} = \begin{bmatrix} A_L^{+} - A_L^{+} a h \\ h \end{bmatrix}.$$

□

In the above relations,

$$\begin{aligned} h &= (1/(q^T M q)) q^T M U \\ &= (1/(q^T q)) q^T U \\ &= \left[1/(1+\nu^T \nu)\right] [\nu^T \ -1] \begin{bmatrix} A_L^+ \\ 0_{1\times m} \end{bmatrix} \\ &= \nu^T A_L^+ /(1+\nu^T \nu). \end{aligned}$$

To the best of our knowledge, the recursive relations for B_L^+ are provided here for the first time.

Corollary 3.3. When $L = \alpha I_m$ and $M = \beta I_n$ for $\alpha, \beta > 0$, the generalized LM-inverse of the matrix $B = [A \mid a]$ becomes the Moore-Penrose (MP) inverse and the recursive relations are

$$\begin{aligned} [A \mid a]^+_{L=\alpha I_m, M=\beta I_n} &= [A \mid a]^+ \\ &= \begin{bmatrix} A^+ - A^+ a d^+ \\ d^+ \end{bmatrix}, \qquad \text{for } d \neq 0, \end{aligned} \tag{48}$$

$$\begin{aligned} [A \quad |a]^+_{L=\alpha I_m, M=\beta I_n} &= [A \mid a]^+ \\ &= \begin{bmatrix} A^+ - \nu\nu^T A^+/(1+\nu^T\nu) \\ \nu^T A^+/(1+\nu^T \nu) \end{bmatrix}, \qquad \text{for } d = 0, \end{aligned} \tag{49}$$

where

$d = (I - AA^+)a$ and $\nu = A^+ a$.

Proof. When $L = \alpha I_m$ and $M = \beta I_n$ for $\alpha, \beta > 0$, we have

$$A^+_{LM_-} = A^+, \quad d_L^+ = d^+, \quad \widetilde{m} = 0, \quad p = (I - A^+ A) M_-^{-1} \widetilde{m} = 0,$$

$$U = \begin{bmatrix} A^+ \\ 0_{1\times m} \end{bmatrix}, \quad \nu = A^+ a, \quad q = \begin{bmatrix} \nu \\ -1 \end{bmatrix}, \quad q^T q = 1 + \nu^T \nu,$$

$$\begin{aligned} h &= q^T U/(q^T q) \\ &= [1/(1+\nu^T\nu)][\nu^T \quad -1] \begin{bmatrix} A^+ \\ 0_{1\times m} \end{bmatrix} \\ &= \nu^T A^+/(1+\nu^T \nu). \end{aligned}$$

Thus, for $d \neq 0$, we have

$$\begin{aligned} [A \mid a]^+_{L=\alpha I_m, M=\beta I_n} &= \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-} a d_L^+ - p d_L^+ \\ d_L^+ \end{bmatrix}_{L=\alpha I_m, M=\beta I_n} \\ &= \begin{bmatrix} A^+ - A^+ a d^+ \\ d^+ \end{bmatrix}. \end{aligned}$$

Similarly, for $d = 0$, we obtain

$$[A \mid a]^{+}_{L=\alpha I_m, M=\beta I_n} = \begin{bmatrix} A^{+}_{LM_-} - A^{+}_{LM_-} ah - ph \\ h \end{bmatrix}_{L=\alpha I_m, M=\beta I_n}$$
$$= \begin{bmatrix} A^{+} - \nu\nu^T A^{+}/(1 + \nu^T \nu) \\ \nu^T A^{+}/(1 + \nu^T \nu) \end{bmatrix}.$$

□

These relations above are the same as the recursive relations for B^+ given in Ref. 9.

4. Conclusions

In this paper, we obtain recursive formulas for the generalized LM-inverse of a columnwise partitioned m by n matrix $B = [A \mid a]$. We consider two separate cases: when the additional column a is a linear combination of the columns of the matrix A and when it is not. We show that, when $L = \alpha I_m$ for $\alpha > 0$, our recursive formulas become identical to those for the generalized M-inverse of a matrix (Ref. 12); when $M = \beta I_n$ for $\beta > 0$, they become those for the generalized L-inverse; and when both $L = \alpha I_m$ and $M = \beta I_n$, they reduce to those for the standard MP inverse of a matrix (Ref. 9). To the best of our knowledge, the recursive formulas for the generalized LM-inverse of a rectangular matrix and those for the generalized L-inverse of a matrix constitute results that are not known hereto.

5. Appendix

In this appendix, we provide some properties that are used in proving our result.

Property 5.1. The column vector z that minimizes $\|Az - \widetilde{b}\|^2_L$, where A is an m by $(n-1)$ matrix, $\widetilde{b} = b - ar$ is an m-vector, and L is a positive-definite m by m matrix is given by

$$z = A^{+}_{LM_-}\tilde{b} + (I - A^{+}_{LM_-}A)t_1$$

for any $(n-1)$-vector t_1.

