Lagrangians and Integrals of Motion for Multi-Degree-of-Freedom Linear Systems with Gyroscopic and Circulatory Forces

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This paper deals with multi-degree-of-freedom linear dynamic systems that may be subjected to damping, gyroscopic, and circulatory forces. Under suitable conditions, Lagrangians are obtained for such systems. These new results include and significantly generalize previous work reported on Lagrangians and invariants of motion for multi-degree-of-freedom linear classically damped systems in that they permit the presence of gyroscopic and circulatory terms when modeling physical systems. To delineate the compass of applicability of these results, the conditions under which the presence of such terms can be included in the dynamic description are provided. An invariant of the motion, or conservation law, is also obtained for such general systems. The invariant is shown to be a natural generalization of the well-known conservation-of-energy principle that is applicable to undamped multi-degree-of-freedom potential systems.

I. Introduction

LAGRANGIANS for dynamic systems provide a concise description that can be unpacked through the use of the Euler–Lagrange equations to yield the equations that govern their evolution in time. In addition, they can be used to help decipher symmetries, study stability, provide useful perturbation results, and find invariants. Arguably, a Lagrangian is therefore the most compact way of describing the information contained in a dynamic system. This is why the inverse problem of Lagrangian mechanics (the search for a suitable Lagrangian that delivers, through the Euler–Lagrange equations, a desired evolution equation for a dynamic system) is so important and has been worked on by numerous investigators (e.g., [1–6]). However, almost all, if not all, of the effort to date has been focused on finding Lagrangians for systems for which the evolution equations are of very low order (one- and two-degree-of-freedom systems), which describe dynamic systems that arise commonly in physics and engineering.

The reason for this is that the inverse problem of Lagrangian mechanics is very difficult to solve for large multi-degree-of-freedom systems. The conditions for the existence of Lagrangians were first obtained in a rigorous manner by Helmholtz back in the 19th century and are called the Helmholtz conditions [7]. However, their use is greatly limited in finding Lagrangians for multi-degree-of-freedom systems because they involve the solution of coupled partial differential equations, a complex and onerous task. And, as mentioned, these coupled equations can at best be handled for systems with a very small number of degrees of freedom, typically one or two. Thus, this approach is very difficult to use for multi-degree-of-freedom systems that typically arise in various fields of engineering, where the number of degrees of freedom can easily be in the hundreds, and often much more. Among multi-degree-of-freedom systems, damped linear systems are perhaps the most often used in routine engineering analysis and design because they describe the small-amplitude vibrations of many naturally occurring and engineered systems. It is these systems that this paper deals with.

In [8], Helmholtz's approach is used to obtain useful general results for a two-degree-of-freedom damped linear system. It shows the difficulties posed by the use of Helmholtz's conditions and why their use has been limited to date to systems with only a few degrees of freedom. General damped linear multi-degree-of-freedom systems using a more direct approach are also studied in [8]. However, the results are limited to systems in which the damping and stiffness matrices are restricted to have certain structures and the parameters in them are required to depend in specific ways on the elements of the mass matrix. In a later paper, general classically damped linear multi-degree-of-freedom systems are dealt with, and Lagrangians and invariants of motion for such systems are explicitly obtained [9]. Such dynamic systems with symmetric stiffness and damping matrices are commonly used to model assemblies and subassemblies in civil, mechanical, and aerospace engineering that undergo small-amplitude oscillations (see, for example, [10,11]). They afford the conceptual and analytical simplicity of possessing classical normal modes of vibration, and their response can be interpreted in terms of those of an uncoupled set of second-order differential equations [12].

The wider class of linear multi-degree-of-freedom dynamic systems, in whose description the matrices multiplying the generalized velocity and generalized displacement n-vectors may not be symmetric, is of much greater interest from the viewpoint of modeling multitudes of physical phenomena and engineered systems. These systems can arise when discretizing continuous vibrating systems and in various real-life engineered ones [e.g., 13–21]. Their oscillatory response is usually more challenging to understand, both conceptually and analytically, than those of classically damped linear multi-degree-of-freedom systems. This paper addresses this broader general class of linear multi-degree-of-freedom dynamic systems, which includes those that may (or may not) have gyroscopic terms and those that may (or may not) have circulatory contributions to their stiffness. Classically damped systems constitute a special class of such systems, which have neither gyroscopic nor circulatory terms, and hence the results obtained herein are generalizations of those given previously in [9].

The organization of this paper is as follows. Under suitable assumptions, Lagrangians for systems that may have gyroscopic and circulatory terms are obtained in Sec. II. An invariant of the motion, or conservation law for the system, is also obtained. Section III deals with providing results that try to delineate the scope and structure of those dynamic systems to which these new results apply, so that they can be relatively easily identified. Numerical examples are provided to illustrate some of the analytical results. Section IV provides the conclusions.

II. Damped Linear Multi-Degree-of-Freedom Systems with Gyroscopic and Circulatory Forces

We consider the problem of finding a Lagrangian for the damped multi-degree-of-freedom linear system described by the equation

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$$\tilde{M}\ddot{z}+(\tilde{D}+\tilde{G})\dot{z}+(\tilde{K}_s+\tilde{N})z=0 \tag{1}$$

where the column vector z has n components (n-vector), and the matrix $\tilde{M}$ is an $n\times n$ constant real positive-definite matrix. The $n\times n$ constant matrices $\tilde{D}$, $\tilde{G}$, $\tilde{K}_s$, and $\tilde{N}$ are the damping (symmetric), gyroscopic (skew-symmetric), stiffness (symmetric), and the circulatory (skew-symmetric) matrices of the dynamic system, respectively. We shall denote for brevity $\tilde{C}\coloneqq\tilde{D}+\tilde{G}$ and $\tilde{K}\coloneqq\tilde{K}_s+\tilde{N}$, which are then constant real nonsymmetric matrices. The matrices $\tilde{M}^{-1}\tilde{K}$ and $\tilde{M}^{-1}\tilde{C}$ are assumed to be 1) diagonalizable matrices and 2) commute with each other.

Premultiplying by $\tilde{M}^{-1}$, Eq. (1) becomes

$$\ddot{z}+\tilde{M}^{-1}\tilde{C}\dot{z}+\tilde{M}^{-1}\tilde{K}z=0 \tag{2}$$

Because the matrices $\tilde{M}^{-1}\tilde{K}$ and $\tilde{M}^{-1}\tilde{C}$ commute, they satisfy the relation $\tilde{C}\tilde{M}^{-1}\tilde{K}=\tilde{K}\tilde{M}^{-1}\tilde{C}$. Furthermore, because they are each diagonalizable, there exists a matrix $\tilde{P}$ such that they can be simultaneously diagonalized [22]. We then have the relations

$$\tilde{P}^{-1}(\tilde{M}^{-1}\tilde{K})\tilde{P}=\tilde{\Lambda},\quad\text{and}\quad\tilde{P}^{-1}(\tilde{M}^{-1}\tilde{C})\tilde{P}=\tilde{\Delta} \tag{3}$$

where $\tilde{\Lambda}$ and $\tilde{\Delta}$ are diagonal matrices.

Setting $z(t)=\tilde{M}^{-1/2}x(t)$ and premultiplying by $\tilde{M}^{-1/2}$, Eq. (1) can also be rewritten as

$$\ddot{x}+C\dot{x}+Kx=0 \tag{4}$$

where

$$C=\tilde{M}^{-1/2}\tilde{C}\tilde{M}^{-1/2}=\tilde{M}^{-1/2}(\tilde{D}+\tilde{G})\tilde{M}^{-1/2}\coloneqq D+G \tag{5}$$

and

$$K=\tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}=\tilde{M}^{-1/2}(\tilde{K}_s+\tilde{N})\tilde{M}^{-1/2}=K_s+N \tag{6}$$

Note that because $\tilde{C}$ and $\tilde{K}$ are nonsymmetric so are the matrices C and K. The matrices $D=M^{-1/2}\tilde{D}M^{-1/2}$ and $K_s=M^{-1/2}\tilde{K}_sM^{-1/2}$ are symmetric, whereas $G=M^{-1/2}\tilde{G}M^{-1/2}$ and $N=M^{-1/2}\tilde{N}M^{-1/2}$ are skew symmetric. The matrices G and N represent the gyroscopic and the circulatory terms, respectively, in the description of the dynamic system given in Eq. (4).

