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Sequential Determination of the {1, 4}-Inverse of a Matrix

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Abstract. In this paper, we provide a set of results for the sequential determination of the $\{1,4\}$-generalized inverse of a matrix. This inverse is of importance in areas where the minimal norm solution of a system of algebraic equations is desired.

Key Words. Generalized inverses, sequential determination, addition of block-column matrices, $\{1,4\}$-inverse.

1. Introduction

The $\{1,4\}$-generalized inverse of a matrix has come into considerable prominence in recent years because it appears in a significant manner in analytical dynamics and in other areas of application such as tomography. In analytical dynamics, it plays a crucial role in the equations of motion that describe mechanical systems that have holonomic and nonholonomic constraints, which may be ideal or nonideal.

Given an m by n matrix B and the consistent linear set of equations $Bx = b$, the $\{1,4\}$ inverse $B^{\{1,4\}}$ gives the shortest length solution

$$x = B^{\{1,4\}}b.$$

Such problems, which require the shortest length solution corresponding to a linear set of equations, are commonly encountered in solving a variety of inverse problems from given observational data. Often, as more and more data are collected, the matrix B increases in size; hence, a sequential determination of $B^{\{1,4\}}$ becomes an important issue. This paper addresses that issue.

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2. Sequential Determination of $B^{\{1,4\}}$

Consider the matrix $B = [A, \mathrm{a}\,]$, where A is an m by r matrix and a is an m by p block that is appended to it. We shall prove the main result in three steps.

Result 2.1. Given the matrix $B = [A, \mathrm{a}\,]$ as above,

$$B^{\{1,4\}} = \begin{bmatrix} A^{\{1,2,4\}}(I - aV) \\ V \end{bmatrix}, \tag{1}$$

where

$$V = Q^{\{1,2,4\}}(Ra)^T R + (I + Z^T Z)^{-1} Z^T A^{\{1,2,4\}}[I - a\,Q^{\{1,2,4\}}(Ra)^T R], \tag{2a}$$

$$R = I - AA^{\{1,2,4\}}, \qquad Q = (Ra)^T Ra, \qquad F = I - Q^{\{1,2,4\}}Q, \tag{2b}$$

and

$$Z = A^{\{1,2,4\}}aF. \tag{2c}$$

Proof. The $\{1,4\}$-inverse provides the unique minimum length solution $x = B^{\{1,4\}}b$ of the consistent equation set

$$Bx = [A, \mathrm{a}\,]\begin{bmatrix} z \\ s \end{bmatrix} = b,$$

where we have partitioned the vector x into an r-vector z and a p-vector s. For any fixed s_o that satisfies the equation

$$Az + as_0 = b,$$

we express the equation $Bx = b$ as

$$Az = (b - as_o), \tag{3}$$

whose solution is

$$\hat{z}(s_0) = A^{\{1,2,4\}}(b - as_0) + [I - A^{\{1,2,4\}}A]\,u, \tag{4}$$

for some arbitrary vector u and any $\{1,2,4\}$-inverse of the m by r matrix A; see Ref. 2. The two vectors on the right-hand side of equation (4) are orthogonal to each other because

$$[I - A^{\{1,2,4\}}A]^T A^{\{1,2,4\}} = [I - A^{\{1,2,4\}}A]\,A^{\{1,2,4\}} = 0.$$

Here, we have used the $\{4\}$-property of $A^{\{1,2,4\}}$. Using equation (4) in equation (3), we obtain

$$Ras_0 = Rb, \tag{5}$$

where we have denoted

$$R = I - AA^{\{1,2,4\}}.$$

We note that the matrix R is not, in general, a symmetric matrix. Now, after premultiplying both sides by $(Ra)^T$, the solution of equation (5) is

$$\begin{aligned}\hat{s}_0(w) &= Q^{\{1,2,4\}}(Ra)^T Rb + (I - Q^{\{1,2,4\}}Q)w\\ &= Q^{\{1,2,4\}}(Ra)^T Rb + Fw,\end{aligned} \tag{6}$$

where the vector w is an arbitrary p-vector,

$$Q = (Ra)^T(Ra) \text{ and } F = I - Q^{\{1,2,4\}}Q.$$

Again, the two vectors on the right-hand side of the last equality in (6) are orthogonal to each other. Furthermore, because of the {4}-property of $Q^{\{1,2,4\}}$,

$$\begin{aligned}F^T &= (I - Q^{\{1,2,4\}}Q)^T\\ &= I - Q^{\{1,2,4\}}Q\\ &= F,\end{aligned}$$

so that F is symmetric as well as idempotent and

$$\begin{aligned}F^T F &= F^2\\ &= F.\end{aligned} \tag{7}$$

We need to find the vectors u and w so that the length

$$K(u, w) = \hat{z}^T(s_0(w), u)\, \hat{z}(s_0(w), u) + \hat{s}_0^T(w)\, \hat{s}_0(w) \tag{8}$$

is a minimum. Using equations (4) and (6), we see that the relation (8) becomes

$$\begin{aligned}K(u, w) = {}&\|A^{\{1,2,4\}}[b - a\{ Q^{\{1,2,4\}}(Ra)^T Rb + Fw\}]\|_2^2 + \|(I - A^{\{1,2,4\}}A)\, u\|_2^2\\ &+ \|Q^{\{1,2,4\}}(Ra)^T Rb\|_2^2 + \|Fw\|_2^2.\end{aligned} \tag{9}$$

