Firdaus E. Udwadia
Department of Aerospace and Mechanical Engineering,
University of Southern California,
430 K Olin Hall,
Los Angeles, CA 90089

George Leitmann
Department of Mechanical Engineering,
University of California,
Berkeley, CA 94720

Hancheol Cho
Department of Aerospace and Mechanical Engineering,
University of Southern California,
Los Angeles, CA 90089

Some Further Remarks on Hamilton's Principle

This paper deals with the inverse problem of finding a suitable integrand so that upon the use of the calculus of variations, one obtains the equations of motion for systems in which the forces are nonpotential. New extensions and generalizations of previous results are obtained. [DOI: 10.1115/1.4002435]

1 Introduction

The development of the equations of motion for a mechanical system from Hamilton's principle can be viewed as a problem in the calculus of variations when the constraints on the system are holonomic and the forces are derivable from a potential function. Rendering stationary the integral of the Lagrangian over a fixed interval of time taken between two fixed points in configuration space, then, yields the equations of motion for the system. However, it is interesting to investigate the types of forces that can be engendered through the use of an appropriately chosen function (integrand) whose integral when rendered stationary, yields the proper equations of motion even when the forces acting on a system do not arise from a potential, that is, when they are nonconservative.

In 1931, Bolza [1] gave a general procedure for finding such an integrand for a single degree-of-freedom system. This was followed by Douglas [2,3] who obtained the necessary and sufficient conditions for the existence of an integrand for multidegree-of-freedom systems. However, it is difficult to obtain the integrands for given, specific forces. In a 1963 note, Leitmann [4] provided some examples of such forces and the corresponding integrands for which a variational principle exists. A single degree-of-freedom system was considered and two examples were provided. Recently, the so-called semi-inverse method [5] has attracted much attention due to its simplicity and applicability to certain cases. However, this method assumes a specific form for the function f, which needs to be obtained from experience, intuition, or both, and utilizes a Lagrange multiplier type approach.

In this paper, we extend the results in Ref. [4] to some more general nonpotential systems and provide a more systematic way of handling the inverse problem of the calculus of variations. The main difficulties lie in performing the necessary integrations explicitly, as will be seen. The examples given in Ref. [4] arise as special cases of the results provided herein. Finally, we apply the general results to some specific systems to indicate the nature of the nonpotential forces, which the results encompass.

2 The Solution to the Inverse Problem

This section deals with the so-called inverse problem of the calculus of variations. Let $G[t,x(t),\dot{x}(t)]$ be a force acting on an unconstrained single degree-of-freedom system whose mass is taken to be unity. We assume that $G(\cdot)$ is of class C^1 on R^3. Our aim is to find the function $f(\cdot):[t_1,t_2]\times R^1\times R^1\rightarrow R^1$ so that the functional

$$\int_{t_1}^{t_2} f[t,x(t),\dot{x}(t)]dt \quad \text{with} \quad x(t_1)=x_1 \quad \text{and} \quad x(t_2)=x_2 \tag{1}$$

when rendered stationary, yields the Euler–Lagrange equation of motion

$$\ddot{x}(t)=G[t,x(t),\dot{x}(t)] \tag{2}$$

We assume that the general solution $x(t)=g(t,a,b)$ of Eq. (2) passes for some values of a and b through $x(t_1)=x_1$ and $x(t_2)=x_2$. Noting that $\dot{x}(t)=g_t(t,a,b)$, we further assume that one can solve for a and b from the expressions for $x(t)$ and $\dot{x}(t)$ so that $a=\phi[t,x(t),\dot{x}(t)]$ and $b=\psi[t,x(t),\dot{x}(t)]$, where $\phi(\cdot)$ and $\psi(\cdot)$ are the continuous functions from $[t_1,t_2]\times R^2\rightarrow R^1$. Then, $\ddot{x}(t)=g_{tt}(t,a,b)$ and $G(\cdot):[t_1,t_2]\times R^2\rightarrow R^1$ takes the values $G(t,x,r)=g_{tt}[t,\phi(t,x,r),\psi(t,x,r)]$, where t, x, and r are considered to be independent variables.