Equation (4) is evidently equivalent to Eq. (1) and, throughout this paper, we will use it instead of Eq. (1).

The two relations given in Eq. (3) can now be recast in terms of the matrices K and C used in Eq. (4) as

$$\underbrace{(\tilde{P}^{-1}\tilde{M}^{-1/2})(\tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2})(\tilde{M}^{1/2}\tilde{P})}_{\tilde{M}^{-1}\tilde{K}}\coloneqq P^{-1}KP=\tilde{\Lambda} \tag{7}$$

and

$$\underbrace{(\tilde{P}^{-1}\tilde{M}^{-1/2})(\tilde{M}^{-1/2}\tilde{C}\tilde{M}^{-1/2})(\tilde{M}^{1/2}\tilde{P})}_{\tilde{M}^{-1}\tilde{C}}\coloneqq P^{-1}CP=\tilde{\Delta} \tag{8}$$

where $P\coloneqq\tilde{M}^{1/2}\tilde{P}$ is a matrix that can have complex entries.

Equations (7) and (8) show that, when the matrices $\tilde{M}^{-1}\tilde{K}$ and $\tilde{M}^{-1}\tilde{C}$ are simultaneously diagonalized by the matrix $\tilde{P}$, the matrices K and C are simultaneously diagonalized by the invertible matrix P. Moreover, the aforementioned condition for the matrices $\tilde{M}^{-1}\tilde{K}$ and $\tilde{M}^{-1}\tilde{C}$ to commute ($\tilde{C}\tilde{M}^{-1}\tilde{K}=\tilde{K}\tilde{M}^{-1}\tilde{C}$) now simplifies to $CK=KC$ (i.e., the matrices K and C commute). Furthermore, the matrices $\tilde{M}^{-1}\tilde{K}$ and $\tilde{M}^{-1}\tilde{C}$ are diagonalizable if and only if the matrices K and C are diagonalizable. Thus, we find that, when the matrices $\tilde{M}^{-1}\tilde{K}$ and $\tilde{M}^{-1}\tilde{C}$ are diagonalizable and commute, the matrices K and C in Eq. (4) are likewise diagonalizable and commute, making K and C then simultaneously diagonalizable [22]. However, it should be noted again that, because the matrices $\tilde{K}$ and $\tilde{C}$ are not symmetric in general, neither are the matrices K and C because $K=\tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$ and $C=\tilde{M}^{-1/2}\tilde{C}\tilde{M}^{-1/2}$.

A. Lagrangians for the Dynamic System

In the following, we provide two different Lagrangians for the dynamic system described by Eq. (4) when the matrices K and C are not symmetric.

Result 1: A Lagrangian for the dynamic system (4) where $K=K_s+N$ and $C=D+G$ are nonsymmetric matrices that are diagonalizable and commute is given by

$$L=\frac{1}{2}\dot{x}^TR^{-1}e^{Ct}\dot{x}-\frac{1}{2}x^TR^{-1}e^{Ct}Kx \tag{9}$$

where the (nonsingular) symmetric matrix $R=PP^T$ and the nonsingular matrix $P=\tilde{M}^{1/2}\tilde{P}$ is as defined in Eqs. (7) and (8).

Proof: We will show that, upon using the aforementioned Lagrangian, the Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right)-\frac{\partial L}{\partial x}=0 \tag{10}$$

yields the equation of motion of the dynamic system given in Eq. (4).

We begin by observing that, since $P^{-1}KP=\Lambda$, by taking the transpose on both sides of the equation, we obtain $P^TK^TP^{-T}=\Lambda^T$, and noting that Λ is a diagonal matrix, one obtains

$$P^TK^TP^{-T}=P^{-1}KP \tag{11}$$

so that

$$K^T=P^{-T}P^{-1}KPP^T=R^{-1}KR \tag{12}$$

and therefore

$$e^{K^Tt}=R^{-1}e^{Kt}R \tag{13}$$

In a similar manner, starting with $P^{-1}CP=D$, where D is a diagonal matrix, we get

$$C^T=R^{-1}CR \tag{14}$$

and

$$e^{C^Tt}=R^{-1}e^{Ct}R \tag{15}$$

Differentiating the Lagrangian L in Eq. (9) (partially) with respect to $\dot{x}$ then yields

$$\frac{\partial L}{\partial\dot{x}}=\frac{1}{2}R^{-1}e^{Ct}\dot{x}+\frac{1}{2}e^{C^Tt}R^{-T}\dot{x}=\frac{1}{2}R^{-1}e^{Ct}\dot{x}+\frac{1}{2}R^{-1}e^{Ct}RR^{-T}\dot{x}$$
$$=R^{-1}e^{Ct}\dot{x} \tag{16}$$

where we have used Eq. (15) and the fact that R^{-1} is a symmetric matrix in the second equality.

A further differentiation with respect to time t then gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right)=R^{-1}e^{Ct}\ddot{x}+R^{-1}e^{Ct}C\dot{x} \tag{17}$$

Also, differentiating L partially with respect to x we get

$$-\frac{\partial L}{\partial x}=\frac{\partial}{\partial x}\left(\frac{1}{2}x^TR^{-1}e^{Ct}Kx\right)=\frac{1}{2}R^{-1}e^{Ct}Kx+\frac{1}{2}K^Te^{C^Tt}R^{-T}x$$
$$=\frac{1}{2}R^{-1}e^{Ct}Kx+\frac{1}{2}(R^{-1}KR)(R^{-1}e^{Ct}R)R^{-T}x$$
$$=\frac{1}{2}R^{-1}e^{Ct}Kx+\frac{1}{2}R^{-1}Ke^{Ct}x=R^{-1}e^{Ct}Kx \tag{18}$$

In the second line of Eq. (18), Eqs. (12) and (14) have been used; the third line follows because R^{-1} is symmetric; and the last equality follows because the matrices K and C commute.

Using relations (17) and (18) in Eq. (10) proves the result. □

Result 2: Another Lagrangian for the dynamic system (4) where K and C are nonsymmetric matrices that are diagonalizable and commute is given by

$$L = \frac{1}{2}\dot{x}^T R^{-1} e^{Ct}\dot{x} + \frac{1}{2}\dot{x}^T R^{-1} e^{Ct} C x + \frac{1}{4} x^T R^{-1} e^{Ct} C^2 x - \frac{1}{2} x^T R^{-1} e^{Ct} K x \tag{19}$$

Proof: We again compute the necessary derivatives that appear in the Euler–Lagrange equation (8) to verify the result.

Noting that the first member on the right-hand sides of Eqs. (9) and (19) are identical, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(R^{-1}e^{Ct}\dot{x} + \frac{1}{2}R^{-1}e^{Ct}Cx\right)\\ &= R^{-1}e^{Ct}\ddot{x} + R^{-1}e^{Ct}C\dot{x} + \frac{1}{2}R^{-1}e^{Ct}C^2x + \frac{1}{2}R^{-1}e^{Ct}C\dot{x}\\ &= R^{-1}e^{Ct}\ddot{x} + \frac{3}{2}R^{-1}e^{Ct}C\dot{x} + \frac{1}{2}R^{-1}e^{Ct}C^2x\end{aligned} \tag{20}$$

Also, noting that $R^{-1} = R^{-T}$ because R is symmetric, we find that

$$\frac{\partial L}{\partial x} = \frac{1}{2}C^T e^{C^T t} R^{-1}\dot{x} + \frac{1}{4}R^{-1}e^{Ct}C^2x + \frac{1}{4}(C^T)^2 e^{C^T t}R^{-T}x - R^{-1}e^{Ct}Kx \tag{21}$$

where the last member on the right-hand side follows from Eq. (18). However, using Eqs. (14) and (15), we find that

$$(C^T)^2 e^{C^T t}R^{-T} = (R^{-1}CR)(R^{-1}CR)R^{-1}e^{Ct}RR^{-1} = R^{-1}e^{Ct}C^2$$

and, similarly, $C^T e^{C^T t}R^{-1} = R^{-1}e^{Ct}C$. The right-hand side of Eq. (21) then simplifies to

$$\frac{\partial L}{\partial x} = \frac{1}{2}R^{-1}e^{Ct}C\dot{x} + \frac{1}{2}R^{-1}e^{Ct}C^2x - R^{-1}e^{Ct}Kx \tag{22}$$

The result follows from relations (20) and (22). □

We observe that the presence of the Lagrangians given in Eqs. (9) and (19) indicates that the equation of motion given by Eq. (4) can be thought of as originating from a holonomically constrained mechanical system with x as the generalized coordinate.