The minimum of (9)with respect to the vector u is obtained obviously when

$$\|[I - A^{\{1,2,4\}}A]\, u\|_2^2 = 0.$$

Hence, we obtain

$$\begin{aligned}\tilde{K}(w) = {}&\|A^{\{1,2,4\}}[b - a\{Q^{\{1,2,4\}}(Ra)^T Rb + Fw\}]\|_2^2\\ &+ \|Q^{\{1,2,4\}}(Ra)^T Rb\|_2^2 + \|Fw\|_2^2,\end{aligned} \tag{10}$$

which needs to be minimized with respect to the vector w. Taking the derivative of the left-hand side with respect to w and setting it to zero yields

$$(Z^T Z + F^T F) w = Z^T A^{\{1,2,4\}}[b - a\, Q^{\{1,2,4\}}(Ra)^T Rb], \tag{11}$$

where we have denoted

$$Z = A^{\{1,2,4\}} aF.$$

Using the relations in (7), the left-hand side of equation (11) can be simplified further to give

$$\begin{aligned}[Z^T(A^{\{1,2,4\}}aF) + F^T F]\, w &= [Z^T(A^{\{1,2,4\}}aF)F + F]\, w \\ &= (Z^T Z + I)\, Fw. \end{aligned} \tag{12}$$

Using this on the left-hand side of equation (11) gives

$$[Z^T Z + I]\, Fw = Z^T A^{\{1,2,4\}}(b - aQ^{\{1,2,4\}}(Ra)^T Rb]. \tag{13}$$

Since the matrix $Z^T Z + I$ is positive definite, equation (13) can be solved for Fw, which gives

$$Fw = [Z^T Z + I]^{-1} Z^T A^{\{1,2,4\}}(b - aQ^{\{1,2,4\}}(Ra)^T Rb]. \tag{14}$$

Using this expression for Fw in equation (6), we obtain

$$\begin{aligned}\hat{s}_0 &= Q^{\{1,2,4\}}(Ra)^T Rb + (I + Z^T Z)^{-1} Z^T A^{\{1,2,4\}}[I - aQ^{\{1,2,4\}}(Ra)^T R]\, b \\ &= Vb, \end{aligned} \tag{15}$$

and hence,

$$\begin{aligned}\hat{z}(s_0) &= A^{\{1,2,4\}}(b - as_0) \\ &= A^{\{1,2,4\}}(I - aV)\, b, \end{aligned} \tag{16}$$

which proves our result. □

Next we establish the connection between any chosen $\{1,4\}$-inverse of any given matrix H and a $\{1,2,4\}$-inverse of H.

Result 2.2. Given *any* m by n matrix H, and given any chosen $\{1,4\}$-inverse of the matrix H, a $\{1,2,4\}$-inverse of H is given by

$$H^{\{1,2,4\}} = H^{\{1,4\}} H H^{\{1,4\}}. \tag{17}$$

Proof. We prove that $H^{\{1,2,4\}}$ as defined in (17) satisfies the $\{1\}$, $\{2\}$, and $\{4\}$ Moore-Penrose properties of generalized inverses (see Ref. 2).

(i) The $\{1\}$-property is satisfied because

$$
\begin{aligned}
HH^{\{1,2,4\}}H &= HH^{\{1,4\}}[HH^{\{1,4\}}H] \\
&= HH^{\{1,4\}}[H] \\
&= H.
\end{aligned} \tag{18}
$$

(ii) The $\{2\}$-property is satisfied because

$$
\begin{aligned}
H^{\{1,2,4\}}HH^{\{1,2,4\}} &= H^{\{1,4\}}[HH^{\{1,4\}}H]\, H^{\{1,4\}}HH^{\{1,4\}} \\
&= H^{\{1,4\}}[H]\, H^{\{1,4\}}HH^{\{1,4\}} \\
&= H^{\{1,4\}}[HH^{\{1,4\}}H]\, H^{\{1,4\}} \\
&= H^{\{1,4\}}[H]\, H^{\{1,4\}} \\
&= H^{\{1,2,4\}}.
\end{aligned} \tag{19}
$$

(iii) The $\{4\}$-property is satisfied because

$$
\begin{aligned}
[H^{\{1,2,4\}}H]^T &= [H^{\{1,4\}}HH^{\{1,4\}}H]^T \\
&= [H^{\{1,4\}}H]^T[H^{\{1,4\}}H]^T \\
&= [H^{\{1,4\}}H]\,[H^{\{1,4\}}H] \\
&= [H^{\{1,4\}}HH^{\{1,4\}}]\, H \\
&= H^{\{1,2,4\}}H.
\end{aligned} \tag{20}
$$

□

The above two results now enable us to obtain a formula for the sequential determination of the $\{1,4\}$-inverse of the augmented matrix B in terms of any chosen $\{1,4\}$-inverse of the matrix A. We state our final result as follows.