Thus, we need to determine the integrand function $f(\cdot)$, i.e., $(t,x,r)\rightarrow f(t,x,r)$, such that Eq. (2) is identical with the Euler–Lagrange equation

$$\frac{\partial^2 f}{\partial t\,\partial r}+\frac{\partial^2 f}{\partial x\,\partial r}\dot{x}(t)+\frac{\partial^2 f}{\partial r^2}\ddot{x}(t)-\frac{\partial f}{\partial x}=0 \tag{3}$$

where the partial derivatives of $f(\cdot)$ are evaluated at $[t,x(t),\dot{x}(t)]$. Substituting $\ddot{x}(t)$ for Eq. (2) into Eq. (3), we get

$$\frac{\partial^2 f}{\partial t\,\partial r}+\frac{\partial^2 f}{\partial x\,\partial r}\dot{x}(t)+\frac{\partial^2 f}{\partial r^2}G-\frac{\partial f}{\partial x}=0 \tag{4}$$

Since Eq. (2) gives the solution for every initial condition pair $[x(t_1),\dot{x}(t_1)]$, the equation

$$\frac{\partial^2 f}{\partial t\,\partial r}+\frac{\partial^2 f}{\partial x\,\partial r}r+\frac{\partial^2 f}{\partial r^2}G(t,x,r)-\frac{\partial f}{\partial x}=0 \tag{5}$$

must be an identity for all $(t,x,r)\in[t_1,t_2]\times R^1\times R^1$. To get an explicit expression for $f(t,x,r)$, we partially differentiate Eq. (5) with respect to r to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 f}{\partial r^2}\right)+\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial r^2}\right)r+\frac{\partial}{\partial r}\left(\frac{\partial^2 f}{\partial r^2}\right)G+\frac{\partial^2 f}{\partial r^2}\frac{\partial G}{\partial r}=0 \tag{6}$$

Contributed by the Applied Mechanics of ASME for publication in the Journal of Applied Mechanics. Manuscript received March 8, 2010; final manuscript received July 27, 2010; accepted manuscript posted August 24, 2010; published online October 22, 2010. Assoc. Editor: Wei-Chau Xie.

Upon defining $M(t,x,r) \triangleq (\partial^2/\partial r^2)f(t,x,r)$, Eq. (6) becomes the partial differential equation

$$\frac{\partial M}{\partial t} + \frac{\partial M}{\partial x} r + \frac{\partial M}{\partial r} G + M \frac{\partial G}{\partial r} = 0 \tag{7}$$

whose solution is obtained as [1]

$$M(t,x,r) = \Phi[\phi(t,x,r), \psi(t,x,r)]\Theta[t, \phi(t,x,r), \psi(t,x,r)] \tag{8}$$

where $\Phi(\cdot)$ is an arbitrary function of t, $\phi(t,x,r)$ and $\psi(t,x,r)$, and

$$\Theta(t,a,b) \triangleq \exp\left(-\int \frac{\partial G}{\partial r}[t, g(t,a,b), g_t(t,a,b)]dt\right) \tag{9}$$

where $\partial G/\partial r[t,g(t,a,b),g_t(t,a,b)] \triangleq (\partial/\partial r)G(t,x,r)|_{\substack{x=g(t,a,b)\\ r=g_t(t,a,b)}} > 0$. Finally, the function $f(\cdot)$ is found by integrating $M(t,x,r)$ twice with respect to r

$$f(t,x,r) = \int\int M(t,x,\tilde{r})d\tilde{r}dr + r\lambda(t,x) + \mu(t,x) \tag{10}$$

where $\lambda(t,x)$ and $\mu(t,x)$ are arbitrary within the requirement of Eq. (5). A similar derivation for the function $f(t,x,r)$ is also given in Refs. [1,4,6]. The notation used here and our treatment closely follows Ref. [6].

