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General Forms for the Recursive Determination of Generalized Inverses: Unified Approach

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Abstract. Results for the recursive determination of different types of generalized inverses of a matrix are presented for the case of the addition of a block-column matrix of arbitrary size. Using a unifying underlying theme, results for the generalized inverse, least-square generalized inverse, minimum norm generalized inverse, and Moore–Penrose inverse are included.

Key Words. Generalized inverses, recursive determination, addition of block-column matrices, unified approach, general forms.

1. Introduction

The recursive determination of a generalized inverse of a matrix finds extensive applications in the fields of statistical inference (Refs. 1–3), filtering theory, estimation theory (Ref. 4), system identification (Ref. 5), and network theory. More recently, generalized inverses have found renewed applicability in the field of analytical dynamics (Ref. 6). The reason for the wide use of recursive relations is that they provide a systematic method to generate updates, whenever sequential addition of data or new information is made available and updated estimates, which take this additional information into account, are required.

The recursive scheme for the computation of the Moore–Penrose (MP) inverse (Refs. 7 and 8) of a matrix when an additional column (or row) is added to it was ingeniously obtained in a paper by Greville in 1960. Because of its extensive applicability, Greville's result is widely stated in almost every book that touches on the subject of generalized inverses of matrices. However, because of the complexity of his solution technique, Greville's proof

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is seldom, if ever, quoted or outlined even in specialized texts, which deal solely with generalized inverses of matrices (e.g., in books like Refs. 4 and 9–11).

Recently, a simpler alternative proof for Greville's result was given (Ref. 12). In this paper, we provide results for the recursive determination of several commonly used generalized inverses of an m by n matrix B, with $m = r + p$, where the matrix B is obtained by the addition of a block of p columns to the m by r matrix A. Thus, we generalize Greville's result to: (a) include different types of generalized inverses; (b) include the addition of a block of columns as opposed to only a single column vector.

The recursive relation for determining the MP inverse of a matrix, when a block is added to it, was determined by Bhimsankaram (Ref. 13). However, in his paper, Bhimsankaram starts with the final result for the MP inverse; his proof is simply a verification that his result satisfies the four conditions for the Moore–Penrose inverse. In this paper, we provide a constructive proof for the determination of the recursive relation for finding the MP inverse of a matrix to which a block of columns is added. Furthermore, we show that the same thread of reasoning runs through the constructive procedure for determining other types of generalized inverses of such matrices as well. Thus, results are also obtained here for other types of generalized inverses. These results will be of considerable use in fields like analytical mechanics (see, for example, Ref. 14).

For convenience, we shall introduce the following notation. Given a real matrix B, its MP inverse G satisfies the following four conditions:

(C1) $BGB = B$,
(C2) $GBG = G$,
(C3) BG is symmetric,
(C4) GB is symmetric.

We shall denote a matrix G which satisfies all four of these conditions by $B^{\{1,2,3,4\}}$; similarly, a matrix which satisfies only the first and fourth condition above shall be denoted as $B^{\{1,4\}}$ and shall be referred to as the $\{1, 4\}$-inverse of B, etc.

The most commonly used generalized inverses are the MP-inverse [also denoted as the $\{1, 2, 3, 4\}$-inverse], the $\{1, 3\}$-inverse, the $\{1, 4\}$-inverse, and the $\{1\}$-inverse because they are relevant to the solution x of the matrix equation $Bx = b$ or the equation $Bx \approx b$. In fact, the MP-inverse $B^{\{1,2,3,4\}}$ is defined as that matrix which, when postmultiplied by b, yields the minimum-length least-square solution x of the possibly inconsistent equation $Bx \approx b$, for any b. Similarly, a $\{1, 3\}$-inverse of B is defined as any matrix $B^{\{1,3\}}$ which, when postmultiplied by b, gives a least-square solution x to the possibly inconsistent equation $Bx \approx b$, for any b; a $\{1, 4\}$-inverse of B is defined as any matrix $B^{\{1,4\}}$ which, when postmultiplied by b, gives the minimum-length solution for any b for which the equation $Bx=b$ is consistent; and the $\{1\}$-inverse of B is defined as any matrix $B^{\{1\}}$ which, when postmultiplied by b, gives a solution to the consistent equation $Bx=b$. This paper will be concerned with these four commonly used generalized inverses, which we shall denote, in general, by B^*.