Proof. We have

$$\|Az - \widetilde{b}\|^2_L = \left\|Az - AA^{+}_{LM_-}\widetilde{b} + AA^{+}_{LM_-}\widetilde{b} - \widetilde{b}\right\|^2_L$$
$$= \left\|A(z - A^{+}_{LM_-}\widetilde{b}) - (I - AA^{+}_{LM_-})\widetilde{b}\right\|^2_L.$$

Using

$$AA^+_{LM_-}A = A \text{ and } \left(AA^+_{LM_-}\right)^T = LAA^+_{LM_-}L^{-1},$$

we obtain

$$\left[(I - AA^+_{LM_-})\widetilde{b}\right]^T L\left[A(z - A^+_{LM_-}\widetilde{b}\right] = \widetilde{b}^T L(I - AA^+_{LM_-})L^{-1}LA(Z - A^+_{LM_-}\widetilde{b}) = 0.$$

As we can see from the above equation, $A(z - A^+_{LM_-}\widetilde{b})$ and $(I - AA^+_{LM_-})\widetilde{b}$ are L-orthogonal. Thus, we have

$$\begin{aligned}\|Az - \widetilde{b}\|^2_L &= \left\|A(z - A^+_{LM_-}\widetilde{b}) - (I - AA^+_{LM_-})\widetilde{b}\right\|^2_L \\ &= \left\|A(z - A^+_{LM_-})\right\|^2_L + \left\|(I - AA^+_{LM_-})\widetilde{b}\right\|^2_L.\end{aligned}$$

When

$$z = A^+_{LM_-}\tilde{b} + (I - A^+_{LM_-}A)t_1,$$

$\|A(z - A^+_{LM_-})\widetilde{b})\|^2_L$ becomes zero and we achieve the minimum value of $\|Az - \widetilde{b}\|^2_L$, which is given by $\|(I - AA^+_{LM_-})\widetilde{b}\|^2_L$. This completes the proof. □

Property 5.2. The column vector $d = 0$ if and only if the column vector a is a linear combination of the columns of the matrix A.

Proof. We need to show that

$$d = 0 \Leftrightarrow a = A\gamma,$$

where γ is an $(n-1)$-vector.
When

$$d = (I - AA^+_{LM_-})a = 0,$$

we have

$$a = A\gamma,$$

where

$$\gamma = A^+_{LM_-}a.$$

When $a = A\gamma$, by premultiplying $a = A\gamma$ by $I - AA^+_{LM_-}$ we have

$$(I - AA^+_{LM_-})a = d = 0.$$

Thus, we get the result. □

Property 5.3. For any column m-vector d, $d^+_L = (d^T Ld)^{-1}d^T L$ for $d \neq 0$ and $d^+_L = 0$ for $d = 0$.

Proof. If $(d^T Ld)^{-1}d^T L$ is the generalized L-inverse of d for $d \neq 0$, it must satisfy all the four properties of the L-inverse (Ref. 1) as follows:

$$
\begin{aligned}
&\text{(i)} \quad dd_L^+ d = d \cdot (d^T Ld)^{-1}d^T L \cdot d = d,\\
&\text{(ii)} \quad d_L^+ dd_L^+ = (d^T Ld)^{-1}d^T L \cdot d \cdot (d^T Ld)^{-1}d^T L = (d^T Ld)^{-1}d^T L = d_L^+,\\
&\text{(iii)} \quad (dd_L^+)^T = \left[d(d^T Ld)^{-1}d^T L\right]^T = L[d(d^T Ld)^{-1}d^T L]L^{-1} = Ldd_L^+ L^{-1},\\
&\text{(iv)} \quad (d_L^+ d)^T = \left[(d^T Ld)^{-1}d^T Ld\right]^T = (d^T Ld)^{-1}d^T Ld = d_L^+ d,
\end{aligned}
$$

where we note that $d^T Ld$ is a positive scalar. Thus, $(d^T Ld)^{-1}d^T L$ is the generalized L-inverse of d when $d \neq 0$. Similarly, to prove that $d_L^+ = 0$ when $d = 0$, we simply note that d_L^+ satisfies the four properties of the generalized L-inverse. □

Property 5.4. For the column m-vector $d = (I - AA_{LM_-}^+)a \neq 0$, we have $d_L^+ A = 0$.

Proof. Since $d_L^+ = (d^T Ld)^{-1}d^T L$ (see Property 5.3 above) and since $d = (I - AA_{LM_-}^+)a$, we have

$$
\begin{aligned}
d_L^+ A &= (d^T Ld)^{-1}d^T LA\\
&= (d^T Ld)^{-1}[(I - AA_{LM_-}^+)a]^T LA\\
&= (d^T Ld)^{-1}a^T (I - AA_{LM_-}^+)^T LA\\
&= (d^T Ld)^{-1}a^T L(I - AA_{LM_-}^+)L^{-1}LA = 0.
\end{aligned}
$$

□

Property 5.5. For any column m-vector $d \neq 0$, $d_L^+ d = 1$.

Proof. Since

$$d_L^+ = (d^T Ld)^{-1}d^T L,$$

we get

$$d_L^+ d = (d^T Ld)^{-1}d^T Ld = 1.$$

□

Property 5.6. $E^T ME = M_-(I - A_{LM_-}^+ A)$.

Proof. Using

$$E = \begin{bmatrix} (I - A_{LM_-}^+ A) \\ 0_{1\times(n-1)} \end{bmatrix} \text{ and } M = \begin{bmatrix} M_- & \widetilde{m} \\ \widetilde{m}^T & \overline{m} \end{bmatrix},$$

we have

$$
\begin{aligned}
E^T M E &= \begin{bmatrix} (I - A^+_{LM_-} A)^T & 0^T_{1\times(n-1)} \end{bmatrix} \begin{bmatrix} M_- & \widetilde{m} \\ \widetilde{m}^T & \overline{m} \end{bmatrix} \begin{bmatrix} (I - A^+_{LM_-} A) \\ 0_{1\times(n-1)} \end{bmatrix} \\
&= \begin{bmatrix} M_-(I - A^+_{LM_-} A)M_-^{-1} & 0^T_{1\times(n-1)} \end{bmatrix} \begin{bmatrix} M_-(I - A^+_{LM_-} A) \\ \widetilde{m}^T (I - A^+_{LM_-} A) \end{bmatrix} \\
&= M_-(I - A^+_{LM_-} A).
\end{aligned}
$$

□

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