As a simple example, let us consider the following linear oscillator equation:

$$\ddot{x}(t) = -cx(t) - 2d\dot{x}(t) \tag{11}$$

where c and d are the positive constants with the initial conditions $x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$. For simplicity, let us assume $d^2-c>0$, the overdamped case. Then, it is straightforward to show that

$$\phi(t,x,r) = \frac{1}{\xi_2 - \xi_1} e^{-(\xi_1+\xi_2)t}[(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})x + (e^{\xi_1 t} - e^{\xi_2 t})r] \tag{12a}$$

$$\psi(t,x,r) = \frac{1}{\xi_2 - \xi_1} e^{-(\xi_1+\xi_2)t}[\xi_1\xi_2(e^{\xi_2 t} - e^{\xi_1 t})x + (\xi_2 e^{\xi_1 t} - \xi_1 e^{\xi_2 t})r] \tag{12b}$$

where $\xi_1 \triangleq -d + \sqrt{d^2-c}$ and $\xi_2 \triangleq -d - \sqrt{d^2-c}$. According to Eq. (9), we have

$$\Theta(t,x,r) = e^{2dt} \tag{13}$$

Using Eq. (10), we have the following integrand:

$$f(t,x,r) = e^{2dt}\int\int \Phi[\phi(t,x,\tilde{r}), \psi(t,x,\tilde{r})]d\tilde{r}dr + r\lambda(t,x) + \mu(t,x) \tag{14}$$

Since $\Phi(\cdot)$ is an arbitrary function of its arguments, we may use

$$\Phi(t,x,r) = \phi(t,x,r) = \frac{1}{\xi_2 - \xi_1} e^{-(\xi_1+\xi_2)t}[(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})x + (e^{\xi_1 t} - e^{\xi_2 t})r] \tag{15}$$

Substituting Eq. (15) into Eq. (14) yields

$$f(t,x,r) = \frac{1}{\xi_2 - \xi_1} e^{4dt}\left[\frac{1}{2}(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})xr^2 + \frac{1}{6}(e^{\xi_1 t} - e^{\xi_2 t})r^3\right] + r\lambda(t,x) + \mu(t,x) \tag{16}$$

where $\lambda(t,x)$ and $\mu(t,x)$ should be chosen so that $f(t,x,r)|_{\substack{x=x(t)\\ r=\dot{x}(t)}}$ in Eq. (14) satisfies the Euler–Lagrange equation. For the sake of simplicity, we can choose the following:

$$\lambda(t,x) = 0, \quad \mu(t,x) = \frac{c}{3(\xi_1 - \xi_2)} e^{4dt}(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})x^3 \tag{17}$$

Finally, the integrand function $f(\cdot)$, which yields the force $G(\cdot)$ in Eq. (11), is given by

$$\begin{aligned} f(t,x,r) &= \frac{1}{\xi_2 - \xi_1} e^{4dt}\left[\frac{1}{2}(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})xr^2 + \frac{1}{6}(e^{\xi_1 t} - e^{\xi_2 t})r^3\right] \\ &\quad - \frac{c}{3(\xi_2 - \xi_1)} e^{4dt}(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})x^3 = \frac{1}{6(\xi_2 - \xi_1)} e^{4dt}[(\xi_2 e^{\xi_2 t} \\ &\quad - \xi_1 e^{\xi_1 t})x(3r^2 - 2cx^2) + (e^{\xi_1 t} - e^{\xi_2 t})r^3] \end{aligned} \tag{18}$$

Likewise, if $\Phi(t,x,r)=\psi(t,x,r)$ for the linear overdamped system, one other possible integrand function is given by

$$\begin{aligned} f(t,x,r) &= \frac{1}{\xi_2 - \xi_1} e^{4dt}\left[\frac{1}{2}\xi_1\xi_2(e^{\xi_2 t} - e^{\xi_1 t})xr^2 + \frac{1}{6}(\xi_2 e^{\xi_1 t} - \xi_1 e^{\xi_2 t})r^3\right] \\ &\quad - \frac{c}{3(\xi_2 - \xi_1)} e^{4dt}\xi_1\xi_2(e^{\xi_2 t} - e^{\xi_1 t})x^3 \\ &= \frac{1}{6(\xi_2 - \xi_1)} e^{4dt}[\xi_1\xi_2(e^{\xi_2 t} - e^{\xi_1 t})x(3r^2 - 2cx^2) + (\xi_2 e^{\xi_1 t} \\ &\quad - \xi_1 e^{\xi_2 t})r^3] \end{aligned} \tag{19}$$