When the matrices $\tilde{K}$ and $\tilde{C}$ are symmetric, then so are the commuting matrices K and C, and being symmetric they can be simultaneously diagonalized by an orthogonal transformation so that the matrix P is then orthogonal. Using this orthogonal matrix P that now simultaneously diagonalizes the symmetric matrices K and C [see Eqs. (7) and (8)], we find that $R := PP^T = PP^{-1} = I$. We then have the following result.

Corollary 1: When $K = K_s$ and $C = D$ where K_s and D are the symmetric stiffness and damping matrices, and K_s and D commute, then Eq. (4) describes the classically damped multi-degree-of-freedom dynamic system

$$\ddot{x} + D\dot{x} + K_s x = 0 \tag{23}$$

Two Lagrangians for this system are

$$L = \frac{1}{2}\dot{x}^T e^{Dt}\dot{x} - \frac{1}{2}x^T e^{Dt}K_s x \tag{24}$$

and

$$L = \frac{1}{2}\dot{x}^T e^{Dt}\dot{x} + \frac{1}{2}\dot{x}^T e^{Dt}Dx + \frac{1}{4}x^T e^{Dt}D^2x - \frac{1}{2}x^T e^{Dt}K_s x \tag{25}$$

Proof: Replacing $R = R^{-1}$ by I, K by K_s, and C by D, in Eqs. (9) and (19), the results follow. □

Thus, when the dynamic system (4) has matrices K and C that are symmetric and that commute, Eqs. (24) and (25) are two Lagrangians for the system. This shows that Results 1 and 2, which are applicable when the matrices K and C are general nonsymmetric matrices, reduce to those obtained earlier in [9], where they were both taken to be symmetric.

Corollary 2: When $C = D + G = 0$ and $K = K_s + N$ is a diagonalizable nonsymmetric matrix that includes circulatory effects, then Eq. (4) describes the multi-degree-of-freedom undamped dynamic system

$$\ddot{x} + Kx = 0 \tag{26}$$

A Lagrangian for this system is given by

$$L = \frac{1}{2}\dot{x}^T R^{-1}\dot{x} - \frac{1}{2}x^T R^{-1}Kx \tag{27}$$

Proof: Because $C = 0$, we get $e^{Ct} = I$. Using Eq. (9) [or Eq. (19)], the result follows. As before, the matrix $R = PP^T$, where P is defined in Eq. (7). □

Thus, for a system where there are no damping or gyroscopic terms, but which is circulatory, the Lagrangian is explicitly given by Eq. (27).

B. Integral of Motion for the Dynamic System

By an integral of motion, we mean here a function $E(x, \dot{x})$ that remains a constant when evaluated along any trajectory of the system. We now give an integral of motion of the dynamic system given in Eq. (4). The usual principle of conservation of energy (which provides an integral of motion) is applicable to undamped potential systems [see Eq. (36)]. The integral of motion obtained here is a generalization of this idea to systems that include dissipative and gyroscopic terms, as well as circulatory forces.

Result 3: An integral of motion of the dynamic system described in Eq. (4) where $K = K_s + N$ and $C = D + G$ are nonsymmetric matrices and K and C commute is

$$\begin{aligned}E &= \frac{1}{2}[E_{\dot{x}\dot{x}} + E_{\dot{x}x} + E_{xx}]\\ &= \frac{1}{2}[\dot{x}(t)^T R^{-1}e^{Ct}\dot{x}(t) + \dot{x}(t)^T R^{-1}e^{Ct}Cx(t) + x(t)^T R^{-1}e^{Ct}Kx(t)]\end{aligned} \tag{28}$$

where

$$E_{\dot{x}\dot{x}} = \dot{x}^T R^{-1}e^{Ct}\dot{x},\quad E_{\dot{x}x} = \dot{x}^T R^{-1}e^{Ct}Cx, \quad \text{and} \quad E_{xx} = x^T R^{-1}e^{Ct}Kx \tag{29}$$

Proof: We will show that the derivative of E with respect to time along the trajectories of the system described by Eq. (4) is zero by computing the derivative of each of the three terms in Eq. (28). Taking the first term, we get

$$\begin{aligned}\frac{\mathrm{d}E_{\dot{x}\dot{x}}}{\mathrm{d}t} &= \dot{x}^T R^{-1}e^{Ct}\ddot{x} + \ddot{x}^T R^{-1}e^{Ct}\dot{x} + \dot{x}^T R^{-1}e^{Ct}C\dot{x}\\ &= \dot{x}^T R^{-1}e^{Ct}\ddot{x} + \dot{x}^T e^{C^T t}R^{-T}\ddot{x} + \dot{x}^T R^{-1}e^{Ct}C\dot{x}\\ &= 2\dot{x}^T R^{-1}e^{Ct}\ddot{x} + \dot{x}^T R^{-1}e^{Ct}C\dot{x}\end{aligned} \tag{30}$$

where we have used Eq. (15) and the fact that R is symmetric in the third equality.

Similarly, we find that

$$\begin{aligned}\frac{\mathrm{d}E_{\dot{x}x}}{\mathrm{d}t} &= \ddot{x}^T R^{-1}e^{Ct}Cx + \dot{x}^T R^{-1}e^{Ct}C\dot{x} + \dot{x}^T R^{-1}e^{Ct}C^2x\\ &= x^T C^T e^{C^T t}R^{-T}\ddot{x} + \dot{x}^T R^{-1}e^{Ct}C\dot{x} + x^T(C^T)^2 e^{C^T t}R^{-T}\dot{x}\\ &= x^T R^{-1}e^{Ct}C\ddot{x} + \dot{x}^T R^{-1}e^{Ct}C\dot{x} + x^T R^{-1}C^2 e^{Ct}\dot{x}\end{aligned} \tag{31}$$

where, from Eqs. (14) and (15), the relations $C^T e^{C^T t}R^{-T} = R^{-1}Ce^{Ct} = R^{-1}e^{Ct}C$ and $(C^T)^2 e^{C^T t}R^{-T} = R^{-1}C^2 e^{Ct}$ are used in the last equality.

Last, we have

$$\begin{aligned}\frac{\mathrm{d}E_{xx}}{\mathrm{d}t} &= \dot{x}^T R^{-1} e^{Ct} K x + x^T R^{-1} e^{Ct} K \dot{x} + x^T R^{-1} e^{Ct} C K x \\ &= \dot{x}^T R^{-1} e^{Ct} K x + \dot{x}^T K^T e^{C^T t} R^{-T} x + x^T R^{-1} e^{Ct} C K x \\ &= \dot{x}^T R^{-1} e^{Ct} K x + \dot{x}^T R^{-1} K e^{Ct} x + x^T R^{-1} e^{Ct} C K x \\ &= 2\dot{x}^T R^{-1} e^{Ct} K x + x^T R^{-1} e^{Ct} C K x \end{aligned} \tag{32}$$

The last equality follows because the matrices K and C commute.

