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Equations of Motion for Constrained Multibody Systems and their Control

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In honor of Bob kalaba, friend, colleague, and mentor.

Abstract. This paper presents some recent advances in the dynamics and control of constrained multibody systems. The constraints considered need not satisfy the D'Alembert principle and therefore the results are of general applicability. They show that, in the presence of constraints, the constraint force acting on the multibody system can always be viewed as made up of the sum of two components whose explicit form is provided. The first of these components consists of the constraint force that would have existed were all the constraints ideal; the second is caused by the nonideal nature of the constraints, and though it needs specification by the mechanician who is modeling the specific system at hand, it has a specific form. The general equations of motion obtained herein provide new insights into the simplicity with which Nature seems to operate. They point toward the development of new and novel approaches for the exact control of complex multibody nonlinear systems.

Key Words. Analytical dynamics, multibody and nonlinear systems, constrained motion, explicit equations of motion, exact control of nonlinear systems.

1. Introduction

The general problem of obtaining the equations of motion of a constrained discrete mechanical system is one of the central issues in multibody dynamics. While formulated at least as far back as Lagrange (Ref. 1), the determination of the explicit equations of motion, even within the restricted compass of Lagrangian dynamics, has been a major hurdle. The Lagrange multiplier method relies on problem-specific approaches to the

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determination of the multipliers, which are often difficult to obtain for systems with a large number of degrees of freedom and many nonintegrable constraints. Formulations offered by Gibbs (Ref. 2), Appell (Ref. 3), and Poincare (Ref. 4) require a felicitous choice of problem specific quasicoordinates and suffer from similar problems in dealing with systems with a large number of degrees of freedom and many nonintegrable constraints. Gauss (Ref. 5) developed a general principle governing the constrained motion for systems that satisfy the D'Alembert principle; Dirac (Ref. 6) offered a formulation for Hamiltonian systems with singular Lagrangians where the constraints do not depend explicitly on time.

The explicit equations of motion obtained by Udwadia and Kalaba (Ref. 7) provide a new and different perspective on the constrained motion of multibody systems. They introduce the notion of generalized inverses in the description of such motion and, through their use, obtain a simple and general explicit equation of motion for constrained multibody mechanical systems without the use of, or any need for, the notion of Lagrange multipliers. Their approach has allowed us, for the first time, to obtain the explicit equations of motion for multibody systems with constraints that may be: (i) nonlinear functions of the velocities, (ii) explicitly dependent on time, and (iii) functionally dependent. However, their equations deal only with systems where the constraints are ideal and satisfy the D'Alembert principle, as do all the other formulations/equations developed so far (e.g. Refs. 1–6 and Refs. 8–9). The D'Alembert principle says that the motion of a constrained mechanical system occurs in such a way that, at every instant of time, the sum total of the work done under virtual displacements by the constraint forces is zero.

In this paper, we extend these results along two directions. First, we extend the D'Alembert principle to include constraints that may be in general nonideal so that the constraint forces may do positive, negative, or zero work under virtual displacements at any given instant of time during the motion of the constrained system. Thus, we expand Lagrangian mechanics to include nonideal constraint forces within its compass. Second, explicit equations of motion are obtained. They lead to deeper insights into the way Nature seems to work. With the help of these equations, we provide a new fundamental, general principle governing constrained multibody dynamics.

2. Fundamental Equation

Consider first an unconstrained multibody system whose configuration is described by the n generalized coordinates

$$q=[q_1, q_2, \dots, q_n]^T.$$

By unconstrained, we mean that the components $\dot{q}_i$ of the velocity of the system can be assigned independently at any given initial time, say $t = t_0$. The system equation of motion can be obtained, using Newtonian or Lagrangian mechanics, by the relations

$$M(q,t)\ddot{q} = Q(q,\dot{q},t), \quad q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0, \tag{1}$$