If $\Phi(t,x,r)=[\phi(t,x,r)]^2$ is chosen, we have

$$\begin{aligned} f(t,x,r) &= \frac{1}{(\xi_2 - \xi_1)^2} e^{6dt}\left[\frac{1}{2}(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})^2 x^2 r^2 + \frac{1}{12}(e^{\xi_2 t} - e^{\xi_1 t})^2 r^4 \right. \\ &\quad \left. - \frac{1}{3}(e^{\xi_2 t} - e^{\xi_1 t})(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})xr^3\right] \\ &\quad - \frac{c}{4(\xi_2 - \xi_1)^2} e^{6dt}(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})^2 x^4 \\ &= \frac{1}{12(\xi_2 - \xi_1)^2} e^{6dt}[3(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})^2 x^2(2r^2 - cx^2) + (e^{\xi_2 t} \\ &\quad - e^{\xi_1 t})^2 r^4 - 4(e^{\xi_2 t} - e^{\xi_1 t})(\xi_2 e^{\xi_2 t} - \xi_1 e^{\xi_1 t})xr^3] \end{aligned} \tag{20}$$

We can verify that the integrand $f(t,x,r)$ given in Eqs. (18)–(20) is correct by directly substituting it into the Euler–Lagrange equation (3). We have, thus, illustrated three (out of an infinite number) different integrand functions $f(\cdot)$ that yield the force $G(\cdot)$ given in Eq. (11).

3 Variational Principle for Systems With Nonpotential Forces

This section deals with the derivation of some special forms of nonpotential forces that can be obtained through the use of the variational calculus by appropriately choosing the integrand functions. Application of the variational calculus will, thus, yield the proper equation of motion for such nonpotential systems. It is noted that in order to find integrand function $f(\cdot)$ in Eq. (1) of the variational problem, the general solution $a=\phi[t,x(t),\dot{x}(t)]$ and $b=\psi[t,x(t),\dot{x}(t)]$ must be known in order to use a prescribed function $\Phi(\cdot)$. However, the use of $\Phi(t,x,r)\equiv 1$ while narrowing down the available functions $f(\cdot)$ averts this difficulty [4]. It is obvious, then, that for $f(t,x,r)$ to be explicitly determined, $\int(\partial G(t,x,r)/\partial r)dt$ in Eq. (9) and $\int\int M(t,x,\tilde{r})d\tilde{r}dr$ in Eq. (10) should be suitably integrable and that the main difficulties in finding the function $f(\cdot)$, therefore, analytically originate from these two integrations.

With this in mind, we next consider a general form for $\partial G(t,x,r)/\partial r$ given by

$$\frac{\partial G(t,x,r)}{\partial r} = g_1(t) + g_2(x)r + g_3(r)G(t,x,r) \tag{21}$$

so that $\int(\partial G(t,x,r)/\partial r)dt$ can be easily obtained and we can avoid the difficulty of integrating $\int\int M(t,x,\tilde{r})d\tilde{r}dr$ by separating the variables. Equation (21) leads us to assume the following form for $G(t,x,r)$:

$$G(t,x,r) = \beta(t)r + \gamma(x)r^2 + \alpha(t,x)\delta(r) + \bar{\alpha}(t,x) \tag{22}$$

where α, $\bar{\alpha}$, β, γ, and δ are the arbitrary (suitably smooth) functions of their arguments. We, then, explore various cases in which $G(t,x,r)$ has the form given in Eq. (22).