Along the trajectories of the dynamic system, we must have $\ddot{x} = -C\dot{x} - Kx$ and hence relations (30) and (31) along these trajectories become

$$\begin{aligned}\frac{\mathrm{d}E_{\dot{x}\dot{x}}}{\mathrm{d}t} &= 2\dot{x}^T R^{-1} e^{Ct}(-C\dot{x} - Kx) + \dot{x}^T R^{-1} e^{Ct} C \dot{x} \\ &= -\dot{x}^T R^{-1} e^{Ct} C \dot{x} - 2\dot{x}^T R^{-1} e^{Ct} K x \end{aligned} \tag{33}$$

and

$$\begin{aligned}\frac{\mathrm{d}E_{\dot{x}x}}{\mathrm{d}t} &= x^T R^{-1} e^{Ct} C(-C\dot{x} - Kx) + \dot{x}^T R^{-1} e^{Ct} C \dot{x} + x^T R^{-1} C^2 e^{Ct} \dot{x} \\ &= -x^T R^{-1} e^{Ct} C K x + \dot{x}^T R^{-1} e^{Ct} C \dot{x} \end{aligned} \tag{34}$$

On adding the right-hand sides of Eqs. (32–34), we find that their sum equals zero, and the result follows. □

Corollary 3: When $K = K_s$ and $C = D$, where K_s and D are symmetric matrices and K_s and D commute, an invariant of the motion of the multi-degree-of-freedom classically damped dynamic system given by Eq. (23) is

$$E = \frac{1}{2}(\dot{x}^T e^{Dt} \dot{x} + \dot{x}^T e^{Dt} D x + x^T e^{Dt} K_s x) \tag{35}$$

Proof: As in Corollary 1, on replacing $R = R^{-1}$ by I, K by K_s, and C by D, in the second equality in Eq. (28), the result follows. □

As before, this shows that the invariant given in Result 3 is a generalization to nonsymmetric matrices of the corresponding result obtained in [9] for commuting symmetric matrices.

Remark 1: When $K = K_s$, where K_s is a symmetric matrix, and $C = D = 0$, then Eq. (23) describes the multi-degree-of-freedom undamped dynamic system

$$\ddot{x} + K_s x = 0 \tag{36}$$

Such a system is also referred to as a potential system. An invariant of motion for the system can be obtained by setting $D = 0$ in Eq. (35) to give

$$E = \frac{1}{2}\dot{x}^T \dot{x} + \frac{1}{2} x^T K_s x \tag{37}$$

The first member on the right in the preceding equation is the kinetic energy of the system described by Eq. (36), and the second member is its potential energy. The fact that E is invariant along the trajectories of the dynamic system then simply expresses the principle of conservation of energy.

Hence, in a sense, one might think of Eq. (35) [from which Eq. (37) is obtained by setting $D = 0$] as an extension of the conservation-of-energy invariant to the situation when the system is classically damped.

Corollary 4: For the system in which $C = 0$ and the diagonalizable matrix $K = K_s + N$ is nonsymmetric as described by Eq. (26), an invariant of the motion of the system is obtained from Eq. (28) by simply setting $C = 0$. One then obtains the integral of motion as

$$E = \frac{1}{2}[\dot{x}^T R^{-1} \dot{x} + x^T R^{-1} K x] \tag{38}$$

This invariant can be similarly thought of as an extension of the conservation-of-energy invariant given in Eq. (37) to dynamic systems described by Eq. (26) in which the stiffness matrix K is nonsymmetric.

Remark 2: Along the trajectories of the motion of the dynamic system described by Eq. (4), the integral of motion E given in Eq. (28) is conserved (i.e., remains a constant). When the system has no damping and its stiffness matrix is symmetric ($K = K_s$), the dynamic system's description reduces to that given in Eq. (36) and the expression for E, as seen in in Eq. (37), reduces to simply the sum of the kinetic and potential energy of the system, which is conserved. One can then interpret the expression given in Eq. (28) as a conservation law that extends the conservation-of-energy invariant to the more general dynamic system described by Eq. (4), which has a general (nonsymmetric) damping matrix C and which has a general (nonsymmetric) stiffness matrix K.

III. Conditions on Nonsymmetric Matrices *K* and *C*

The dynamic system of Eq. (4) can be alternatively written as

$$\ddot{x} + \underbrace{(D + G)}_{C}\dot{x} + \underbrace{(K_s + N)}_{K}x = 0 \tag{39}$$

In structural dynamics, D is interpreted as the symmetric damping matrix, G represents the gyroscopic skew-symmetric matrix, K_s is the symmetric structural stiffness matrix that comes from a potential (and is usually assumed to be positive definite), and the matrix N accounts for circulatory forces.

In this section, we investigate the general dynamic system described Eq. (39) and obtain sufficient conditions that the matrices K_s, D, N, and G need to satisfy so that the Lagrangians and the invariant of motion obtained in Sec. II are applicable.

Recall that the nonsymmetric matrices K and C in Sec. II are required to satisfy the following two conditions [see Eq. (4)]:

$$\begin{aligned}&1)\ K \text{ and } C \text{ are each diagonalizable, and} \\ &2)\ K \text{ and } C \text{ commute with each other}\end{aligned} \tag{40}$$

so that they can be simultaneously diagonalized. Furthermore, if K and C can be simultaneously diagonalized, then they are obviously each diagonalizable; also, there exists a matrix P such that $P^{-1}KP = \Lambda$ and $P^{-1}CP = \Delta$, where Λ and Δ are diagonal matrices. Thus, $\Lambda\Delta = \Delta\Lambda$, from which it follows that the matrices K and C commute. Thus, the matrices K and C are simultaneously diagonalizable if and only if conditions 1 and 2 [see (40)] are satisfied.

Depending on the nature of the physical system being modeled, in practical applications some of the matrices K_s, D, N, and G may be reasonably assumed to be zero. For example, when the physical system is modeled as a classically damped system, then 1) $N = G = 0$, so that $K = K_s$ and $C = D$, and 2) the matrices K_s and D commute. We note that then both the aforementioned conditions are satisfied, the first condition (40) being satisfied because the two matrices are symmetric, and therefore diagonalizable. Lagrangians and invariants for such classically damped linear multi-degree-of-freedom systems have been obtained in [9] and can also be obtained from the results given in Sec. II by particularizing them to the case when K and C are symmetric matrices that commute (see Corollaries 1 and 3 in Sec. II).

There are many structural and mechanical systems in which there are no circulatory terms ($N = 0$), and so we begin by considering Eq. (39) when $K = K_s \neq 0$, a symmetric matrix.

Lemma 1: If the $n \times n$ symmetric matrix K_s has distinct eigenvalues and the matrices K_s and U commute, then the matrix U must be a symmetric matrix and therefore diagonalizable.

Proof: Because K_s is symmetric, it can be diagonalized by an orthogonal matrix T, and we have

$$K_s = T\Lambda T^T \tag{41}$$

where Λ is a diagonal matrix along whose diagonal are the distinct eigenvalues λ_i of K_s. Because K_s and U commute, we have $K_s U = U K_s$, which can be rewritten using Eq. (41) as

$T\Lambda T^T U = UT\Lambda T^T$. Premultiplying both sides of this equation by T^T and postmultiplying them by T, yields the relation

$$\Lambda Q = Q\Lambda \tag{42}$$

where $Q = T^T UT$. But relation (42) implies that $(\lambda_i - \lambda_j)Q(i, j) = 0$, $\forall\ i, j$, where $Q(i, j)$ is the (i, j)th element of the matrix Q. Because the eigenvalues of K_s are distinct, $\lambda_i \neq \lambda_j$ for $i \neq j$, and therefore $Q(i, j) = 0$ for $i \neq j$. Hence, the matrix Q is diagonal, and the matrix $U = TQT^T$ is therefore a symmetric matrix. □

Result 4: If, for the multi-degree-of-freedom linear dynamic system

$$\ddot{x} + C\dot{x} + K_s x = 0 \tag{43}$$

in which 1) K_s is symmetric and commutes with C, and 2) K_s has distinct eigenvalues, the matrix $C = D + G$ must be a symmetric matrix, implying that $G = 0$, and therefore $C = D$.