Given a real m by $(r+p)$ matrix B, one can partition it as $[A, a]$ where A consists of the first r columns of the matrix B and a denotes its last p columns. The m by p matrix a comprises new or additional information, while the m by r matrix A comprises accumulated, or past, data. The generalized inverse B^* of the updated matrix B is then sought in terms of the generalized inverse A^* of the matrix A, which corresponds to past accumulated data, the matrix a containing new or additional information. The MP-inverse of a matrix is unique; the other generalized inverses, which we shall deal with here, are not unique; so, by say $B^{\{1,4\}}$, we shall mean any one of the set of $\{1, 4\}$-inverses of the matrix B, etc.

2. Main Result

Let $B=[A, a]$ be an m by $(r+p)$ matrix whose last p columns are denoted by a. Let

$$R=I-AA^*, \qquad Q=(Ra)^T Ra, \qquad F=I-Q^*Q, \qquad Z=A^*aF.$$

Then,

$$B^* = \begin{bmatrix} A^*(I-aV)+(I-A^*A)T \\ V \end{bmatrix}. \tag{1}$$

(i) When $*=\{1,2,3,4\}$,

$$V=Q^*a^TR+[I+Z^TZ]^{-1}Z^TA^*(I-aQ^*a^TR), \qquad T=0. \tag{2a}$$

(ii) When $*=\{1,3\}$,

$$V=Q^*a^TR+FP, \tag{2b}$$

P is an arbitrary p by m matrix,
T is an arbitrary r by m matrix.

(iii) When $*=\{1,4\}$,

$$V=Q^*(Ra)^TR+[I+Z^TZ]^{-1}Z^TA^*[I-aQ^*(Ra)^TR], \quad \mathrm{T}=0. \tag{2c}$$

(iv) When $* = \{1\}$,

$$V = Q^*(Ra)^T R + FP \tag{2d}$$

P is an arbitrary p by m matrix,
T is an arbitrary r by m matrix.

We note that, when $Ra = 0$, that is, when the columns of a belong to the range space of A, then

$$Q = 0, \qquad F = I.$$

Then, for (ii) and (iv) above, the matrix V is any arbitrary p by m matrix.

Proof.

(i) We consider the least-square problem

$$Bx = [A, a]\begin{bmatrix} z \\ s \end{bmatrix} = Az + as \approx b, \tag{3}$$

where we have partitioned the vector x into the r-vector z and the p-vector s. To determine the minimum-length least-square solution x of $Bx \approx b$, we consider all those pairs (z, s), which minimize

$$J(z, s) = \|Az + as - b\|_2^2$$

and, from these pairs, select the one whose length $z^T z + s^T s$ is a minimum.

We begin by setting $s = s_0$, where s_0 is some fixed p-vector. Then, we write

$$J(z, s_0) = \|Az - (b - as_0)\|_2^2. \tag{4}$$

Minimizing (4) over all r-vectors z, we obtain, from the definition of the MP-inverse,

$$\hat{z}(s_0, u) = A^{\{1,2,3,4\}}(b - as_0) + (I - A^{\{1,2,3,4\}}A)u, \tag{5}$$

where u is some arbitrary r-vector. We note that the two vectors given by the two right-hand members of Eq. (5) are orthogonal to each other. Equation (5) shows that, for a given p-vector s_0, the r-vector $\hat{z}$, which minimizes (4), is a function of s_0. Thus, (4) can be written as

$$\begin{aligned} J(\hat{z}(s_0), s_0) &= \|AA^{\{1,2,3,4\}}(b - as_0) + as_0 - b\|_2^2 \\ &= \|(I - AA^{\{1,2,3,4\}})as_0 - (I - AA^{\{1,2,3,4\}})b\|_2^2 \\ &= \|R(as_0 - b)\|_2^2, \end{aligned} \tag{6}$$

where we have used the fact that

$$AA^{\{1,2,3,4\}}A = A$$

in the first equality and we have denoted

$$R = I - AA^{\{1,2,3,4\}},$$

which is a symmetric idempotent matrix. We next need to determine the p-vector s_0, which minimizes (6). This vector is given by

$$\begin{aligned}\hat{s}_0(w) &= Q^{\{1,2,3,4\}}(Ra)^T Rb + (I - Q^{\{1,2,3,4\}}Q)w\\ &= Q^{\{1,2,3,4\}}(Ra)^T Rb + Fw,\end{aligned} \tag{7}$$

where we have denoted

$$Q = (Ra)^T Ra, \qquad F = I - Q^{\{1,2,3,4\}}Q.$$

The p-vector w is an arbitrary vector. Again, the two vectors given by the two right-hand side members of Eq. (7) are orthogonal to each other.

