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Hamel's paradox and the foundations of analytical dynamics

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ABSTRACT

This paper deals with an explanation of a paradox posed by Hamel in his 1949 book on Theoretical Mechanics. The explanation deals with the foundations of mechanics and points to new insights into analytical dynamics.

1. Introduction

We begin by stating the problem considered by Hamel and the subsequent paradox posed by him in his book on theoretical mechanics (Hamel [1]). Hamel considers in one of his examples the planar motion of the blade of an ice skate of mass m modeled as an edge resting on a horizontal ice surface (see Fig. 1). The center of mass of the blade is at the point M, and it makes contact with the ice surface at the point B which is at a distance s from M. The blade makes an angle $\vartheta(t)$ with the x-axis of an inertial coordinate frame at the instant of time t, and the coordinates of the point B at that instant are denoted by (x,y). The blade's motion is restricted so that its velocity is always along the line along which it is inclined in the x–y plane.

Following Hamel, the kinetic energy of the system is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + ms\dot{\vartheta}[\dot{y}\cos\vartheta - \dot{x}\sin\vartheta] + \frac{1}{2}I_B\dot{\vartheta}^2, \tag{1.1}$$

where I_B is the moment of inertia of the blade about the point B. The nonholonomic constraint equation that requires that the velocity of the blade be along the direction of its length is given by

$$\dot{y}\cos\vartheta - \dot{x}\sin\vartheta = 0. \tag{1.2}$$

Hamel obtains the generalized (momenta) impulses as

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x} - ms\sin\vartheta\dot{\vartheta}, \tag{1.3}$$

$$p_y = \frac{\partial T}{\partial \dot{y}} = m\dot{y} + ms\cos\vartheta\dot{\vartheta}, \tag{1.4}$$

and

$$p_\vartheta = \frac{\partial T}{\partial \dot{\vartheta}} = ms(\dot{y}\cos\vartheta - \dot{x}\sin\vartheta) + I_B\dot{\vartheta}, \tag{1.5}$$

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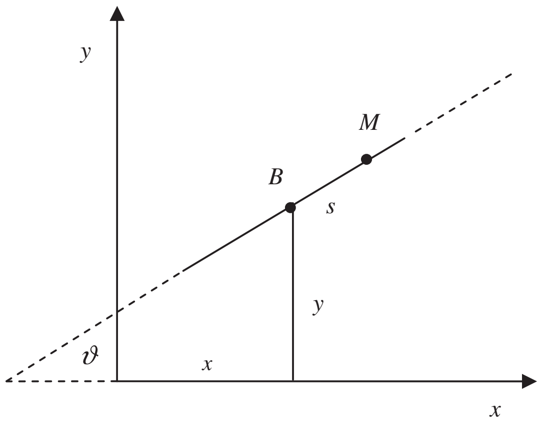

Fig. 1. Taken from Hamel [1], this figure shows the blade inclined at an angle ϑ to the x-axis of an inertial frame of reference.

and gets

$$W_x = \frac{d}{dt}(m\dot{x} - ms\sin\vartheta\dot{\vartheta}), \tag{1.6}$$

$$W_y = \frac{d}{dt}(m\dot{y} + ms\cos\vartheta\dot{\vartheta}), \tag{1.7}$$

and

$$W_\vartheta = \frac{d}{dt}(ms(\dot{y}\cos\vartheta - \dot{x}\sin\vartheta)) + I_B\ddot{\vartheta} + ms\dot{\vartheta}(\dot{y}\sin\vartheta + \dot{x}\cos\vartheta). \tag{1.8}$$

Using Lagrange's equations and the Lagrange multiplier, λ, Hamel then writes the equations of motion of the constrained system as

$$\frac{d}{dt}(m\dot{x} - ms\sin\vartheta\dot{\vartheta}) = -\lambda\sin\vartheta + X, \tag{1.9}$$

$$\frac{d}{dt}(m\dot{y} + ms\cos\vartheta\dot{\vartheta}) = \lambda\cos\vartheta + Y, \tag{1.10}$$

$$\frac{d}{dt}(ms(\dot{y}\cos\vartheta - \dot{x}\sin\vartheta)) + I_B\ddot{\vartheta} + ms\dot{\vartheta}(\dot{y}\sin\vartheta + \dot{x}\cos\vartheta) = M, \tag{1.11}$$

where X and Y are the components in the x and y directions of the impressed (or 'given') force on the blade at B, and M is the impressed moment. In order to simplify our discussion, we shall assume that these impressed forces and moments are zero, and that the mass m of the blade is unity.

