NORTH-HOLLAND

Inequality Constraints in the Process of Jumping

Cinthia Itiki*

Biomedical Engineering Department
University of Southern California
Los Angeles, California 90089-1453

Robert Kalaba

Biomedical Engineering and Economics Departments
University of Southern California
Los Angeles, California 90089-1453

and

Firdaus Udwadia

Civil and Mechanical Engineering Departments
University of Southern California
Los Angeles, California 90089-1453

Transmitted by F. E. Udwadia

ABSTRACT

This work presents a study of jumping through a biomechanical model of a leg, which is subjected to an inequality constraint. The activation and deactivation of an equality constraint reproduce the inequality constraint. Within the framework of the generalized inverse equations of motion, it is shown that the activation of this additional constraint can be implemented by Greville's formulae.

* On leave of absence from Universidade de São Paulo, EPUSP, DEE.

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INTRODUCTION

Consider one leg represented by a mechanical system with three particles, as in Fig. 1. The foot is represented by mass m_1 and coordinates (x_1, y_1). The shank has mass m_2 and its middle point has coordinates (x_2, y_2). The thigh is described by its middle point coordinates (x_3, y_3) and mass m_3. The knee has coordinates $(2x_2 - x_1, 2y_2 - y_1)$. Gravity pulls all the three masses down.

The jump is such that both, the middle point of the thigh and the foot, go straight up. These conditions are described by the constraint equations

$$x_3(t) = 0, \tag{1}$$

and

$$x_1(t) = 0. \tag{2}$$

Two distance constraints define the lengths of the shank (L_S) and thigh (L_T)

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - \left(\frac{L_S}{2}\right)^2 = 0, \tag{3}$$

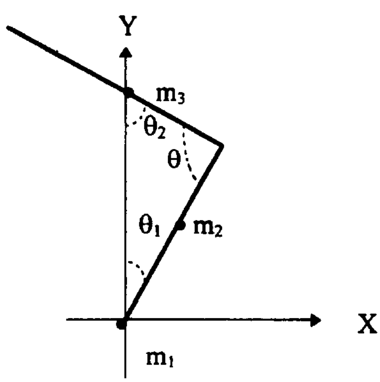

FIG. 1. Biomechanical model of the leg.

and

$$\{x_3 - (2x_2 - x_1)\}^2 + \{y_3 - (2y_2 - y_1)\}^2 - \left(\frac{L_T}{2}\right)^2 = 0. \tag{4}$$

The angle between thigh and shank (θ) is given by the following function of time

$$\theta(t) = \pi - 0.7\pi\left(1 + \tau t + \frac{\tau^2}{2}t^2\right)e^{-\tau t}. \tag{5}$$

Since $\theta = \pi - \theta_1 - \theta_2$, we obtain

$$\sin\theta_1 \sin\theta_2 - \cos\theta_1 \cos\theta_2 = \cos\theta, \tag{6}$$

and the resultant constraint equation is

$$(x_1 - x_2)(x_3 - 2x_2 + x_1) + (y_1 - y_2)(y_3 - 2y_2 + y_1) = \frac{L_T L_S}{4}\cos\theta. \tag{7}$$

The last constraint is an inequality constraint on the vertical coordinate of the foot

$$y_1 \geqslant 0. \tag{8}$$

The problem consists in handling the inequality, so that we may predict when the foot m_1 lifts off. While the foot stays on the ground, this is a kinematic problem. At and after lift-off, it becomes a dynamic one. This work deals with the problem of the foot lifting off, and it does not include the problem of impact, when the foot comes back to the ground.

INEQUALITY CONSTRAINTS AND GREVILLE'S FORMULAE

The inequality constraint plays two different roles, depending on the position of the foot (on the ground or in the air). When the foot tries to go below the ground, the constraint becomes an equality $y_1 = 0$, for the foot will stay on the ground and not go below it. Whenever the foot is above the ground, there is no constraint regarding its vertical coordinate. In other words, y_1 could have any value, as long as the foot stays in the air.