As stated before, although the general solution of the nonlinear differential equation (2) with $G(t,x,r)$ given by Eq. (22) is hard to find, we can avoid the difficulty by setting $\Phi(t,x,r)\equiv 1$ in Eq. (8) [4]. If we choose $G(t,x,r)=\beta(t)r$, then, $(\partial/\partial r)G(t,x,r)=\beta(t)$, which is of the form of Eq. (21) in which $g_1(t)=\beta(t)$, $g_2(x)=0$, and $g_3(r)=0$. Thus, for a nonpotential force of the form $G(t,x,r)=\beta(t)r$ acting on a particle of unit mass, we can find the integrand $f(t,x,r)=\exp(-\int\beta(t)dt)r^2/2$, which when used to make the integral given in Eq. (1) stationary, yields the Euler–Lagrange equation $\ddot{x}(t)=G[t,x(t),\dot{x}(t)]=\beta(t)\dot{x}(t)$. In a similar manner, $G(t,x,r)=\gamma(x)r^2$ is another possible nonpotential force since $(\partial/\partial r)G(t,x,r)=2\gamma(x)r$ has the form of Eq. (21) if we let $g_1(t)=0$, $g_2(x)=2\gamma(x)$, and $g_3(r)=0$. As another example, for $G(t,x,r)=\alpha(t,x)\delta(r)$, $\delta(r)\neq 0$, we have $(\partial/\partial r)G(t,x,r)=\alpha(t,x)\times(d\delta(r)/dr)=[(1/\delta(r))(d\delta(r)/dr)][\alpha(t,x)\delta(r)]$, which is of the form of Eq. (21) with $g_3(r)=(1/\delta(r))(d\delta(r)/dr)$. This allows us to solve the inverse problem of finding the integrand $f(t,x,r)$, which when used to render stationary the integral given in Eq. (1), yields the equation of motion $\ddot{x}(t)=\alpha[t,x(t)]\delta[\dot{x}(t)]$. Likewise, exploring other cases, such as $G(t,x,r)=\beta(t)r+\gamma(x)r^2$, $G(t,x,r)=\beta(t)r+\alpha(t,x)\delta(r)$, $G(t,x,r)=\beta(t)r+\bar{\alpha}(t,x)$, $G(t,x,r)=\gamma(x)r^2+\alpha(t,x)\delta(r)$, $G(t,x,r)=\gamma(x)r^2+\bar{\alpha}(t,x)$, $G(t,x,r)=\alpha(t,x)\delta(r)+\bar{\alpha}(t,x)$, $G(t,x,r)=\beta(t)r+\gamma(x)r^2+\alpha(t,x)\delta(r)$, $G(t,x,r)=\beta(t)r+\gamma(x)r^2+\bar{\alpha}(t,x)$, $G(t,x,r)=\beta(t)r+\alpha(t,x)\delta(r)+\bar{\alpha}(t,x)$, $G(t,x,r)=\gamma(x)r^2+\alpha(t,x)\delta(r)+\bar{\alpha}(t,x)$, and $G(t,x,r)=\beta(t)r+\gamma(x)r^2+\alpha(t,x)\delta(r)+\bar{\alpha}(t,x)$, a set of nonpotential forces can be found for which a suitable function $f(t,x,r)|_{\substack{x=x(t)\\ r=\dot{x}(t)}}$ whose integral when made stationary, yields the correct equation of motion. The following six sets of nonpotential forces can be adduced from a suitable function $f(\cdot)$ via the variational calculus:

$$G_1[t,x(t),\dot{x}(t)] = \alpha_1[t,x(t)] + \beta_1(t)\dot{x}(t) + \gamma_1[x(t)]\dot{x}^2(t) \tag{23a}$$

$$G_2[t,x(t),\dot{x}(t)] = \alpha_2[t,x(t)]\delta_2[\dot{x}(t)], \quad \delta_2[\dot{x}(t)] \neq 0 \tag{23b}$$

$$G_3[t,x(t),\dot{x}(t)] = \beta_3(t)\dot{x}(t) + \gamma_3[x(t)]\dot{x}^2(t) + \alpha_3[t,x(t)]\dot{x}^p(t), \quad \dot{x}(t) > 0 \tag{23c}$$

$$G_4[t,x(t),\dot{x}(t)] = \beta_4(t)\dot{x}(t) + \gamma_4[x(t)]\dot{x}^2(t) + \alpha_4(t)\dot{x}(t)\ln\dot{x}(t), \quad \dot{x}(t) > 0 \tag{23d}$$

$$G_5[t,x(t),\dot{x}(t)] = \bar{\alpha}_5(t) + \alpha_5[t,x(t)]e^{q\dot{x}(t)}, \quad q \neq 0 \tag{23e}$$

$$G_6[t,x(t),\dot{x}(t)] = \bar{\alpha}_6[x(t)] + \alpha_6[t,x(t)]e^{q\dot{x}^2(t)}, \quad q \neq 0 \tag{23f}$$