Proof: By the previous Lemma, the result follows directly. □

Thus, if in a structural or mechanical system, K is a symmetric matrix and has distinct eigenvalues, and it commutes with the matrix C, then this matrix C must be symmetric. Furthermore, because C is now symmetric, it is diagonalizable, and because K and C commute, they can be simultaneously diagonalized by an orthogonal matrix T. This is indeed the case of classically damped multi-degree-of-freedom systems, which was referred to earlier.

Lemma 2: If the $n \times n$ symmetric matrix K_s has distinct eigenvalues, the only skew-symmetric matrix U that commutes with K_s is the zero matrix.

Proof: Because K_s and U commute and K_s is symmetric with distinct eigenvalues, from the proof of Lemma 1, the matrix $U = TQT^T$ is a symmetric matrix, because Q is diagonal. If U is also skew symmetric, then it can only be the zero matrix. □

We have shown that, if K_s has distinct eigenvalues and commutes with a skew-symmetric matrix, then that skew-symmetric matrix is the zero matrix. In other words, if a skew-symmetric matrix is nonzero, it cannot commute with a symmetric matrix K_s that has distinct eigenvalues. If a nonzero skew-symmetric matrix commutes with a symmetric matrix K_s, then K_s has at least one repeated eigenvalue. If K_s has at least one repeated eigenvalue, it will be shown later that there are an uncountably infinite number of skew-symmetric matrices with which it will commute.

Result 5: If $D = N = 0$, K_s is symmetric with distinct eigenvalues and the gyroscopic matrix $G \neq 0$, so that the equation of motion of a multi-degree-of freedom linear dynamic system is

$$\ddot{x} + G\dot{x} + K_s x = 0 \tag{44}$$

then K_s and G cannot commute and therefore cannot be simultaneously diagonalized.

Proof: By the previous Lemma, the result follows directly, because if $G \neq 0$, the matrices K_s and G cannot commute. □

Thus, the multi-degree-of-freedom linear dynamic system described by Eq. (4) in which the matrix K is symmetric and has distinct eigenvalues cannot commute with a nonzero matrix C that is purely gyroscopic (skew symmetric) in nature. The results in Sec. II are hence not applicable to systems described by Eq. (44).

In what follows, we will be dealing with block diagonal matrices, and we now define the concept of two $n \times n$ block diagonal matrices having the same diagonal structure in the following remark.

Remark 3: Consider an $n \times n$ block diagonal matrix

$$A = \mathrm{diag}(A_1, A_2, \ldots, A_k) \tag{45}$$

in which the sth (square) diagonal block A_s has dimensions $i_s \times i_s$ with $i_1 \geq i_2, \ldots, \geq i_k$, and another $n \times n$ block diagonal matrix

$$B = \mathrm{diag}(B_1, B_2, \ldots, B_r) \tag{46}$$

for which the pth (square) diagonal block B_p has dimensions $j_p \times j_p$ with $j_1 \geq j_2, \ldots, \geq j_r$.

We shall say that matrices A and B have the same block diagonal structure if

$$k = r, \quad \text{and} \quad i_s = j_s, \qquad s = 1, \ldots, k \tag{47}$$

That is, if 1) matrices A and B have the same number of blocks along their diagonals ($k = r$), and 2) the corresponding (square) diagonal blocks of the two matrices have the same dimensions as we go down their respective diagonals (from top-left to bottom-right), then we say that A and B have the same block diagonal structure. We will need to use this concept as we go along.

Lemma 3: Let the $n \times n$ symmetric matrix K_s have repeated eigenvalues, $k < n$ of which are distinct. Let the orthogonal transformation that diagonalizes it be T, so that $\Lambda = T^T K_s T$, where Λ is a block diagonal matrix $\mathrm{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k)$, and the dimension of the (square) identity matrix I_j equals the multiplicity of the eigenvalue λ_j.

1) The matrices K_s and U commute if and only if the matrices Λ and $Q = T^T UT$ have the same block diagonal structure.

2) If U is diagonalizable, so is each diagonal block Q_j of the matrix Q.

3) In particular, U is symmetric (and therefore diagonalizable) if and only if each diagonal block Q_j is symmetric; U is skew symmetric (and again therefore diagonalizable) if and only if each diagonal block Q_j is skew symmetric.

Proof:

1) Let K_s have $k < n$ distinct eigenvalues. Without loss of generality, we can assume that any repeated eigenvalues of K_s lie continuously along the diagonal of Λ and are arranged so that

$$\Lambda = \begin{bmatrix} \lambda_1 I_1 & & & \\ & \lambda_2 I_2 & & \\ & & \ddots & \\ & & & \lambda_k I_k \end{bmatrix}, \qquad i_1 \geq i_2, \ldots, \geq i_k \tag{48}$$

where I_j, $j = 1, \ldots, k$ denotes the $i_j \times i_j$ identity matrix, and i_j is the multiplicity of the repeated eigenvalue λ_j.

Assume that K_s and U commute, proceeding as in the proof of Lemma 1, the matrix $Q = T^T UT$ must be a block diagonal matrix that has the form

$$Q = \begin{bmatrix} Q_1 & & & \\ & Q_2 & & \\ & & \ddots & \\ & & & Q_k \end{bmatrix} \tag{49}$$

where Q_j is an arbitrary $i_j \times i_j$ matrix. The corresponding square blocks along the diagonals of the matrices Λ and Q have the same dimensions, and hence the matrices Λ and Q have the same block diagonal structure. We note in passing that, because $U = TQT^T$, it need not be symmetric.

Conversely, if Λ and Q have the same block diagonal structure shown in Eqs. (48) and (49), respectively, then they clearly commute, because on carrying out their multiplication, we find that $\Lambda Q = Q\Lambda$ or $(T^T K_s T)(T^T UT) = (T^T UT)(T^T K_s T)$, from which it follows that the matrices K and U commute, because the matrix T is orthogonal.

2) If U is diagonalizable, so that $U = Y\Xi Y^{-1}$, where Ξ is a diagonal matrix, then $Q = T^T UT = T^T Y\Xi Y^{-1} T$, and so Q is diagonalized by the matrix $T^T Y$. However, Q is a block diagonal matrix and it is diagonalizable if and only if every diagonal block Q_j of Q is diagonalizable.

3) If U is symmetric, then $Q = T^T UT$ is symmetric, and if Q is symmetric, $U = TQT^T$ is symmetric, similarly, for when U is skew symmetric. □

Lemma 4: Let the $n \times n$ symmetric matrix K_s have $k < n$ distinct eigenvalues with $\Lambda = T^T K_s T$. The matrix Λ is the block diagonal matrix $\mathrm{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k)$, where the dimension of the (square) identity matrix I_j equals the multiplicity of the eigenvalue λ_j of K_s. If K_s commutes with a diagonalizable matrix U, then every diagonal block Q_j of the matrix $Q = T^T UT$ [see Eq. (49)] is diagonalizable so that there exist matrices W_j, $j = 1, \ldots, k$, such that $Q_j = W_j \Xi_j W_j^{-1}$.

The matrices K_s and U can be simultaneously diagonalized by the matrix TW, where the block diagonal matrix $W = \text{diag}(W_1, W_2, \ldots, W_k)$.