Using the constraint (1.2) the Lagrange multiplier, λ, can be eliminated from Eqs. (1.9)–(1.11), as Hamel does in his book, and the equations of motion for the system can be written as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ I_B\ddot{\vartheta} \end{bmatrix} = \begin{bmatrix} s\dot{\vartheta}^2\cos\vartheta - \dot{x}\dot{\vartheta}\tan\vartheta \\ s\dot{\vartheta}^2\sin\vartheta + \dot{x}\dot{\vartheta} \\ -s\dot{\vartheta}(\dot{y}\sin\vartheta + \dot{x}\cos\vartheta) \end{bmatrix}. \tag{1.12}$$

Hamel in his book goes on to state (and we quote):

"Having come up with the equations of motion, we can obviously make use of the equation of constraint and therefore simplify the third equation [i.e., (1.11)] by eliminating the first term; we would not have been permitted to work with the kinetic energy

$$T^+ = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_B\dot{\vartheta}^2, \tag{1.13}$$

which is obtained by using the equation of constraint [in (1.1)]. Very obviously this would have given the wrong equations." (The square brackets above are placed by the authors for clarity.)

This quote points out that were we to substitute the constraint (1.2) in the expression for the kinetic energy given in (1.1) and thereby eliminate the second term on the right hand side in this relation, we would obtain the expression T^+ given in Eq. (1.13). The use of this amended kinetic energy, T^+, along with the constraint (1.2) would then have led us – in a manner similar to what we did before in using Lagrange's equations along with the multiplier λ – to the following equation of motion

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\vartheta} \end{bmatrix} = \begin{bmatrix} -\dot{x}\dot{\vartheta}\tan\vartheta \\ \dot{x}\dot{\vartheta} \\ 0 \end{bmatrix}, \tag{1.14}$$

which, as stated above by Hamel, would indeed be incorrect!

While Hamel is correct in what he writes, he leaves the reader unclear as to why such a seemingly harmless substitution of the constraint Eq. (1.2) in the kinetic energy expression T leads to the wrong equations of motion; for, after all, the system must satisfy this constraint at *each and every* instant of time.

More importantly, from his statements that we have quoted above, he leaves the reader unclear whether this situation is unique to this specific problem, or perhaps that it occurs in only certain special problems. And if the latter were to be true, he leaves open the question of identifying those special circumstances in which this seemingly harmless substitution of the constraint equation(s) – which in any case must always be satisfied by the system – in the expression for the kinetic energy of the system would lead to erroneous results upon the application of Lagrange's equations using the usual multiplier method.

It is this paradox that Hamel brings up in his book that has led us to investigate it in greater detail, and as shown below, its understanding will lead us to the very foundations of analytical dynamics.

2. On the dynamics of constrained systems

As a prelude to explaining what might be going on here it is useful to move from the Lagrangian view of mechanics to the one developed by Gauss [2]. Both views ultimately rest on d'Alembert's principle (assumption) that states that the total work done by all the forces of constraint under virtual displacements sum to zero (Udwadia and Kalaba [3]). However, Gauss provides a clearer picture of the manner in which constrained mechanical systems need to be conceptualized, so that the equations of motion that are then obtained properly reflect physical observational evidence. His conceptualization is done in three distinct steps.