One way of determining the time when the foot lifts off is based on the estimation of the position of the foot. Let us assume that the foot is on the ground at time t. Based on the foot's current position and velocity, we can obtain its acceleration for the case when there is no constraint on its vertical coordinate. Using the acceleration value, we may estimate the next vertical coordinate of the foot $y_1(t+h)$, which can be obtained by using the approximation $y_1(t+h) = y_1(t) + h\,\dot{y}_1(t) + (h^2/2)\,\ddot{y}_1(t)$. The obtained estimate may take any value. If it is a positive value, then the foot lifts off, and there is no constraint on its vertical coordinate. However, if the predicted position of the foot is negative, we should disregard the calculated acceleration and the foot position estimate. This is due to the inequality constraint, which states that the foot cannot go below the ground. A new acceleration should be obtained by including an equality constraint on the vertical coordinate of the foot $y_1 = 0$. The resultant foot position would satisfy all the constraints.

The acceleration of constrained systems may be obtained by several methods. In our work, we use the generalized inverse method [1]. The constraint equations are differentiated twice, so that linear relationships on the acceleration components ($A\ddot{X} = B$) are obtained. To handle the inequality constraint, we define two systems of constraints. The first one does not include a constraint on the vertical coordinate of the foot, and the second one does. The first set of constraints results in the following linear relationships on the acceleration components

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 \\ a_{4,1} & a_{4,2} & -2\,a_{4,1} & -2\,a_{4,2} & a_{4,1} & a_{4,2} \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{x}_2 \\ \ddot{y}_2 \\ \ddot{x}_3 \\ \ddot{y}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -(\dot{x}_2 - \dot{x}_1)^2 - (\dot{y}_2 - \dot{y}_1)^2 \\ -(\dot{x}_3 - 2\,\dot{x}_2 + \dot{x}_1)^2 - (\dot{y}_3 - 2\,\dot{y}_2 + \dot{y}_1)^2 \\ b_5 \end{bmatrix} \tag{9}$$

where

$$a_{4,1} = (x_3 - 2x_2 + x_1),$$

$$a_{4,2} = (y_3 - 2y_2 + y_1),$$

$$a_{5,1} = (x_3 - 3x_2 + 2x_1),$$

$$a_{5,2} = (y_3 - 3y_2 + 2y_1),$$

$$a_{5,3} = (-x_3 + 4x_2 - 3x_1),$$

$$a_{5,4} = (-y_3 + 4y_2 - 3y_1),$$

$$a_{5,5} = (x_1 - x_2),$$

$$a_{5,4} = (y_1 - y_2),$$

and

$$b_5 = -2\{(\dot{x}_1 - \dot{x}_2)(\dot{x}_1 - 2\dot{x}_2 + \dot{x}_3) + (\dot{y}_1 - \dot{y}_2)(\dot{y}_1 - 2\dot{y}_2 + \dot{y}_3)\}$$
$$- \frac{L_S L_T}{4}(\dot{\theta}^2 \cos\theta + \ddot{\theta}\sin\theta).$$

Let us define A_5 and B_5 as the matrix A and vector B given by Eq. (9), for the system with five constraints.

The second set of constraint equations includes the constraint on the vertical coordinate of the foot. This last constraint inserts a new row a_6 on matrix A and a new scalar b_6 on vector B. Matrix A_6 and vector B_6 are given by

$$A_6 = \begin{bmatrix} A_5 \\ a_6 \end{bmatrix}, \tag{10}$$

and

$$B_6 = \begin{bmatrix} B_5 \\ b_6 \end{bmatrix}, \tag{11}$$

where $a_6 = [0 \ 1 \ 0 \ 0 \ 0 \ 0]$ and $b_6 = 0$.

The generalized inverse method involves the calculation of the Moore–Penrose inverse matrix $(AM^{-1/2})^{+}$, where the mass matrix M is a diagonal matrix with elements $\{m_1, m_1, m_2, m_2, m_3, m_3\}$. Since we have to obtain the Moore–Penrose inverse matrix for both systems of constraint equations, it would be highly desirable to have a formula to calculate $(A_6 M^{-1/2})^{+}$ based on the value of $(A_5 M^{-1/2})^{+}$.