Proof: In Lemma 3, we have already shown that Λ and Q have the same block diagonal structure and that each diagonal block Q_j of Q is diagonalizable. Let W_j be the similarity transformation that diagonalizes the $i_j \times i_j$ block Q_j so that $Q_j = W_j \Xi_j W_j^{-1}$, where Ξ_j is a diagonal $i_j \times i_j$ matrix containing the eigenvalues of Q_j. Then the block diagonal matrix $W = \text{diag}(W_1, W_2, \ldots, W_k)$ is such that $Q = W \Xi W^{-1}$, where $\Xi = \text{diag}(\Xi_1, \Xi_2, \ldots, \Xi_k)$ and $\Xi_j, j = 1, \ldots, k$ are diagonal matrices. are diagonal matrices. The matrix U can then be expressed as $U = TQT^T = TW\Xi W^{-1}T^T$. This shows that the matrix Y in Lemma 3 is simply $Y = TW$. Then Y is the similarity transformation that renders both K and U diagonal, because

$$(TW)^{-1}KTW = W^{-1}(T^TKT)W = W^{-1}\Lambda W = \Lambda \tag{50}$$

Note that the matrices Λ and W are block diagonal matrices. The last equality follows because corresponding to each $i_j \times i_j$ diagonal block $\lambda_j I_j$ of the matrix Λ there is an $i_j \times i_j$ diagonal block W_j of the matrix W such that $W_j^{-1}\lambda_j I_j W_j = \lambda_j W_j^{-1} W_j = \lambda_j I_j$. Also,

$$(TW)^{-1}UTW = W^{-1}(T^TUT)W = W^{-1}QW = \Xi$$

which is a diagonal matrix. □

Result 6: If 1) $D = N = 0$ in Eq. (39) so that the equation of motion of a multi-degree-of freedom linear dynamic system is

$$\ddot{x} + G\dot{x} + K_s x = 0 \tag{51}$$

where $G \neq 0$ is a gyroscopic (skew-symmetric) matrix, and 2) K_s and G commute, then K_s must have repeated eigenvalues, $k < n$ of which are distinct, and the matrices $T^TK_sT = \Lambda = \text{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k)$ and $Q = T^TGT$ must have the same block diagonal structure [see Eqs. (48) and (49)], and every diagonal block Q_j of Q must be skew symmetric. Furthermore, because the matrices K_s and G are each diagonalizable and commute, they can be simultaneously diagonalized, thus making their sum $K_s + G$ also diagonalizable.

Proof: By Lemma 2, the matrix K_s must have repeated roots or else G would have to be zero. The orthogonal matrix T diagonalizes K_s. That the matrices Λ and G are required to have the same block diagonal structure is proved in Lemma 3. Because Q is skew symmetric, it is a normal matrix ($Q^TQ = QQ^T$) and can be diagonalized by a unitary matrix. In fact, each diagonal block Q_j of Q is skew symmetric, and so is a normal matrix, and as shown in Lemma 4, the matrices K_s and G are simultaneously diagonalized by the matrix TW. Hence, the matrix $K_s + G$ is diagonalizable. □

Remark 4: Consider any two $n \times n$ matrices $\Lambda = \text{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k)$ (with $k < n$ distinct eigenvalues) and Q that have the same block diagonal structure shown in Eqs. (48) and (49), with Q skew symmetric. Then the matrices

$$K_s = T\Lambda T^T \quad \text{and} \quad G = TQT^T \tag{52}$$

are always diagonalizable and simultaneously diagonalizable for all orthogonal matrices T.

Proof: This follows directly from Lemmas 3 and 4 and Result 6. □

Numerical Example 1: At this point, it might be useful to provide a numerical example to clarify some of the results obtained so far. Consider the 6×6 block diagonal matrix

$$\begin{aligned}\Lambda &= \text{diag}(200I_1, 300I_2, 100I_3)\\ &= \text{diag}(200, 200, 200, 300, 300, 100)\end{aligned} \tag{53}$$

where $i_1 = 3, i_2 = 2$, and $i_3 = 1$, as described in Remark 3. Consider also the skew-symmetric matrix

$$Q = \begin{bmatrix} 0 & -3 & 2 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} := \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} \tag{54}$$

as in Eqs. (48) and (49). The matrix Λ has repeated eigenvalues, three of which are distinct. The diagonal blocks Q_1, Q_2, and Q_3 of the matrix Q are skew symmetric and have dimensions 3×3, 2×2, and 1×1, respectively, with the same block diagonal structure as Λ. From Remark 4, we know that, given any orthogonal matrix T, the matrices $K = T\Lambda T^T$ and $G = TQT^T$ will be simultaneously diagonalizable.

Consider, for example, any unit vector $x = [x_1, \tilde{x}]^T$, $x_1 \neq 1$. Then the matrix

$$T = \begin{bmatrix} x_1 & \tilde{x} \\ \tilde{x}^T & I - \alpha\tilde{x}^T\tilde{x} \end{bmatrix}, \qquad \alpha = \frac{1}{1 - x_1} \tag{55}$$

is an orthogonal matrix. Taking

$$x_1 = a = 1/\sqrt{6} \quad \text{and} \quad \tilde{x} = [-a, a, a, -a, a] \tag{56}$$

so that $\|x\| = 1$, we obtain, using Eq. (55), the matrices

$$K_s = T\Lambda T^T = \begin{bmatrix} 216.6667 & 11.4983 & -11.4983 & 29.3265 & -29.3265 & -52.3231 \\ 11.4983 & 207.9327 & -7.9327 & 20.2323 & -20.2323 & -36.0976 \\ -11.4983 & -7.9327 & 207.9327 & -20.2323 & 20.2323 & 36.0976 \\ 29.3265 & 20.2323 & -20.2323 & 251.6027 & 48.3973 & 7.9327 \\ -29.3265 & -20.2323 & 20.2323 & 48.3973 & 251.6027 & -7.9327 \\ -52.3231 & -36.0976 & 36.0976 & 7.9327 & -7.9327 & 164.2626 \end{bmatrix} \tag{57}$$

and

$$C = G = TQT^T = \begin{bmatrix} 0.0000 & -0.2247 & -1.0000 & -0.5918 & 2.2247 & -1.4082 \\ 0.2247 & -0.0000 & 0.8449 & 1.5064 & -0.3798 & 0.9431 \\ 1.0000 & -0.8449 & 0.0000 & -1.6614 & 0.5348 & -1.0981 \\ 0.5918 & -1.5064 & 1.6614 & 0.0000 & 0.8734 & 0.5633 \\ -2.2247 & 0.3798 & -0.5348 & -0.8734 & 0.0000 & 0.5633 \\ 1.4082 & -0.9431 & 1.0981 & -0.5633 & -0.5633 & 0.0000 \end{bmatrix} \tag{58}$$

The dynamic system described by Eq. (51) in Result 6 has matrices K_s and G that are diagonalizable and can be simultaneously diagonalized, and hence the expressions for the Lagrangians and the invariant given in Sec. II are applicable to such systems.

that must commute. Any other orthogonal matrix T [for example, the one obtained by a suitably different unit vector x instead of that given in Eq. (56)], and any other skew symmetric Q instead of that in Eq. (54), which has the same block diagonal structure as the matrix Λ

given in Eq. (53), will yield matrices $K_s = T\Lambda T^T$ and $G = TQT^T$ that will always commute. The matrices C and K_s are simultaneously diagonalizable. The results in Sec. II will then be applicable to systems described in Result 6 for all such matrices.

Remark 5: Using Eq. (39) and considering the dynamic system

$$\ddot{x} + D\dot{x} + Nx = 0 \tag{59}$$

where D is symmetric and N is skew symmetric, Results 4–6 and Remark 4 would correspondingly apply to such systems, by simply replacing K_s by D, and G by N in these results.

Many real-life dynamic systems have gyroscopic contributions to their motion. Gyroscopic matrices also arise when Routhian elimination of cyclic coordinates is done. Such systems are described by

$$\ddot{x} + (D + G)\dot{x} + K_s x = 0, \qquad D, G, K_s \neq 0 \tag{60}$$

in which the circulatory term has again been excluded.

Lemma 5: The symmetric $n \times n$ matrix K_s commutes with the matrix $C = D + G$ if and only if K_s commutes with both D and G.

Proof: If K_s commutes with both D and G, then $K_s D = DK_s$ and $K_s G = GK_s$ so that

$$K_s D + K_s G = DK_s + GK_s \tag{61}$$

which yields

$$K_s\underbrace{(D + G)}_{C} = \underbrace{(D + G)}_{C}K_s \tag{62}$$

Hence, K_s commutes with C.