2.1. Description of the unconstrained system

The unconstrained system is conceptually the mechanical system in which the virtual displacements are assumed to be all independent of one another. One obtains the equations of motion that govern this unconstrained system by: (1) starting with the assumption that no constraints exist between the coordinates that describe the configuration of the system, (2) writing down the kinetic energy of the system under this assumption, and then (3) using Lagrange's equation, taking into account all the impressed forces (both conservative and non-conservative) that are acting on the system. We note that the inclusion of these impressed forces in Lagrange's equations arises through a determination of the work that they do under virtual displacements; and these displacements are assumed to be independent when describing the unconstrained system. Using Lagrange's equations we can write the equations of motion of this unconstrained system as

$$M(q,t)\ddot{q} = F(q,\dot{q},t), \tag{2.1}$$

where q is the n-vector of coordinates specifying the configuration of the unconstrained system, the n by n matrix $M > 0$ is the mass matrix, and F is the impressed force n-vector that acts on the system. From Eq. (2.1) we find that the acceleration of the unconstrained system is given by

$$a := M(q,t)^{-1}F(q,\dot{q},t). \tag{2.2}$$

In Hamel's example, $q = [x, y, \vartheta]^T$. Eqs. (1.9)–(1.11) with λ set to zero are the unconstrained equations of motion of the system; they correspond to Eq. (2.1). This is because setting λ to zero signifies that the constraint (1.2) is assumed to be non-existent, and hence that the coordinates (and the corresponding virtual displacements) are independent.

2.2. Description of the constraints

One now conceptualizes that the necessary m equality constraints are then imposed on this unconstrained system that is described in our previous step. These constraints, which may be holonomic or nonholonomic, if sufficiently smooth, can be differentiated appropriately with respect to time to yield the equation

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t), \tag{2.3}$$

where the matrix A is an m by n matrix, and b is an m-vector. Each row of A arises by appropriately differentiating one of the m constraint equations.

In Hamel's example, the constraint that needs to be imposed on the unconstrained system described earlier is given by Eq. (1.2). Upon differentiating this equation once with respect to time we can express it in the form of Eq. (2.3) where

$$A = [\,-\sin\vartheta \quad \cos\vartheta \quad 0\,], \tag{2.4}$$

and

$$b = \dot{y}\dot{\vartheta}\sin\vartheta + \dot{x}\dot{\vartheta}\cos\vartheta. \tag{2.5}$$

2.3. Description of the constrained system

Gauss in his landmark paper goes on to show that the motion of the constrained system is obtained by ensuring that the deviation of its acceleration from that of the unconstrained system (weighted by the matrix M) is a global minimum at each instant of time (Gauss [2]). It can be shown that this condition gives the explicit, so-called fundamental equation of motion of the constrained system (Udwadia and Kalaba [4]) in the form

$$M\ddot{q} = F(q,\dot{q},t) + A^T(AM^{-1}A^T)^+(b - Aa), \tag{2.6}$$

wherein the various quantities have been defined in the previous two steps, and the superscript '+' denotes the Moore-Penrose inverse of the matrix. We note that relation (2.6) is valid:

(a) whether or not the equality constraints are holonomic or nonholonomic,
(b) whether or not they are nonlinear functions of the generalized velocities and coordinates, and
(c) whether or not they are functionally dependent.

When the unconstrained description (given by (2.1)) of the constrained system is such that the n by n matrix M is singular, then Eq. (2.6) cannot be used since M^{-1} does not exist. In that case Eq. (2.6) needs to be replaced by the equation

$$\ddot{q} = \begin{bmatrix} (I - A^+A)M \\ A \end{bmatrix}^+ \begin{bmatrix} F \\ b \end{bmatrix}, \tag{2.7}$$

under the proviso that the rank of the matrix $[M|A^T]$ is n. This rank condition is a necessary and sufficient condition for the constrained system to have a unique acceleration – a consequence of physical observation of the motion of bodies. (Udwadia and Phohomsiri [5])

While this step can be carried out through the use of Lagrange multipliers, Eqs. (2.6) and (2.7) are more convenient, and work in situations (such as, when the constraints are not functionally independent) in which the Lagrange multiplier method breaks down. They obviate the difficulties in finding the Lagrange multipliers, especially when the constraints are nonlinear functions of their arguments.