where the n by n matrix M is symmetric and positive definite. The matrix $M(q,t)$ and the generalized force n-vector (n by 1 matrix) $Q(q,\dot{q},t)$ are known. In this paper, by 'known' we mean known functions of their arguments. The generalized acceleration of the unconstrained system, which we denote by the n-vector a, is then given by

$$\ddot{q} = M^{-1}Q = a(q,\dot{q},t). \tag{2}$$

We suppose next that the system is subjected to h holonomic constraints of the form

$$\varphi_i(q,t) = 0, \quad i = 1, 2, \ldots, h, \tag{3}$$

and m–h nonholonomic constraints of the form

$$\varphi_i(q,\dot{q},t) = 0, \quad i = h+1, h+2, \ldots, m. \tag{4}$$

The initial conditions $q_0 = q(t = t_0)$ and $\dot{q}_0 = \dot{q}(t = t_0)$ are assumed to satisfy these constraints so that

$$\varphi_i(q_0, t_0) = 0, \quad i = 1, 2, \ldots, h,$$

and

$$\varphi_i(q_0, \dot{q}_0, t_0) = 0, \quad i = h+1, h+2, \ldots, m.$$

These constraints encompass all the usual holonomic and nonholonomic constraints (or combinations thereof) to which the multibody system may be subjected. We note that the constraints may be also explicit functions of the time and that the nonholonomic constraints may be nonlinear in the velocity components $\dot{q}_i$. Under the assumption of sufficient smoothness, we can differentiate equations (3) twice with respect to time and equations (4) once with respect to time to obtain the consistent equation set

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t), \tag{5}$$

where the constraint matrix A is a known m by n matrix and b is a known m-vector. It is important to note that, for a given set of initial conditions, the equation set (5) is equivalent to equations (3) and (4), which can be obtained by appropriately integrating the set (5).

The presence of the constraints (5) imposes additional constraint forces on the multibody system that alter its acceleration, so that the explicit equation of motion of the constrained system becomes

$$M\ddot{q} = Q(q, \dot{q}, t) + Q^c(q, \dot{q}, t). \tag{6}$$

The additional term Q^c on the right-hand side arises by virtue of the imposed constraints prescribed by equations (5).

We begin by generalizing the D'Alembert principle to include constraint forces that may do positive, negative, or zero work under virtual displacements.

We assume that, for any virtual displacement vector $v(t)$, the total work

$$W = v^T(t) Q^c(q, \dot{q}, t),$$

done by the constraint forces at each instant of time t, is prescribed (for the given, specific dynamical system under consideration) through the specification of a known n-vector $C(q, \dot{q}, t)$ such that

$$W = v^T(t) C(q, \dot{q}, t). \tag{7}$$

Equation (7) reduces to the usual D'Alembert principle when $C(t) \equiv 0$, for then the total work done under virtual displacements is prescribed to be zero and the constraints are then said to be ideal. In general, the prescription of C is the task of the mechanician who is modeling the specific constrained system whose equation of motion is to be found. It may be determined for the specific system at hand through experimentation, analogy with other systems, or otherwise. We include the situation here when the constraints may be ideal over certain intervals of time and nonideal over other intervals. Also, W at any given instant of time may be negative, positive, or zero, allowing us to include multibody systems where energy may be extracted from, or fed into, them through the presence of the constraints. We shall denote the acceleration of the unconstrained system subjected to this prescribed force C by

$$c(q, \dot{q}, t) = M^{-1} C.$$

In what follows, we shall omit the arguments of the various quantities, except when needed for clarity.

We begin by stating our result for the constrained multibody system described above. For convenience, we state it in two equivalent forms (Refs. 10–11).