Greville [2] calculates the Moore–Penrose inverse matrix by inserting one column of the original matrix at a time. In our case, each new constraint adds a new row to the matrix A. To use Greville's formulae, we need to make small alterations to the procedure. Let us define a new matrix C, which is equal to $(AM^{-1/2})$. Matrices C_5 and C_6 for the systems with five and six constraints respectively, are given by

$$C_5 = \left(A_5 M^{-1/2} \right) \tag{12}$$

and

$$C_6 = \left(A_6 M^{-1/2} \right) = \begin{bmatrix} C_5 \\ c_6 \end{bmatrix}, \tag{13}$$

where $c_6 = (a_6 M^{-1/2})$.

Applying Greville's formulae [3] we obtain

$$C_6^{+} = \left[C_5^{+} - d_6^{+} c_6 C_5^{+},\ d_6^{+} \right], \tag{14}$$

where $d_6 = c_6(I - C_5^{+} C_5)$, if $c_6 \neq c_6 C_5^{+} C_5$; or

$$d_6 = c_6\left(C_5^T C_5\right)^{+} \left\{1 + c_6\left(C_5^T C_5\right)^{+} c_6^T\right\} / \left\{c_6\left(C_5^T C_5\right)^{+} \left(C_5^T C_5\right)^{+} c_6^T\right\},$$

$$\text{if } c_6 = c_6 C_5^{+} C_5. \tag{15}$$

NUMERICAL INTEGRATION

Numerical integrations were performed, using a test on the estimate of the vertical coordinate of the foot. The time step size was 0.002 sec. The foot, shank, and thigh masses were taken to be 0.5, 1.5, and 3.0 kg, respectively. The shank and thigh lengths were 0.4 and 0.5 m, respectively.

Given the initial value of the angle $\theta(0) = 0.3\pi$, we may obtain the initial values of the angles

$$\theta_1(0) = \mathrm{atan}((\sin(\theta(0))\, L_T/2)/(L_S - \cos(\theta(0))\, L_T/2))$$

and

$$\theta_2(0) = \mathrm{atan}((\sin(\theta(0))\, L_S)/(L_T/2 - \cos(\theta(0))\, L_S)).$$

The initial positions are given in terms of $\theta_1(0)$, $\theta_2(0)$, and the lengths of the shank (L_S) and thigh (L_T):

$$x_1(0) = 0; \tag{16}$$

$$y_1(0) = 0; \tag{17}$$

$$x_2(0) = x_1(0) + (L_S/2)\sin(\theta_1(0)); \tag{18}$$

$$y_2(0) = y_1(0) + (L_S/2)\cos(\theta_1(0)); \tag{19}$$

$$x_3(0) = 0; \tag{20}$$

$$y_3(0) = (2\,y_2(0) - y_1(0)) + (L_T/2)\cos(\theta_2(0)). \tag{21}$$

The function $\theta(t)$ is such that $\dot{\theta}(0) = 0$. Consequently, all the initial velocities $\dot{x}_1(0)$, $\dot{y}_1(0)$, $\dot{x}_2(0)$, $\dot{y}_2(0)$, $\dot{x}_3(0)$, and $\dot{y}_3(0)$ are equal to zero. The integration was performed for $\tau = 25$ sec^{-1} and $\tau = 25$ sec^{-1}.

Figures 2 and 3 show the horizontal components of the trajectories of the foot (x_1) and thigh (x_3), for $\tau = 25$ sec^{-1}. Observe that the maximum

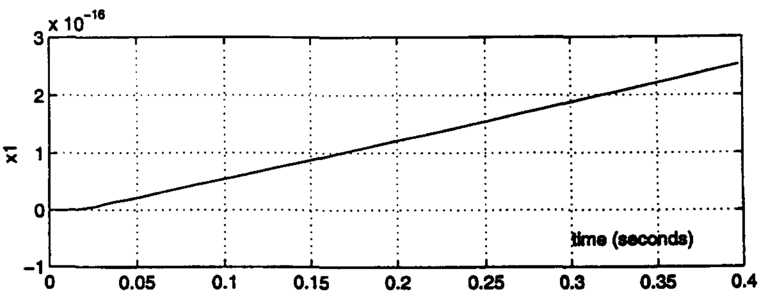

FIG. 2. Horizontal coordinate of the foot.