Of crucial importance in this three-step conceptualization of constrained systems is the need to construe the unconstrained system in Step 1 above as one on which no constraints are assumed to be imposed. Conflating the unconstrained system with the constrained system – by imposing the constraints while obtaining the equations of motion for the supposedly unconstrained system – would naturally then, in general, lead to the wrong final set of equations for the constrained system.

In Hamel's example, substitution of Eq. (1.2) into the kinetic energy of the system given by Eq. (1.1), and use of the amended kinetic energy T^+ so obtained as in relation (1.13), conflates the unconstrained system with the constrained system. Lagrange's equations obtained by using T^+ as the kinetic energy of the system therefore cannot, in general, give us a correct description of the unconstrained equations of motion of the system which we are supposedly looking for in Step 1 above, and also on which we want to further impose our constraint given by (1.2), in order to finally get the equations of motion of the constrained system.

We see then that it is this fundamental misunderstanding on how a general, constrained mechanical system is to be conceptualized in analytical dynamics that leads us to this seeming paradox, and eventually to the wrong results when using T^+ to obtain the correct equations of motion.

If things are as stated above, then this should always be so. Let us explore if that is indeed the case. In the spirit of Hamel's example we do this through the use of four examples.

3. Four illustrative examples

Our aim is to keep the examples as simple as possible while still attempting to cull out as much of the fundamental analytical dynamics from them as possible. This will help to elucidate the somewhat interesting way in which one needs to conceptualize constrained motion in mechanical systems so as to obtain the correct equations of motion that describe them.

3.1. Example 1

Consider a particle of unit mass moving in three-dimensional space. Using an inertial right handed coordinate system $OXYZ$ let us assume that this particle's position at any time t is described by its coordinates (x,y,z) and that it is subjected to an impressed force whose components along the X-, Y- and Z-directions are respectively $F_x(x,y,z,\dot{x},\dot{y},\dot{z},t)$, $F_y(x,y,z,\dot{x},\dot{y},\dot{z},t)$, and $F_z(x,y,z,\dot{x},\dot{y},\dot{z},t)$. The particle is subjected to the nonholonomic constraint $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$.

Our aim is to find the equation of motion of this constrained system. This happens to be, in fact, a significant, problem in the history of analytical dynamics, first proposed by Appell in 1911.

In what follows we shall first provide the three-step, correct approach to conceptualizing the constrained motion of this simple system and so obtain its proper equation of motion. Then we shall see what happens when we conflate the aforementioned distinction between the description of the unconstrained system and the constrained system.

3.1.1. The correct approach

Let us denote $F := [F_x, F_y, F_z]^T$. We first conceptualize the unconstrained system in which all the coordinates that describe the configuration of the particle are independent of one another. The kinetic energy of the particle is

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{z}^2, \tag{3.1}$$

and Lagrange's equation for this unconstrained system *in which all the virtual displacements are assumed independent* of one another is trivially obtained as

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} F_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \\ F_y(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \\ F_z(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \end{bmatrix}. \tag{3.2}$$

Note that the right hand side of Eq. (3.2) arises while applying Lagrange's equation through a consideration of the virtual work done by the force F under virtual displacements, the components of which are assumed independent of one another. From this relation we find that the acceleration of the unconstrained system $a = [F_x, F_y, F_z]^T$.

Our next conceptual step is to impose the constraint

$$\dot{z}^2 = \dot{x}^2 + \dot{y}^2, \tag{3.3}$$

on this unconstrained system that is described by relation (3.2).