Result 2.3. First Main Result. Given the augmented matrix $B=[A,a]$, where A is m by r and a is m by p,

$$
B^{\{1,4\}} = \begin{bmatrix} A^*(I-aV) \\ V \end{bmatrix}, \tag{21}
$$

where

$$
V = Q^*(Ra)^TR + [I+Z^TZ]^{-1}Z^TA^*[I-aQ^*(Ra)^TR], \tag{22a}
$$

$$
R = I-AA^*, \qquad Q=(Ra)^TRa, \qquad F=I-Q^*Q, \qquad Z=A^*aF, \tag{22b}
$$

and

$$A^* = A^{\{1,4\}}AA^{\{1,4\}}, \qquad \text{with} \quad Q^* = Q^{\{1,4\}}QQ^{\{1,4\}}. \tag{22c}$$

Proof. Using Result 2.1 and Result 2.2, this formula follows. □

To any particular $\{1,4\}$-inverse of B found using the result above, one can add any matrix P such that $PB = 0$. The sum of the particular $\{1,4\}$-inverse and the matrix P now yields a new $\{1,4\}$-inverse of B.

Result 2.4. Second Main Result. An alternative formula for $B^{\{1,4\}}$ is

$$B^{\{1,4\}} = \begin{bmatrix} A^*(I - aV) \\ V \end{bmatrix}, \tag{23}$$

where

$$V = Q^*R + (I + Z^T Z)^{-1} Z^T A^*[I - aQ^*R], \tag{24a}$$

$$R = I - AA^*, \qquad Q = Ra, \qquad F = I - Q^*Q, \qquad Z = A^*aF, \tag{24b}$$

and

$$A^* = A^{\{1,4\}}AA^{\{1,4\}}, \qquad \text{with } Q^* = Q^{\{1,4\}}QQ^{\{1,4\}}. \tag{24c}$$

Proof. Here, we simply solve equation (5) without premultiplication by $(Ra)^{\mathrm{T}}$ as

$$\begin{aligned} \hat{s}_0(w) &= Q^{\{1,2,4\}}Rb + (I - Q^{\{1,2,4\}}Q)w \\ &= Q^{\{1,2,4\}}Rb + Fw, \end{aligned} \tag{25}$$

where $Q = Ra$ and as before

$$F = I - Q^{\{1,2,4\}}Q.$$

Following exactly the same steps as given in the proofs of Results 2.1 and 2.2, the above formula is obtained. □

Remark 2.1. If an m by n matrix A has rank m, then

$$\begin{aligned} A^{\{1,2,4\}} &= A^{\{1,4\}}AA^{\{1,4\}} \\ &= A^{\{1,4\}} \\ &= A^{\{1,2,3,4\}}. \end{aligned}$$

This result follows from the general forms of the $\{1,2,3,4\}$-inverse, the $\{1,2,4\}$-inverse, and the $\{1,4\}$-inverse (see Ref. 2) when the rank of the m by n matrix A is m.

An alternative way of looking at it is as follows. When the rank of A is m, the right-hand vector b in the relation $Ax = b$ must lie in the range space of A. The unique minimum length least-squares solution given by

$$x = A^{\{1,2,3,4\}}b$$

must then be identical to the minimum length solution given by

$$x = A^{\{1,4\}}b,$$

for every vector b. Hence, the result follows.

Remark 2.2. Our first main result modifies the formula for $A^{\{1,4\}}$ given in Ref. 1, which is valid when the m by n matrix A has rank m (Ref. 3).

Remark 2.3. When the rank of the m by n matrix A is m, A^* and Q^* in the two main results above can be taken to be simply $A^{\{1,4\}}$ and $Q^{\{1,4\}}$ respectively. This follows directly from Remark 2.1, since a $\{1,4\}$-inverse is also now a $\{1,2,4\}$-inverse in that case.

Remark 2.4. From a computational standpoint, the formula for the sequential determination of $B^{\{1,4\}}$ given in equations (23)–(24) appears superior to that given in equations (21)–(22) (and also superior to that in Ref. 1).

3. Conclusions

The $\{1,4\}$-inverse plays an important role in applied mathematics and mechanics. This paper provides a sequential approach for obtaining it.

References

1. Udwadia, F. E., and Kalaba, R. E., *General Forms for the Recursive Determination of Generalized Inverses: A Unified Approach*, Journal of Optimization Theory and Applications, Vol. 101, pp. 509–521, 1999.
2. Udwadia, F. E., and Kalaba, R. E., *Analytical Dynamics: A New Approach*, Cambridge University Press, Cambridge, England, pp. 62–64, 1996.
3. Wang, G., and Zheng, B., *Personal Communication*, Shanghai Normal University, Shanghai, PRC, 2002.