We now endeavor to determine the vectors u and w so that the length

$$K(u, w) = \hat{z}^T(s_0(w), u)\hat{z}(s_0(w), u) + \hat{s}_0^T(w)\hat{s}_0(w) \tag{8}$$

is minimized. Using Eqs. (5) and (7), we then obtain

$$\begin{aligned}K(u, w) = {}&\|A^{\{1,2,3,4\}}[b - a\{Q^{\{1,2,3,4\}}(Ra)^T Rb + Fw\}]\|_2^2 + \|(I - A^{\{1,2,3,4\}}A)u\|_2^2\\ &+ \|Q^{\{1,2,3,4\}}(Ra)^T Rb\|_2^2 + \|Fw\|_2^2.\end{aligned} \tag{9}$$

We shall now present the following lemma related to the structure of the matrix F.

Lemma 2.1. Let $Q = M^T M$ be a symmetric p by p matrix. Then, for any p-vector w, the vector Fw, where

$$F = I - Q^{\{1,4\}}Q,$$

can always be expressed as

$$U\begin{bmatrix} r \\ 0 \end{bmatrix} = [U_1, U_2]\begin{bmatrix} r \\ 0 \end{bmatrix},$$

where U is an orthogonal matrix and the dimension of the vector r is the same as that of the null space of the matrix Q. Furthermore, the matrix F can be written as $U_1 U_1^T$.

Proof. Since Q is symmetric and positive semidefinite, there exists an orthogonal matrix U such that

$$Q = U \begin{bmatrix} 0_{j \times j} & 0 \\ 0 & \Lambda \end{bmatrix} U^T, \tag{10}$$

where the dimension of the zero matrix in the upper left-hand corner on the right side of Eq. (10) is j, the dimension of the null space of Q. Hence (Ref. 14),

$$Q^{\{1,2,3,4\}} = U \begin{bmatrix} L & 0 \\ M & \Lambda^{-1} \end{bmatrix} U^T, \tag{11}$$

where L and M are arbitrary matrices of the proper dimensions. Using Eqs. (10) and (11), we get

$$\begin{aligned} F &= I - Q^{\{1,4\}}Q \\ &= I - U \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} U^T \\ &= [U_1, U_2] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \\ &= U_1 U_1^T. \end{aligned} \tag{12}$$

Furthermore, since U is orthogonal, we can express the vector w as Ut, so that

$$\begin{aligned} Fw &= \left(I - U \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} U^T \right) Ut \\ &= U_1 U_1^T [U_1, U_2] \begin{bmatrix} r \\ s \end{bmatrix} \\ &= [U_1, 0] \begin{bmatrix} r \\ s \end{bmatrix} \\ &= U_1 r \\ &= [U_1, U_2] \begin{bmatrix} r \\ 0 \end{bmatrix} \\ &= U \begin{bmatrix} r \\ 0 \end{bmatrix}, \end{aligned} \tag{13}$$

where we have partitioned the vector t appropriately into the j-vector r and the $(p\text{-}j)$-vector 0. □

Lemma 2.2. The results of Lemma 2.1 apply when the matrix F is defined as

$$F = I - Q^{\{1,2,3,4\}}Q.$$

Proof. Since $Q^{\{1,2,3,4\}}$ is a special case of $Q^{\{1,4\}}$, when the matrices L and M are zero, the result follows. □