Differentiating the Eq. (3.3) with respect to time once, we obtain

$$\dot{z}\ddot{z} = \dot{x}\ddot{x} + \dot{y}\ddot{y}, \tag{3.4}$$

so that

$$A = [\dot{x} \;\; \dot{y} \;\; -\dot{z}], \tag{3.5}$$

and

$$b = 0. \tag{3.6}$$

The equation of motion of the constrained system – our last conceptual step – is obtained by simply using relation (2.6), which trivially gives ($M = I_3$)[1]

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} + \begin{bmatrix} \dot{x} \\ \dot{y} \\ -\dot{z} \end{bmatrix} \frac{-\dot{x}F_x - \dot{y}F_y + \dot{z}F_z}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \begin{bmatrix} \frac{(\dot{y}^2+\dot{z}^2)F_x - \dot{x}\dot{y}F_y + \dot{x}\dot{z}F_z}{2\dot{z}^2} \\ \frac{-\dot{x}\dot{y}F_x + (\dot{x}^2+\dot{z}^2)F_y + \dot{y}\dot{z}F_z}{2\dot{z}^2} \\ \frac{\dot{x}\dot{z}F_x + \dot{y}\dot{z}F_y + (\dot{x}^2+\dot{y}^2)F_z}{2\dot{z}^2} \end{bmatrix}, \tag{3.7}$$

assuming that $\dot{z} \neq 0$.

3.1.2. The Incorrect Approach

Were we to conflate the unconstrained system with the constrained system, and substitute for $\dot{z}^2$ from Eq. (3.3) in the expression for the kinetic energy T given in (3.1), we would then obtain

$$T^+ = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \dot{x}^2 + \dot{y}^2. \tag{3.8}$$

We note that this substitution has eliminated $\dot{z}$ from the expression for the kinetic energy T^+! Were we to subsume that the system is constrained (since we have used the equation describing the constraint in getting T^+), we would need to then say that the virtual displacements are no longer independent and that the relation

$$\dot{z}\delta z = \dot{x}\delta x + \dot{y}\delta y, \tag{3.9}$$

must be satisfied by them so that the work done by the force F under these virtual displacements becomes

$$W = F_x\delta x + F_y\delta y + F_z\delta z = \left(F_x + \frac{\dot{x}}{\dot{z}}F_z\right)\delta x + \left(F_y + \frac{\dot{y}}{\dot{z}}F_z\right)\delta y. \tag{3.10}$$

[1] The Moore-Penrose inverse of a row vector $a := [a_1, a_2, \ldots, a_n]$ is simply $(aa^T)^{-1}a^T$.

Assuming, as before, that $\dot{z} \neq 0$. Using Eqs. (3.8) and (3.10) in Lagrange's equation, then equations of motion of the constrained system become

$$\ddot{x} = \frac{\dot{z}F_x + \dot{x}F_z}{2\dot{z}}, \tag{3.11}$$

and

$$\ddot{y} = \frac{\dot{z}F_y + \dot{y}F_z}{2\dot{z}}. \tag{3.12}$$

Using these expressions in (3.4) we then obtain

$$\ddot{z} = \frac{\dot{x}\dot{z}F_x + \dot{y}\dot{z}F_y + (\dot{x}^2 + \dot{y}^2)F_z}{2\dot{z}^2}. \tag{3.13}$$

Eqs. (3.11) and (3.12) are not the same as the first two in (3.7). In fact, we have obtained the wrong set of equations of motion! We note that the root cause of this error is obfuscation of the difference between the unconstrained and the constrained descriptions of the mechanical system.

The reader may think that we have performed a slight sleight-of-hand here in using Eq. (3.9) to eliminate δz from the virtual work expression in (3.10). While we might justify our actions by saying that our elimination of δz was prompted by the fact that now T^+ concerns only two coordinates as shown in (3.8), and that we have already assumed that the constraint is active since we have used it to get T^+, we need to be more careful and stick more closely to our three-step conceptualization of constrained motion stated in Section 2 to see what exactly went wrong in the procedure we followed. So let's do that.