Here, α_i, $\bar{\alpha}_i$, β_i, γ_i, and δ_i $(i=1,2,3,4,5,6)$ are the arbitrary functions of their arguments, as indicated, and p and q are the real numbers. One could relax the condition $\dot{x}(t)>0$ to $\dot{x}(t)\neq 0$ in Eq. (23c) if p is restricted to being any real number such that $\dot{x}^p(t)$ is a real, well-defined number. Equation (23a) and a special form of Eq. (23b), namely, $G[t,x(t),\dot{x}(t)]=\alpha[t,x(t)]\dot{x}^2(t)$, are already known and given in Ref. [4]. Furthermore, the general form given in Eq. (23a) includes the special examples (given in propositions 1–3 and 5) in Ref. [7]. We can determine $f(t,x,r)$ of Eq. (1) using Eq. (10). The resulting integrand $f(t,x,r)$ of Eq. (1) for the forces described in Eq. (23) is, respectively, given by

$$f_1(t,x,r) = \frac{1}{2}\Theta_1(t,x)r^2 + \int \alpha_1(t,x)\Theta_1(t,x)dx \tag{24a}$$

$$f_2(t,x,r) = \int\int \frac{1}{\delta_2(\tilde{r})}d\tilde{r}dr + \int \alpha_2(t,x)dx \tag{24b}$$

$$f_3(t,x,r) = \frac{1}{(1-p)(2-p)}\Theta_3(t)\Psi_3(x)r^{2-p} + \Theta_3(t)\int \alpha_3(t,x)\Psi_3(x)dx, \quad p\neq 1 \quad \text{and} \quad p\neq 2 \tag{24c}$$

$$f_4(t,x,r) = \Theta_4(t)\Psi_4(x)(r\ln r - r) + \beta_4(t)\Theta_4(t)\int \Psi_4(x)dx \tag{24d}$$

$$f_5(t,x,r) = \frac{1}{q^2}\Theta_5(t)e^{-q\dot{x}} + \Theta_5(t)\int \alpha_5(t,x)dx \tag{24e}$$

$$f_6(t,x,r) = \frac{1}{2}\sqrt{\frac{\pi}{q}}\Theta_6(x)\left[r\,\mathrm{erf}(\sqrt{q}r) + \frac{e^{-qr^2}}{\sqrt{q\pi}}\right] + \int \alpha_6(t,x)\Theta_6(x)dx, \quad q>0 \tag{24f}$$

$$f_6(t,x,r) = \frac{1}{2}\sqrt{\frac{\pi}{-q}}\Theta_6(x)\left[r\,\mathrm{erfi}(\sqrt{-q}r) - \frac{e^{-qr^2}}{\sqrt{-q\pi}}\right] + \int \alpha_6(t,x)\Theta_6(x)dx, \quad q<0 \tag{24g}$$

where

$$\Theta_1(t,x) = \exp\left[-\int \beta_1(t)dt - 2\int \gamma_1(x)dx\right] \tag{25a}$$

$$\Theta_3(t) = \exp\left[(p-1)\int \beta_3(t)dt\right], \quad \Psi_3(x) = \exp\left[(p-2)\int \gamma_3(x)dx\right] \tag{25b}$$

$$\Theta_4(t) = \exp\left[-\int \alpha_4(t)dt\right], \quad \Psi_4(x) = \exp\left[-\int \gamma_4(x)dx\right] \tag{25c}$$

$$\Theta_5(t) = \exp\left[q\int \bar{\alpha}_5(t)dt\right] \tag{25d}$$

$$\Theta_6(x) = \exp\left[2q\int \bar{\alpha}_6(x)dx\right] \tag{25e}$$

and $\mathrm{erf}(\cdot)$ and $\mathrm{erfi}(\cdot)$ denote the error function and the imaginary error function, respectively. Equations (24f) and (24g) pertain to $G_6(\cdot)$ given in Eq. (23f). Equation (24c) does not hold when $p=1$ or $p=2$. When $p=1$, Eq. (23c) becomes