Using the expression for Fw in Eq. (9) now gives (see Lemmas 2.1 and 2.2)

$$K(u, r) = \|A^{\{1,2,3,4\}}[b - a\{Q^{\{1,2,3,4\}}(Ra)^T Rb + U_1 r\}]\|_2^2 + \|(I - A^{\{1,2,3,4\}}A)u\|_2^2 + \|Q^{\{1,2,3,4\}}(Ra)^T Rb\|_2^2 + \|r\|_2^2, \quad (14)$$

where U_1 and r are as defined in Lemma 2.1. Minimizing $K(u, r)$ with respect to u requires that we choose u such that

$$(I - A^{\{1,2,3,4\}}A)u = 0.$$

Furthermore, differentiating K with respect to r and setting it to zero yields

$$[I + (A^{\{1,2,3,4\}}aU_1)^T(A^{\{1,2,3,4\}}aU_1)]r = (A^{\{1,2,3,4\}}aU_1)^T A^{\{1,2,3,4\}}[I - aQ^{\{1,2,3,4\}}(Ra)^T R]b, \quad (15)$$

from which we obtain

$$r = [I + (A^{\{1,2,3,4\}}aU_1)^T(A^{\{1,2,3,4\}}aU_1)]^{-1}(A^{\{1,2,3,4\}}aU_1)^T A^{\{1,2,3,4\}} \times [I - aQ^{\{1,2,3,4\}}(Ra)^T R]b. \quad (16)$$

Noting that

$$\partial^2 K/\partial r^2 = I + (A^{\{1,2,3,4\}}aU_1)^T(A^{\{1,2,3,4\}}aU_1)$$

is a positive-definite matrix, the value of r given by Eq. (16) indeed minimizes K.

However, the minimization of K requires the determination of the matrix U_1 and therefore the eigenvectors of the matrix Q corresponding to its zero eigenvalues. We now show, by the following lemma, that this is not necessary. In fact, the vector Fw in Eq. (9), to which the value of r given in Eq. (16) leads, can be obtained directly without the need to determine U_1.

Lemma 2.3. Equation (15), which gives the value of r, is equivalent to the relation

$$[I+(A^{\{1,2,3,4\}}aF)^T(A^{\{1,2,3,4\}}aF)]Fw$$
$$=(A^{\{1,2,3,4\}}aF)^T A^{\{1,2,3,4\}}[I-aQ^{\{1,2,3,4\}}(Ra)^T R]b. \tag{17}$$

Proof. We shall prove that relation (15) implies relation (17). We begin with Eq. (15) written in the form

$$[U_1^T+U_1^T(A^{\{1,2,3,4\}}a)^T(A^{\{1,2,3,4\}}a)]U_1 r$$
$$=U_1^T(A^{\{1,2,3,4\}}a)^T A^{\{1,2,3,4\}}[I-aQ^{\{1,2,3,4\}}(Ra)^T R]b. \tag{18}$$

Premultiplying by U_1 and noting Eqs. (12) and (13) yields

$$[F+F^T(A^{\{1,2,3,4\}}a)^T(A^{\{1,2,3,4\}}a)]Fw$$
$$=(A^{\{1,2,3,4\}}aF)^T A^{\{1,2,3,4\}}[I-aQ^{\{1,2,3,4\}}(Ra)^T R]b. \tag{19}$$

Since F is idempotent, Eq. (19) can be rewritten as

$$[F+F^T(A^{\{1,2,3,4\}}a)^T(A^{\{1,2,3,4\}}a)]F^2 w$$
$$=(A^{\{1,2,3,4\}}aF)^T A^{\{1,2,3,4\}}[I-aQ^{\{1,2,3,4\}}(Ra)^T R]b, \tag{20}$$

or

$$[F^2+(A^{\{1,2,3,4\}}aF)^T(A^{\{1,2,3,4\}}aF)]Fw$$
$$=(A^{\{1,2,3,4\}}aF)^T A^{\{1,2,3,4\}}[I-aQ^{\{1,2,3,4\}}(Ra)^T R]b, \tag{21}$$

from which it follows that

$$[F+(A^{\{1,2,3,4\}}aF)^T(A^{\{1,2,3,4\}}aF^2)]Fw$$
$$=(A^{\{1,2,3,4\}}aF)^T A^{\{1,2,3,4\}}[I-aQ^{\{1,2,3,4\}}(Ra)^T R]b. \tag{22}$$