To prove the converse, assume that K_s and C commute. Then Eq. (62) is true. Taking its transpose, one obtains

$$K_s C^T = C^T K_s \tag{63}$$

Adding Eqs. (62) and (63) together gets $K_s(C + C^T) = (C + C^T)K_s$ or $2K_s D = 2DK_s$, hence $K_s D = DK_s$. Substituting this in Eq. (62) gives $K_s G = GK_s$. Hence, if K_s and C commute, then K_s commutes with both D and with G. □

Result 7: Consider the $n \times n$ matrix $K = K_s \neq 0$, with $N = 0$, so that there are no circulatory terms, and $C = D + G$, with $D, G \neq 0$, as described in Eq. (60). For K and C to each be diagonalizable and for them to commute, it is necessary and sufficient that 1) D and K_s must commute and C must be diagonalizable, and 2) the diagonal matrix $\Lambda = T^T K_s T = \mathrm{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k)$, $k < n$, must have the same block diagonal structure as $\Gamma = T^T GT$. The dimension of the (square) identity matrix I_j equals the multiplicity of the eigenvalue λ_j.

Proof: Assume that C is diagonalizable, and C and K commute. K_s is always diagonalizable because it is symmetric. For $C = D + G$ and K_s to commute by Lemma 5, this implies that K_s and D commute (so that the orthogonal matrix T simultaneously diagonalizes both D and K_s) and K_s and G commute. The latter implies that K_s must have repeated eigenvalues (else G would have to be the zero matrix, which it is not), and also Λ and Γ must have the same block diagonal structure. Furthermore, $C = D + G$ must be diagonalizable.

Assume now that points 1 and 2 in Result 7 are true. Then point 2 implies that K_s and G commute by Lemma 3. Also, because by point 1, K_s and D commute, from Lemma 5 we see that K_s and C commute. Because K_s is symmetric, it is diagonalizable, and by point 1, C is diagonalizable. Hence, K_s and C are diagonalizable and they commute. □

Remark 6: A sufficient condition for $C = D + G$ to be diagonalizable in Result 7 is that D and $G \neq 0$ be simultaneously diagonalizable. For that to occur, D and G must be diagonalizable (which they are) and they must commute. Because G is a nonzero skew-symmetric matrix, commutation would imply that D must have at least one repeated eigenvalue so that $\Delta = T^T DT = \mathrm{diag}(\mu_1 I_1, \mu_2 I_2, \ldots, \mu_k I_k)$, $k < n$, and $\Gamma = T^T GT$ have the same block diagonal structure. The dimension of the (square) identity matrix I_j equals the multiplicity of the eigenvalue μ_j of the matrix D. Because G must commute with K_s also and in view of point 2 of Result 7, this would mean that Δ, Λ, and Γ would have the same block diagonal structure.

Simultaneous diagonalizability of D and G is clearly not a necessary condition for C to be diagonalizable. For example,

$$D = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}$$

for which $C = D + G$ is diagonalizable, though D and G do not commute. D here has distinct eigenvalues.

Remark 7: The symmetric $n \times n$ matrix D commutes with the matrix $K = K_s + N$ if and only if D commutes with both K_s and with N.

Proof: The proof follows along the same lines as that given for Lemma 5. Replace K_s in that proof by D and vice versa, replace G by N, and K by C. □

For physical systems modeled by the multi-degree-of-freedom dynamic system

$$\ddot{x} + D\dot{x} + (K_s + N)x = 0, \qquad D, K_s, N \neq 0 \tag{64}$$

in which the gyroscopic skew-symmetric matrix is absent but circulatory effects are included, Remark 7 and Result 7 lead to the following conclusions.

Result 8: Consider the $n \times n$ matrix $C = D \neq 0$, so that $G = 0$, and $K = K_s + N$, with $K_s, N \neq 0$ as described in Eq. (64). For K and C to each be diagonalizable and for them to commute, it is necessary and sufficient that 1) D and K_s must commute, and K must be diagonalizable, and 2) the diagonal matrix $\Delta = T^T DT = \mathrm{diag}(\mu_1 I_1, \mu_2 I_2, \ldots, \mu_k I_k)$, $k < n$, must have the same block diagonal structure as $\mathrm{X} = T^T NT$. The dimension of the (square) identity matrix I_j equals the multiplicity of the eigenvalue μ_j.

Proof: The proof is along lines similar to that of Result 7. The roles of D and K_s are interchanged, and G is replaced by N. □

Remark 8: Along the same lines as Remark 6, a sufficient condition for K to be diagonalizable in Result 8 is that K_s and N commute. Because $N \neq 0$, this would require that $\Lambda = T^T K_s T = \mathrm{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k)$, $k < n$, and $\mathrm{X} = T^T NT$ have the same block diagonal structure. The dimension of the (square) identity matrix I_j equals the multiplicity of the eigenvalue λ_j. In view of item 2 in Result 8, this would mean that Δ, Λ, and X have the same block diagonal structure.

Last, we consider the general dynamic system

$$\ddot{x} + \underbrace{(D + G)}_{C}\dot{x} + \underbrace{(K_s + N)}_{K}x = 0 \tag{65}$$

where $C = D + G$ and $K = K_s + N$, in which the four matrices D, G, K_s, and N are all nonzero matrices. We then have the following result.

Lemma 6: If either 1) K_s commutes with C (see Lemma 5) or 2) D commutes with K (see Remark 7), then $K = K_s + N$ and $C = D + G$ commute, if and only if

$$\begin{aligned} DK_s &= K_s D, \qquad DN = ND, \\ GN &= NG, \quad \text{and} \quad GK_s = K_s G \end{aligned} \tag{66}$$

that is, D commutes with K_s and N, and G commutes with K_s and N.

Proof: When the relations in Eq. (66) are true, we have

$$K_s D + K_s G + ND + NG = DK_s + GK_s + DN + GN$$

from which we get

$$\underbrace{(K_s + N)}_{K}\underbrace{(D + G)}_{C} = \underbrace{(D + G)}_{C}\underbrace{(K_s + N)}_{K} \tag{67}$$

Thus, the relations given in Eq. (66) imply that K and C commute.

To prove the converse, assume that K and C commute, so that relation (67) is true. Take the transpose of Eq. (67), which yields

$$K_sD - K_sG - ND + NG = DK_s - GK_s - DN + GN \tag{68}$$

Adding relations (67) and (68) gives

$$K_sD + NG = DK_s + GN \tag{69}$$

and subtracting Eq. (68) from Eq. (67) gives

$$K_sG + ND = GK_s + DN \tag{70}$$

If, further, K_s commutes with C as in item 1, then $K_sD = DK_s$ and $K_sG = GK_s$ (see Lemma 5). Hence, from Eq. (69) it follows that $NG = GN$ and from Eq. (70) that $ND = DN$.

Similarly, if D commutes with K, it follows from Remark 7 that $K_sD = DK_s$ and $ND = DN$. Upon using Eqs. (69) and (70), we again find that $NG = GN$ and $GK_s = K_sG$, respectively, hence the result. □

Result 9: A sufficient condition for the matrices $C = D + G$ and $K = K_s + N$ (with the $n \times n$ matrices D, G, K_s, $N \neq 0$) in the general dynamic system given in Eq. (65) to be diagonalizable and for them to commute is as follows:

1) The $n \times n$ symmetric matrices D and K_s have repeated eigenvalues, $k < n$ of which are distinct.

2) The diagonal matrices $T^TK_sT = \Lambda = \mathrm{diag}(\lambda_1 I_1, \lambda_2 I_2, \ldots, \lambda_k I_k)$ and $\Delta = T^TDT = \mathrm{diag}(\mu_1 I_1, \mu_2 I_2, \ldots, \mu_k I_k)$ have the same block diagonal structure as each of the matrices $\Gamma = T^TGT$ and $\mathrm{X} = T^TNT$. The dimension of the (square) identity matrix I_j equals the multiplicity of the eigenvalue λ_j of the matrix K_s.

3) The matrix product $\Gamma\mathrm{X}$ is symmetric.