Having got the kinetic energy T^+ in (3.8) by conflating the unconstrained and constrained systems, what would the Lagrange equations of motion of the 'presumably unconstrained' system be? Recall that by 'unconstrained' we always mean that the virtual displacements are independent. Using T^+, the Lagrange equations we would get are

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}. \tag{3.14}$$

The last equation must strike some alarm in the reader's mind, but we must recall that these are not the equations of motion of the actual system yet! They are the equations of motion of our presumed unconstrained system. Now we need to go to our next step in the conceptualization process and differentiate our constraint as we did in (3.4). Our final step is to obtain the equations of motion of the constrained system by imposing this constraint on our presumed unconstrained system, which is described by (3.14). This last step as we saw in Section 2 is done usually through the use of relation (2.6), which gives the description of the constrained system.

But the mass matrix M describing this unconstrained system (Eq. (3.14)) is now singular! And so the unconstrained acceleration a is undefined! Luckily, we do have a way of getting the unique equations of motion in this predicament, provided the matrix $[M|A^T]$ has full rank (Udwadia and Phohomsiri [5]); and this it does, when $\dot{z} \neq 0$. In fact the equations of motion of the constrained system are given now by (2.7) and they yield exactly the same equations as (3.11)–(3.13), which are, of course, wrong!

Thus we see that confusing the constrained and the unconstrained system by imposing the constraint in the kinetic energy (more, generally in the Lagrangian, as we shall see later on) of the unconstrained system before applying the constraints as described in Step 2 of our discussion in Section 2 causes us to get the wrong equations of motion.

The root cause, as before, is that we conceptualized the problem of constrained motion incorrectly.

3.2. Example 2

Consider a particle of unit mass moving in three-dimensional space subjected to no externally impressed forces. The position of the particle at time t is given by its coordinates (x,y,z) measured in an inertial frame of reference. The particle is subjected to the nonholonomic constraint $\dot{y} = cz\dot{x}$, where c is a fixed constant.

We shall find the equation of motion of this particle and go a step further than we did in our previous example by illustrating what might result if we progressively 'mix up' the concepts of unconstrained and constrained dynamical systems, as we shall shortly explain. We begin with the correct approach for conceptualizing the constrained system.

3.2.1. The correct approach

The kinetic energy of the unconstrained system is again given by

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{z}^2, \tag{3.15}$$

and since the impressed forces are all zero, Lagrange's equation, under the assumption that the coordinates are all independent and no constraints exist between them, is given trivially by the equation

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.16}$$

This completes our first step in the conceptualization process; defining the unconstrained system. Next, we impose the constraint

$$\dot{y} = c\,z\,\dot{x}, \tag{3.17}$$

which when differentiated with respect to time gives

$$\ddot{y} = c\,z\,\ddot{x} + c\,\dot{x}\,\dot{z}, \tag{3.18}$$

so that (see (2.3))

$$A = [-cz\ 1\ 0] \quad \text{and} \quad b = c\,\dot{x}\,\dot{z}. \tag{3.19}$$

The last step of the conceptualization process is formed by using the fundamental Eq. (2.6), which then gives

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -\frac{c^2\dot{x}z\dot{z}}{1+c^2z^2} \\ \frac{c\dot{x}\dot{z}}{1+c^2z^2} \\ 0 \end{bmatrix}, \tag{3.20}$$

as the correct description of the motion of our constrained system.

3.2.2. The incorrect approach

Let us again rewrite the kinetic energy T of the unconstrained particle, where we assume that all the coordinates are independent, in the form

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha\dot{y}^2 + \frac{1}{2}(1-\alpha)\dot{y}^2 + \frac{1}{2}\dot{z}^2, \tag{3.21}$$

where $0 \leqslant \alpha < 1$. Note that we have split the contribution of the term $(1/2)\dot{y}^2$ in T given by (3.15) in two terms now. We now 'mix up' the concepts of the unconstrained and constrained descriptions of the system by using the constraint to replace only *part* of the kinetic energy. Thus we have

$$T^{+}(\alpha) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha(c\,z\,\dot{x})^2 + \frac{1}{2}(1-\alpha)\dot{y}^2 + \frac{1}{2}\dot{z}^2, \tag{3.22}$$

where we notice that the term $\frac{1}{2}\alpha\dot{y}^2$ on the right hand side of (3.21) has been replaced through the use of the constraint Eq. (3.17). Note that when $\alpha = 0,\ T^{+} = T$.