Equation (22) implies that

$$[I+(A^{\{1,2,3,4\}}aF)^T(A^{\{1,2,3,4\}}aF)]Fw$$
$$=(A^{\{1,2,3,4\}}aF)^T A^{\{1,2,3,4\}}[I-aQ^{\{1,2,3,4\}}(Ra)^T R]b, \tag{23}$$

where we have used again the fact that F is idempotent. Equation (23) permits us to determine the vector Fw as

$$Fw=[I+(A^{\{1,2,3,4\}}aF)^T(A^{\{1,2,3,4\}}aF)]^{-1}(A^{\{1,2,3,4\}}aF)^T A^{\{1,2,3,4\}}$$
$$\times[I-aQ^{\{1,2,3,4\}}(Ra)^T R]b. \tag{24}$$

Denoting

$$A^{\{1,2,3,4\}}aF=Z,$$

we obtain

$$Fw = [I + Z^T Z]^{-1} Z^T A^{\{1,2,3,4\}} [I - aQ^{\{1,2,3,4\}} (Ra)^T R] b. \tag{25}$$

We note that the result of this lemma hinges solely on the fact that the matrix F is idempotent and that it can be expressed as $U_1 U_1^T$.

The converse can be proved easily by retracing our path backward from Eq. (25). □

Using Eq. (25), Eqs. (7) and (5) now give

$$\hat{s}_0 = Q^{\{1,2,3,4\}} (Ra)^T Rb + [I + Z^T Z]^{-1} Z^T A^{\{1,2,3,4\}} [I - aQ^{\{1,2,3,4\}} (Ra)^T R] b$$

$$= Vb, \tag{26}$$

$$\hat{z}(s_0) = A^{\{1,2,3,4\}} (I - aV) b. \tag{27}$$

Equations (26) and (27) simplify, because R is a symmetric idempotent matrix, and hence,

$$(Ra)^T R = (Ra)^T.$$

Since b is an arbitrary m-vector, the result in (i) now follows. □

(ii) The $\{1, 3\}$-inverse provides a least-square solution $x = B^{\{1,3\}} b$ of the relation $Bx \approx b$. As before, we partition the vector x into an r-vector z and a p-vector s. For a fixed s_0, we minimize

$$J(z, s_0) = \|Az - (b - as_0)\|_2^2 \tag{28}$$

to yield

$$\hat{z}(s_0) = A^{\{1,3\}} (b - as_0) + (I - A^{\{1,3\}} A) u, \tag{29}$$

for some arbitrary vector u and some $\{1, 3\}$-inverse of the m by r matrix A. The arbitrary vector u can be expressed as an arbitrary r by m matrix T multiplied by the vector b, so that

$$\hat{z}(s_0) = A^{\{1,3\}} (b - as_0) + (I - A^{\{1,3\}} A) Tb. \tag{30}$$

Using Eq. (30) in Eq. (28), we next minimize

$$J(\hat{z}(s_0), s_0) = \|(I - AA^{\{1,3\}}) as_0 - (I - AA^{\{1,3\}}) b\|_2^2$$

$$= \|R(as_0 - b)\|_2^2, \tag{31}$$

with respect to s_0, where we have denoted

$$I - AA^{\{1,3\}} = R.$$

As before, this yields

$$\hat{s}_0(w) = Q^{\{1,3\}}(Ra)^T Rb + (I - Q^{\{1,3\}}Q)w = Q^{\{1,3\}}a^T Rb + Fw, \qquad (32)$$

where the vector w is an arbitrary p-vector and

$$I - Q^{\{1,3\}}Q = F.$$

In the second equality above, we have used the fact that R is symmetric and idempotent. Once again, because the vector w is arbitrary, we can express it as an arbitrary p by m matrix P times the vector b, yielding

$$\begin{aligned}\hat{s}_0(w) &= (Q^{\{1,3\}}a^T R + FP)b \\ &= Vb. \end{aligned} \qquad (33)$$

Noting that b is an arbitrary m-vector, the result for this case follows.