Table 1 The function $G[t,x(t),\dot{x}(t)]$ of Eq. (2) and the corresponding function $f(t,x,r)$ of Eq. (10)

$G[t,x(t),\dot{x}(t)]$	$f(t,x,r)$
$x^2(t)\sqrt{\eta^2+\dot{x}^2(t)}$	$r\ln(r+\sqrt{\eta^2+r^2})-\sqrt{\eta^2+r^2}+\frac{1}{3}x^3$
$h[t,x(t)]e^{-\eta\dot{x}(t)}$	$\frac{1}{\eta^2}e^{\eta r}+\int h(t,x)dx$
$\frac{h[t,x(t)]}{\eta^2+\dot{x}^2(t)}$	$\frac{1}{12}r^4+\frac{1}{2}\eta^2r^2+\int h(t,x)dx$
$h[t,x(t)]e^{-\eta^2\dot{x}^2(t)}$	$\frac{\sqrt{\pi}}{2\eta}\left[r\,\mathrm{erfi}(\eta r)-\frac{e^{\eta^2r^2}}{\eta\sqrt{\pi}}\right]+\int h(t,x)dx$, where $\mathrm{erfi}(z):=-i\,\mathrm{erf}(iz)$

$$G_3[t,x(t),\dot{x}(t)]=\alpha_3[t,x(t)]\dot{x}(t)+\gamma_3[x(t)]\dot{x}^2(t) \tag{26}$$

and

$$f_3(t,x,r)=\Theta_3(x)(r\ln r-r)+\int\alpha_3(t,x)\Theta_3(x)dx \tag{27}$$

where $\Theta_3(x)=\exp[-\int\gamma_3(x)dx]$. In addition, when $p=2$, Eq. (23*c*) becomes

$$G_3[t,x(t),\dot{x}(t)]=\beta_3(t)\dot{x}(t)+\alpha_3[t,x(t)]\dot{x}^2(t) \tag{28}$$

and

$$f_3(t,x,r)=-\Theta_3(t)\ln r+\Theta_3(t)\int\alpha_3(t,x)dx \tag{29}$$

where $\Theta_3(t)=\exp[\int\beta_3(t)dt]$.

By using Eq. (24*b*), we provide in Table 1 some nonpotential force $G[t,x(t),\dot{x}(t)]$ that can be obtained (η is constant) by using the calculus of variations along with the corresponding integrand $f(t,x,r)$. It is noted that if we substitute each integrand $f(t,x,r)|_{x=x(t),\,r=\dot{x}(t)}$ into the Euler–Lagrange equation (Eq. (3)), we get the corresponding force $G[t,x(t),\dot{x}(t)]$.

Other special cases, such as $G[t,x(t),\dot{x}(t)]=\alpha\dot{x}^3(t)+\beta\dot{x}(t)$, $\dot{x}(t)>0$ $(\alpha,\beta<0)$, can be obtained from the integrand $f(t,x,r)=(1/2)(e^{2\beta t}/r)+\alpha e^{2\beta t}x$ by using Eq. (24*c*). Another integrand using Eq. (24*b*) for the same force $G(\cdot)$ is given by $f(t,x,r)=r\ln|r|/\beta-(1/2)(r\ln(-\alpha r^2-\beta)/\beta)-(\tan^{-1}(\alpha r/\sqrt{\beta\alpha})/\sqrt{\beta\alpha})+x$, $r>0$.