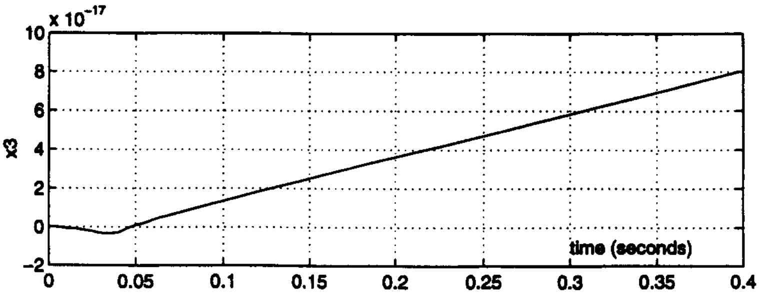

FIG. 3. Horizontal coordinate of the thigh.

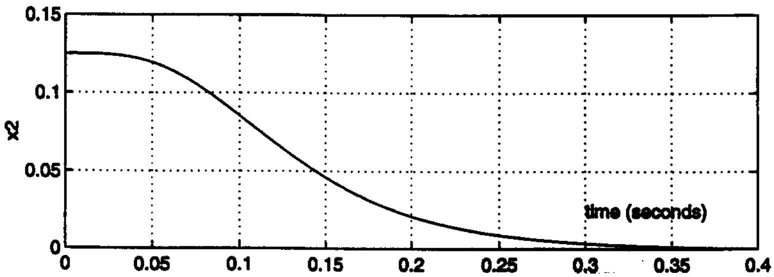

FIG. 4. Horizontal coordinate of the shank.

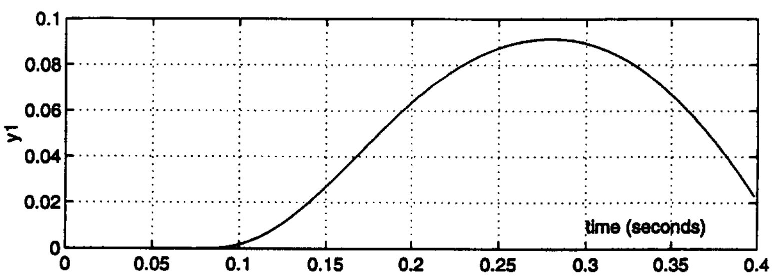

FIG. 5. Vertical coordinate of the foot in jumping.

values of x_1 and x_3 do not exceed 3×10^{-16}, which is about the machine precision. This means that both the foot and the thigh go straight up, without horizontal oscillations. The horizontal trajectory of the shank is given in Fig. 4. The middle point of the shank approximates the vertical axis, as time passes. In Fig. 5, the vertical coordinate of the foot is zero for a

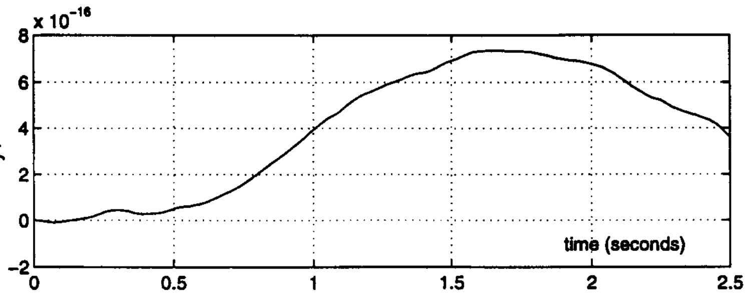

FIG. 6. Vertical coordinate of the foot in standing up.