(i) The explicit equation of motion that governs the evolution of the constrained system is

$$M\ddot{q} = Q + Q_i^c + Q_{ni}^c = Q + M^{1/2}B^+(b - Aa) + M^{1/2}(I - B^+B)M^{-1/2}C \tag{8}$$

$$\ddot{q} = a + M^{-1/2}B^+(b - Aa) + M^{-1/2}(I - B^+B)M^{1/2}c. \tag{9}$$

Equation (9) can be expressed also as

$$\Delta = \ddot{q} - a = M^{-1/2}B^+e + M^{-1/2}(I - B^+B)M^{1/2}c. \tag{10}$$

In Equations (8)–(10),

$$B = AM^{-1/2},$$

and B^+ denotes the Moore-Penrose inverse of the constraint matrix A (Ref. 12); $\Delta(t)$ denotes the deviation of the acceleration of the constrained system $\ddot{q}$ at time t from its unconstrained value $a(t)$ at that time; the quantity $e(t) := (b - Aa)$ represents the extent to which the acceleration a, at the time t, corresponding to the unconstrained motion does not satisfy the constraint equation (5). Later on, from a controls perspective, we will call $e(t)$ the error signal.

(ii) At each instant of time t, the total constraint force Q^c is made up of two additive parts. The first part Q_i^c is the constraint force that would have been generated were the constraints ideal at the time t; the second part Q_{ni}^c is created by the nonideal nature of the constraints at the time t. These two contributions to the total constraint force are given explicitly by

$$Q_i^c = M^{1/2}B^+(b - Aa), \tag{11}$$

$$Q_{ni}^c = M^{1/2}(I - B^+B)M^{-1/2}C, \tag{12}$$

where

$$Q^c = Q_i^c + Q_{ni}^c.$$

The subscripts i and ni refer to ideal and non ideal, respectively. When $C(t) \equiv 0$, the constraints are all ideal and then $Q^c = Q_i^c$.

2.1. Fundamental Principle Governing Motion. Equation (10) leads to the following new fundamental principle of motion for constrained multibody mechanical systems: "The motion of a discrete dynamical system subjected to constraints evolves, at each instant in time, in such a way that the deviation in its acceleration from what it would have at that instant if there were no constraints on it, is the sum of two M-orthogonal components; the first component is directly proportional to the extent e to which the accelerations corresponding to its unconstrained motion, at that instant, do not satisfy the constraints, the matrix of proportionality being $M^{-1/2}B^+$; the second component is proportional to the given n-vector c, the matrix of proportionality being $M^{-1/2}(I-B^+B)M^{1/2}$."

We define two n-vectors u and w to be M-orthogonal if

$$u^T M w = 0.$$

Since the Moore-Penrose inverse of a matrix B^+ may be unfamiliar to some, we provide here some of its properties, which will be used later on. Given an m by n matrix B, the n by m matrix B^+ is a unique matrix that satisfies the following four relations:

$$\text{(i)} \quad BB^+B = B, \tag{13a}$$

$$\text{(ii)} \quad B^+BB^+ = B^+, \tag{13b}$$

$$\text{(iii)} \quad (BB^+)^T = BB^+, \tag{13c}$$

$$\text{(iv)} \quad (B^+B)^T = B^+B. \tag{13d}$$

As stated in our fundamental principle above, the two components of the acceleration engendered by the presence of the constraints are given explicitly by the last two members on the right-hand side of equation (9). The M-orthogonality of these two members follows from the relations

$$\{(I-B^+B)^T M^{-1/2}\}M\{M^{-1/2}(B^+)\} = (I-B^+B)^T B^+ = (I-B^+B)B^+ = 0,$$

where we have used relation (13c) in the second equality and Equation (13b) in the last.

2.2. Derivation of the Fundamental Equation. The derivation of our result is as follows. The acceleration $\ddot{q}$ of the constrained system must satisfy two requirements. It must be such that:

(a) at each instant of time t, it must satisfy the constraints given by equation (5);

(b) the work W done under any virtual displacement by the constraint force Q^c must, at each instant of time t, be as prescribed by the relation (7).