As an important example, let us consider a model of the Coulomb friction force approximated by

$$G[t,x(t),\dot{x}(t)]=\alpha\tanh[\beta\dot{x}(t)],\quad \dot{x}(t)>0 \tag{30}$$

where $\alpha<0$ and $\beta>0$ are the constants. Comparing with Eq. (23*b*), it is obvious that $\alpha_2[t,x(t)]=\alpha$ and $\delta_2[\dot{x}(t)]=\tanh[\beta\dot{x}(t)]$. Then, Eq. (24*b*) yields the following integrand:

$$f(t,x,r)=\int\int\coth(\beta\tilde{r})d\tilde{r}dr+\int\alpha dx=\frac{1}{2\beta^2}[Li_2(e^{-2\beta r})-\beta^2r^2-2\beta r\ln|1-e^{-2\beta r}|+2\beta r\ln|\sinh(\beta r)|]+\alpha x,\quad r>0 \tag{31}$$

where Li_2 is the polylogarithm function, also known as the Jonquière function, defined as

$$Li_n(z)\triangleq\sum_{k=1}^{\infty}\frac{z^k}{k^n} \tag{32}$$

where n is an integer. To verify that the $f(t,x,r)$ in Eq. (31) does yield the Coulomb friction force given by Eq. (30), we directly substitute it into the Euler–Lagrange equation (3). First, we have

$$\frac{\partial f}{\partial r}=\frac{1}{2\beta^2}\left[-2\beta Li_1(e^{-2\beta r})-2\beta^2r-2\beta\ln|1-e^{-2\beta r}|-4\beta^2\frac{re^{-2\beta r}}{1-e^{-2\beta r}}+2\beta\ln|\sinh(\beta r)|+2\beta^2r\coth(\beta r)\right] \tag{33}$$

where we have used the fact $(d/dz)Li_n(z)=(1/z)Li_{n-1}(z)$. The total differentiation of Eq. (33) with respect to time gives us, after some calculations

$$\frac{d}{dt}\left(\frac{\partial}{\partial r}f[t,x(t),\dot{x}(t)]\right)=\ddot{x}(t)\coth[\beta\dot{x}(t)] \tag{34}$$

Next, it is obvious from Eq. (31) that

$$\frac{\partial f}{\partial x}=\alpha \tag{35}$$

With Eqs. (34) and (35), the Euler–Lagrange equation, then, yields the force $G[t,x(t),\dot{x}(t)]$ that is given by Eq. (30) when $\dot{x}(t)>0$.

4 Conclusions

We have discussed here extensions of variational principles for nonpotential forces, employing the inverse problem of the calculus of variations. While a problem of some considerable interest, few new, general results appear to have been obtained to date beyond those obtained in Refs. [1,4,5]. This paper obtains new integrand functions for some general nonpotential forces and develops a simple and systematic approach for getting them. We note that our discussion is restricted to the scalar case (single degree-of-freedom systems). The inverse problem for the vector case is considerably more difficult. Thus, the extension to nonpotential forces in the multidimensional case remains an open problem.

References

[1] Bolza, O., 1931, *Vorlesungen über Varationsrechnung*, Koehler und Amelang, Leipzig.

[2] Douglas, J., 1939, "Solution of the Inverse Problem of the Calculus of Variations," Proc. Natl. Acad. Sci. U.S.A., **25**, pp. 631–637.

[3] Douglas, J., 1940, "Theorems in the Inverse Problem in the Calculus of Variations," Proc. Natl. Acad. Sci. U.S.A., **26**, pp. 215–221.

[4] Leitmann, G., 1963, "Some Remarks on Hamilton's Principle," ASME J. Appl. Mech., **30**, pp. 623–625.

[5] He, J.-H., 2004, "Variational Principles for Some Nonlinear Partial Differential Equations With Variable Coefficients," Chaos, Solitons Fractals, **19**, pp. 847–851.

[6] Leitmann, G., 1986, *The Calculus of Variations and Optimal Control: An Introduction*, Springer, New York.

[7] Musielak, Z. E., 2008, "Standard and Non-Standard Lagrangians for Dissipative Dynamical Systems With Variable Coefficients," J. Phys. A: Math. Theor., **41**, p. 055205.