To any particular $\{1, 3\}$-inverse of B found using Eq. (2b), one can add any matrix G such that $BG = 0$. The sum of the particular $\{1, 3\}$-inverse and the matrix G now yields a new $\{1, 3\}$-inverse of B. □

(iii) The $\{1, 4\}$-inverse provides a minimum length solution $x = B^{\{1,4\}}b$ of the consistent equation $Bx = b$. Partitioning the vector x into an r-vector z and a p-vector s, for a fixed s_0, we express the equation as

$$Az = (b - as_0), \qquad (34)$$

whose solution is

$$\hat{z}(s_0) = A^{\{1,4\}}(b - as_0) + (I - A^{\{1,4\}}A)u, \qquad (35)$$

for some arbitrary vector u and some $\{1, 4\}$-inverse of the m by r matrix A. Using Eq. (35) in (34), we obtain

$$Ras_0 = Rb, \qquad (36)$$

where we have denoted

$$I - AA^{\{1,4\}} = R.$$

In general, the matrix R here is not a symmetric matrix. The solution of Eq. (36) is

$$\begin{aligned}\hat{s}_0(w) &= Q^{\{1,4\}}(Ra)^T Rb + (I - Q^{\{1,4\}}Q)w \\ &= Q^{\{1,4\}}(Ra)^T Rb + Fw, \end{aligned} \qquad (37)$$

where the vector w is an arbitrary p-vector and

$$I - Q^{\{1,4\}}Q = F.$$

We now need to find the vectors u and w so that the length

$$K(u, w) = \hat{z}^T(s_0(w), u)\hat{z}(s_0(w), u) + \hat{s}_0^T(w)\hat{s}_0(w) \tag{38}$$

is minimized. Using Eqs. (36) and (37), this yields an equation similar to Eq. (9), whose minimization can be carried out similarly after noting that Fw can be expressed as $U_1 r$, as before (see Lemma 2.1). This yields an expression for the vector r similar to Eq. (16). Since Lemma 2.3 is still valid, with the replacement of $\{1, 2, 3, 4\}$-inverses by $\{1, 4\}$-inverses, the result given for (iii) follows.

To any particular $\{1, 4\}$-inverse of B found using Eq. (2c), one can add any matrix H such that $HB = 0$. The sum of the particular $\{1, 4\}$-inverse and the matrix H now yields a new $\{1, 4\}$-inverse of B. □

(iv) A solution to the equation $Bx = b$ can be obtained by partitioning the vector x into an r-vector z and a p-vector s. For a fixed s_0, we express the equation as

$$Az = b - as_0, \tag{39}$$

to obtain as before the solution

$$\hat{z}(s_0) = A^{\{1\}}(b - as_0) + (I - A^{\{1\}}A)u, \tag{40}$$

where u is an arbitrary vector and $A^{\{1\}}$ is any $\{1\}$-inverse of the matrix A. The arbitrary vector u can be expressed as in (ii) by an arbitrary r by m matrix T multiplied by the vector b, so that

$$\hat{z}(s_0) = A^{\{1\}}(b - as_0) + (I - A^{\{1\}}A)Tb. \tag{41}$$

Substituting this into Eq. (39) yields

$$Ras_0 = Rb. \tag{42}$$

Solving Eq. (42) for s_0, we obtain

$$\begin{aligned} \hat{s}_0(w) &= Q^{\{1\}}(Ra)^T Rb + (I - Q^{\{1\}}Q)w \\ &= Q^{\{1\}}(Ra)^T Rb + Fw, \end{aligned} \tag{43}$$

where the vector w is an arbitrary p-vector and

$$I - Q^{\{1\}}Q = F.$$

As in (ii), $\hat{s}_0(w)$ can be expressed as

$$\hat{s}_0(w) = (Q^{\{1\}}(Ra)^T Rb + FP)b = Vb, \tag{44}$$

where P is an arbitrary p by m matrix, and the result follows.

As before, to any particular $\{1\}$-inverse of B found using Eq. (2d), one can add any matrix $H+G$, where

$$HB=0 \quad \text{and} \quad BG=0.$$

The sum of the particular $\{1\}$-inverse and the matrix $H+G$ now yields a new $\{1\}$-inverse of B. □

3. Conclusions

We present in this paper a general formula for the recursive determination of several of the commonly used generalized inverses of an m by $(r+p)$ matrix B. Our results show that similar lines of reasoning can be used to obtain recursive relations for (i) the Moore–Penrose inverse $B^{\{1,2,3,4\}}$, (ii) a least-square generalized inverse $B^{\{1,3\}}$, (iii) a minimum-length generalized inverse $B^{\{1,4\}}$, and (iv) a generalized inverse of a matrix $B^{\{1\}}$.

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