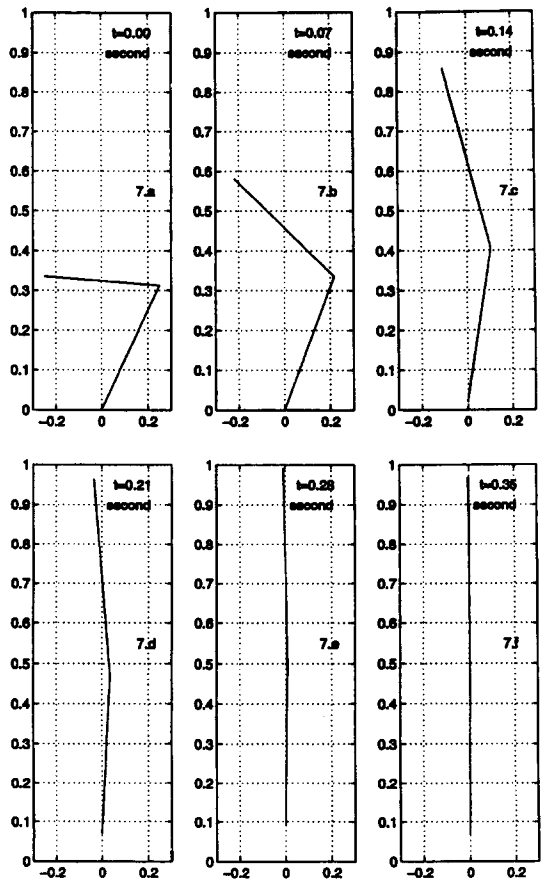

FIG. 7. Jumping frames.

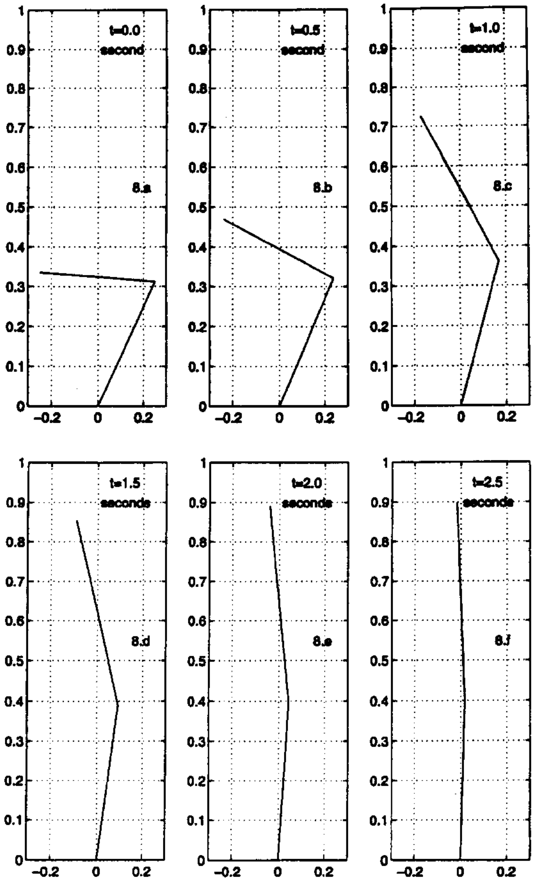

FIG. 8. Standing up frames.

while, then the foot lifts off and reaches a maximum height of about 9 cm. It is interesting to observe that for slower changes in θ ($\tau = 2.5\ \mathrm{sec}^{-1}$), the foot does not lift off. In Fig. 6, the maximum value of the vertical coordinate y_1 is about 8×10^{-16}.

Figures 7a to 7f show how the leg behaves during the jumping process. One may see clearly that the foot lifts off. In contrast, Figs, 8a to 8f show the frames for standing up, when the foot does not lift off.

CONCLUSION

The generalized inverse method allows us to use Greville's formulae, which is an elegant way of inserting a new constraint. This methodology is specially useful in the presence of an inequality constraint, which can be expressed as an equality constraint that is active or inactive, depending on the state of the system.

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REFERENCES

1 F. E. Udwadia and R. E. Kalaba, A new perspective on constrained motion, *Proc. Roy. Soc. London.* 439:407–410 (1992).

2 T. N. E. Greville, Some applications of the pseudoinverse of a matrix, *SIAM Rev.* 2(1):15–22 (1960).

3 F. A. Graybill, *Matrices with Applications in Statistics*, 2nd ed., Wadsworth, Pacific Grove, 1983, pp. 118–119.