Proof: We need to show that the aforementioned three conditions are sufficient to ensure that the matrices C and K are each diagonalizable and that they commute. That is,

$$(D + G)(K_s + N) = (K_s + N)(D + G) \tag{71}$$

Expanding this yields

$$DK_s + DN + GK_s + GN = K_sD + ND + K_sG + NG \tag{72}$$

A sufficient condition for Eq. (72) to be true is that the following equalities hold:

$$DK_s = K_sD, DN = ND, GK_s = K_sG, \quad \text{and} \quad GN = NG \tag{73}$$

If K_s and N commute, then their sum can be simultaneously diagonalized, and hence K is rendered diagonalizable. For this to happen, K_s must have repeated eigenvalues and Λ and X must have the same block diagonal structure. In a similar manner, for C to be diagonalizable, it is sufficient that D has repeated eigenvalues and Δ has the same block diagonal structure as Γ.

Furthermore, as mentioned earlier, for K and C to commute it is sufficient that each of the products K_sD, K_sG, DN, and GN, commute. The first of these products ensures that an orthogonal matrix T simultaneously diagonalizes K_s and D; the second ensures that Λ and Γ have the same block diagonal structure and K_s has repeated eigenvalues; and the third ensures that Δ and X have the same block diagonal structure and D has repeated eigenvalues.

Last, two skew-symmetric matrices commute if and only if their product is a symmetric matrix [21]. Furthermore, the skew-symmetric matrices N and G commute if and only if the skew-symmetric matrices Γ and X commute. This is because if $NG = GN$, then $(TXT^T)(T\Gamma T^T) = (T\Gamma T^T)(TXT^T)$, from which it follows that Γ and X commute, because T is orthogonal. It can also be similarly proved that, if Γ and X commute, then so do N and G. Hence, for N and G to commute a necessary and sufficient condition is that the matrix $\Gamma\mathrm{X}$ must be symmetric. These conditions are easily seen to be tantamount to the three given. □

Under the conditions given in Result 9, the multi-degree-of-freedom dynamic system described by Eq. (65) has the Lagrangians that are given in Sec. II and also the invariant of motion therein.

Remark 9: As proved in Lemma 6, if either 1) K_s commutes with C, or if 2) D commutes with K, then K and C commute if and only if all the relations given in Eq. (73) are satisfied.

Numerical Example 2: Consider the matrices

$$\Lambda = \mathrm{diag}(200, 200, 200, 300, 300, 100) \quad \text{and}$$
$$\Delta = \mathrm{diag}(1, 1, 1, 2, 2, 3) \tag{74}$$

Note that Λ and Δ are both diagonal matrices with the same block diagonal structure.

Using the orthogonal matrix given in Eq. (55) by taking the unit vector

$$x = [x_1, \tilde{x}]^T \tag{75}$$

with $x_1 = a = 1/\sqrt{6}$ and $\tilde{x} = [-a, a, a, -a, a]$ as before, we obtain the matrices K_s given in Eq. (57) and

$$D = T\Delta T^T = \begin{bmatrix} 1.6667 & 0.4599 & -0.4599 & -0.0517 & 0.0517 & 0.3566 \\ 0.4599 & 1.3173 & -0.3173 & 0.0357 & 0.0357 & -0.2460 \\ -0.4599 & -0.3173 & 1.3173 & 0.0357 & -0.0357 & 0.2460 \\ -0.0517 & -0.0357 & 0.0357 & 1.7540 & 0.2460 & -0.5276 \\ 0.0517 & 0.0357 & -0.0357 & 0.2460 & 1.7540 & 0.5276 \\ 0.3566 & 0.2460 & -0.2460 & -0.5276 & 0.5276 & 2.1907 \end{bmatrix} \tag{76}$$

The matrices

$$X = \begin{bmatrix} 0 & 20 & 30 & 0 & 0 & 0 \\ -20 & 0 & -10 & 0 & 0 & 0 \\ -30 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -40 & 0 \\ 0 & 0 & 0 & 40 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and}$$
$$\Gamma = \begin{bmatrix} 0 & -1.0000 & -1.5000 & 0 & 0 & 0 \\ 1.0000 & 0 & 0.5000 & 0 & 0 & 0 \\ 1.5000 & -0.5000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.2857 & 0 \\ 0 & 0 & 0 & -2.2857 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{77}$$

have the same block diagonal structure as Λ (and Δ) and yield

$$N = TXT^T = \begin{bmatrix} 0.0000 & 15.0639 & 13.5134 & -19.1464 & -13.5134 & -2.8165 \\ -15.0639 & 0.0000 & -24.7794 & -16.6145 & -5.9175 & -5.3485 \\ -13.5134 & 24.7794 & 0.0000 & 1.8350 & 20.6969 & -9.4310 \\ 19.1464 & 16.6145 & -1.8350 & 0.0000 & -17.4680 & -11.2660 \\ 13.5134 & 5.9175 & -20.6969 & 17.4680 & 0.0000 & -11.2660 \\ 2.8165 & 5.3485 & 9.4310 & 11.2660 & 11.2660 & 0.0000 \end{bmatrix} \tag{78}$$

and

$$G = T\Gamma T^T = \begin{bmatrix} 0.0000 & -0.7532 & -0.6757 & 1.0740 & 0.7923 & 0.1408 \\ 0.7532 & 0.0000 & 1.2390 & 0.9112 & 0.3763 & 0.2674 \\ 0.6757 & -1.2390 & 0.0000 & -0.1722 & -1.1153 & 0.4715 \\ -1.0740 & -0.9112 & 0.1722 & 0.0000 & 0.9982 & 0.6438 \\ -0.7923 & -0.3763 & 1.1153 & -0.9982 & 0.0000 & 0.6438 \\ -0.1408 & -0.2674 & -0.4715 & -0.6438 & -0.6438 & 0.0000 \end{bmatrix} \tag{79}$$

Note that the matrices Γ and X commute because their product is a symmetric matrix, and therefore so do the matrices N and G. The matrices K_s, D, N, and G, given in Eqs. (57), (76), (78), and (79), respectively, satisfy the conditions given in Result 9. Therefore, the dynamic system described by Eq. (65) with these matrices can be described by the Lagrangians and the invariant of motion given in Sec. II.

Remark 10: Result 9 shows the following. Consider the matrices Λ, Δ, X, and Γ, which all have the same block diagonal structure with ΓX symmetric. The diagonal matrices Λ and Δ have repeated eigenvalues. Then for all orthogonal matrices T, the matrices $K_s = T\Lambda T^T$, $D = T\Delta T^T$, $G = T\Gamma T^T$, and $N = TXT^T$ with ΓX symmetric will be such that the dynamic system described by Eq. (65) with the matrices $C = D + G$ and $K = K_s + N$ possesses Lagrangians given by Eqs. (9) and (19) and an invariant of motion given in Eq. (28). For example, we could have used any other unit vector x in Eq. (75) (and therefore a different orthogonal matrix T), for which the first component $x_1 \neq 1$. Similarly, we could have used any skew-symmetric matrices X and Γ that have the same block diagonal structure as Λ (and Δ), provided the product ΓX is symmetric, to generate matrices G and N and have the results in Sec. II apply, thereby obtaining Lagrangians and invariants of motion for such systems.

IV. Conclusions

This paper deals with the inverse problem of Lagrangian mechanics for multi-degree-of-freedom linear systems in which the mass matrix $\tilde{M}$ is positive definite and the matrices that multiply the generalized velocity n vector and those that multiply the generalized displacement n vector ($\tilde{C}$ and $\tilde{K}$, respectively) may each be nonsymmetric, in general.

Under the assumptions that 1) $\tilde{M}^{-1}\tilde{K}$ and $\tilde{M}^{-1}\tilde{C}$ are each diagonalizable and 2) they commute, explicit Lagrangians for such systems are obtained. These new results are derived in a simple, straightforward manner without the use of Helmholtz's conditions. An invariant of the motion for such general systems is also obtained. The invariant provides a generalization of the standard conservation-of-energy principle that is well known for undamped multi-degree-of-freedom potential systems.

Conditions for the aforementioned two assumptions to hold in the presence of gyroscopic and/or circulatory terms are explicitly provided to help delineate the scope of applicability of these results.

Significant generalizations of the work reported in [9] are achieved because Lagrangians and invariants are now found for multi-degree-of-freedom linear systems that may have damping and/or gyroscopic terms, as well as stiffness and/or circulatory terms.

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