Since we require the acceleration of the constrained system to satisfy the consistent set of equations

$$A\ddot{q} = A(\Delta + a) = B(M^{1/2}\Delta) + Aa = b,$$

we have, from the theory of generalized inverses,

$$M^{1/2}\Delta = B^+(b - Aa) + (I - B^+B)z, \tag{14}$$

where z is any arbitrary n-vector and B^+ is the Moore-Penrose inverse of the matrix $B = AM^{-1/2}$, whose properties are described in equations (13). From equation (14), we then have

$$\begin{aligned} M\ddot{q} &= Ma + M\Delta \\ &= Q + M^{1/2}B^+(b - Aa) + M^{1/2}(I - B^+B)z \\ &= Q + Q^c, \end{aligned} \tag{15}$$

so that

$$Q^c = M^{1/2}B^+(b - Aa) + M^{1/2}(I - B^+B)z. \tag{16}$$

To explicitly find Q^c, we determine next the second member on the right in equation (16) in such a way as to ensure that the second of the above-mentioned requirements is satisfied.

A virtual displacement at time t is any displacement that satisfies the relation

$$Av = 0$$

at that time (Ref. 13). Since

$$Av = B(M^{1/2}v) := B\mu,$$

the explicit solution of the homogeneous set of equations

$$B\mu = 0$$

is simply

$$M^{1/2}v = \mu = (I - B^+B)y, \tag{17a}$$

or

$$v = M^{-1/2}(I - B^+B)y, \tag{17b}$$

where y is any arbitrary n-vector. And so, from relation (7), we require that

$$\begin{aligned} W &= v^T Q^c \\ &= v^T [M^{1/2} B^+ (b - Aa) + M^{1/2} (I - B^+ B) z] \\ &= v^T C, \end{aligned} \tag{18}$$

where, at each instant of time, C is specified by the mechanician who is modeling the specific mechanical system. Using Equation (17) in the last equality in (18), we get

$$\begin{aligned} & y^T (I - B^+ B)^T M^{-1/2} [M^{1/2} B^+ (b - Aa) + M^{1/2} (I - B^+ B) z] \\ &= y^T (I - B^+ B)^T M^{-1/2} C, \end{aligned} \tag{19}$$

which yields, because y is arbitrary,

$$\begin{aligned} (I - B^+ B) z &= (I - B^+ B)^T M^{-1/2} C \\ &= (I - B^+ B) M^{-1/2} C. \end{aligned} \tag{20}$$

Relation (20) follows from (19) through the use of relations (13d) and (13b) because

$$\begin{aligned} (I - B^+ B)^T M^{-1/2} M^{1/2} B^+ &= [I - (B^+ B)^T] B^+ \\ &= [I - (B^+ B)] B^+ = 0, \end{aligned} \tag{21}$$

and

$$\begin{aligned} (I - B^+ B)^T (I - B^+ B) &= (I - B^+ B)(I - B^+ B) \\ &= (I - B^+ B). \end{aligned} \tag{22}$$

Using (20), we get

$$M^{1/2} (I - B^+ B) z = M^{1/2} (I - B^+ B) M^{-1/2} C, \tag{23}$$

which when used in the second member on the right in Equation (16) gives

$$Q^c = M^{1/2} B^+ (b - Aa) + M^{1/2} (I - B^+ B) M^{-1/2} C, \tag{24}$$

and the result given by Equation (8) now follows from Equation (15).

The explicit equations of motion obtained herein, like those obtained earlier for ideal constraints (Ref. 7), are completely innocent of the notion of Lagrange multipliers. Over the last 200 years, Lagrange multipliers have been so widely used in the development of the equations of motion of constrained multibody systems that it is sometimes tempting to mistakenly believe that they have an instrinsic presence in the description of constrained motion. This is not true. As shown in this paper, neither in the formulation of the physical problem of the motion of constrained multibody systems nor in the equations governing their motion are any Lagrange multipliers involved. The use of Lagrange multipliers [a mathematical tool invented by Lagrange (Ref. 1)] constitutes just one of the several intermediary mathematical devices invented for handling constraints. In fact, the direct use of this device appears difficult when the constraints are functionally dependent. Lagrange multipliers do not appear in the physical description of constrained motion; therefore, they cannot and do not ultimately appear in the equations governing such motion.

3. Discussion and Conclusions

The simplicity of the general explicit equation of motion obtained herein relies on the interplay of four central observations:

(i) No transformation of coordinates or their elimination is undertaken when constraints are present; the coordinates in which the unconstrained multibody system is described are the same as those used to describe the constrained system. At first, this appears to be counterintuitive and indeed goes against a 200 year-old, well-accepted current of practice in dynamics and theoretical physics that was first initiated by Lagrange. Such coordinate transformations and eliminations are often useful in handling problems of mathematical physics. However, it is the fact that we do not use them that appears to be ultimately responsible for the simplicity of the explicit equation obtained herein and the fundamental insights about the nature of constrained motion provided by it.

(ii) The constraints are described in their differentiated form by equation (5); this a consequence of the realization that, at any instant of time t, the state of the system $(q(t), \dot{q}(t))$ is assumed known, and it is the state immediately following this instant that must then be the focus of our inquiry. Our attention must then naturally focus on the system acceleration $\ddot{q}$.

(iii) For a physical system where the constraint forces do work, the equations of motion cannot be obtained solely through knowledge of the kinematical nature of the constraints as described by equations (3) and (4); one needs to have an additional dynamical characterization of the constraints given by the extension of the D'Alembert principle or some equivalent of it, as stated in Equation (7). Such a characterization yields a unique equation of motion, as expected from and consistent with practical observation.

(iv) The Moore-Penrose inverse of a matrix shows an intrinsic presence in the equations of motion. It manages to sort out the manner in which the constraints interact with the given forces [known acceleration $a(t)$] to yield an equation of motion that is both simple and provides new physical insights.

Lastly, it is worth mentioning that the general equations of motion obtained here have immediate application to the tracking control of nonlinear multibody systems (Refs. 14–15), a problem that has been worked on for many decades, with weak success, by control theorists. The constraint force Q^c can be interpreted as the control force required to be applied to the nonlinear multibody system which is described by Equation (1) so that it exactly satisfies the trajectory requirements imposed by equation (5) [equivalently, by equations (3) and (4)] at each instant of time. One then obtains a closed-form control force given by equation (24). And this for a general, nonlinear multibody system! In fact, this control force is exactly what Nature would use, were it required to satisfy the constraint equations (3) and (4) (also thought of now as the trajectory requirements!) along with relation (7). Furthermore, were we to set $C \equiv 0$ (the ideal constraint case), we would obtain the force Nature would employ to control the nonlinear multibody system described by equation (1) with (ideal) constraints described by equations (3) and (4). We would then have

$$M\ddot{q} = Q + Q^{\text{control}} = Q + M^{1/2}B^{+}(b - Aa). \tag{25}$$

And so we see that Nature appears to be actually behaving much as a control engineer would! The second member on the right in Equation (25) can be thought of as providing a feedback control, using a feedback proportional to the error signal $e(t) := (b - Aa)$, which measures the extent to which the acceleration that we know at time t, namely $a(t)$, does not satisfy our trajectory requirement (5). However, it is in the choice of the gain matrix $M^{1/2}(q,t)B^{+}(q,\dot{q},t)$ that Nature seems to really excel! She picks the control gain with incredible ingenuity so as to exactly satisfy the trajectory requirement (5) at each instant of time. It is the choice of this matrix, which in general is a highly nonlinear function of $q, \dot{q}, t$, that would most likely baffle our best control theorists! Such reinterpretations of the equations in this paper within the framework of control theory show their considerable scope of applicability and utility. The details of this approach to the control of multibody systems (accuracy and robustness, etc.) would take us far afield here. The interested reader may find them in Refs. 14 and 15.

In conclusion, we have extended the Lagrangian formulation of mechanics to include constraints that may be ideal and/or nonideal, and the equations of motion presented in this paper are applicable to multibody mechanical systems that include such constraints. They appear to be the simplest and most comprehensive equations of motion so far discovered for such systems. They point to new and novel ways of controlling complex, nonlinear mechanical systems.

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