Under the assumption that all the coordinates are independent, the Lagrange equation of motion for the presumed unconstrained system, for a fixed value of α, becomes

$$\begin{bmatrix} 1+c^2\alpha z^2 & 0 & 0 \\ 0 & 1-\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -2c^2\alpha\dot{x}\,z\,\dot{z} \\ 0 \\ c^2\alpha\dot{x}^2 z \end{bmatrix}. \tag{3.23}$$

We use the word 'presumed' because we have, conceptually speaking, 'partially' confused the unconstrained and constrained descriptions of the system to the extent of replacing the term $\frac{1}{2}\alpha\dot{y}^2$ in T by using the constraint Eq. (3.17). Hence, these are indeed not the correct equations of motion of the unconstrained system!

In fact, one can think of the parameter α as a measure of the extent to which we have conceptually conflated the unconstrained description with the constrained description of the mechanical system. For, in a manner of speaking, one can say that if we chose $\alpha = 0$ then $T^{+} = T$ and we have not conceptually engaged in conflating the two descriptions of the system. On the other hand, when $\alpha = 1$, we have 'completely' confused the unconstrained and the constrained descriptions, having completely replaced the entire contribution made by the $\dot{y}^2$ term to the kinetic energy, T, by using the constraint Eq. (3.17). For $0 < \alpha < 1$, we could consider that we have done a 'partial' conflation, its extent increasing as α moves closer to unity.

From Eq. (3.23) we find that the acceleration of the presumed unconstrained system is given by

$$a = \begin{bmatrix} \frac{-2c^2\alpha\dot{x}z\dot{z}}{1+c^2\alpha z^2} \\ 0 \\ c^2\alpha\dot{x}^2 z \end{bmatrix}. \tag{3.24}$$

Noting that the constraint Eq. (3.17) upon differentiation can again be expressed by relations (3.18) and (3.19), we now need to impose this constraint on our 'presumably' unconstrained system, which is described by (3.23). This is done through the use of the fundamental Eq. (2.6), which then gives the relations (note, $\alpha \neq 1$)

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -\frac{c^2(\alpha+1)\dot{x}\,z\,\dot{z}}{1+c^2z^2} \\ \frac{c(1-c^2\alpha z^2)\dot{x}\,\dot{z}}{1+c^2z^2} \\ c^2\alpha\dot{x}^2z \end{bmatrix}. \tag{3.25}$$

Not surprisingly, the equation of motion depends on the parameter α that describes the extent to which we conflated the unconstrained and the constrained descriptions of our system. When $\alpha = 0$ and $T^+ = T$, Eqs. (3.20) and (3.25) are identical, and we get the correct equations of motion for the constrained system. But for other values of α they are not the same. The substitution of the constraint in the kinetic energy leads to incorrect equations of motion since the descriptions of the unconstrained and the constrained system (described in Section 2) are then 'partially' confused.

Lastly, we point out another erroneous way of conceptualizing constrained systems. For $\alpha \neq 0$, since we have already used the constraint equation in describing part of the kinetic energy of the system so as to obtain T^+, one might think that the virtual displacements are no longer independent. This reasoning is, in fact, again not true. For from the Lagrangian mechanics point of view, instead of the relations (3.23), we would then get not a set of equations, but the sum of three terms adding up to zero. We would obtain

$$\{(1 + c^2\alpha z^2)\ddot{x} + 2c^2\alpha\dot{x}\,z\,\dot{z}\}\delta x + \{(1-\alpha)\ddot{y}\}\delta y + \{\ddot{z} - c^2\alpha\dot{x}^2z\}\delta z = 0. \tag{3.26}$$

The virtual displacements, being no longer independent, must now satisfy the relation

$$\delta y = cz\delta x. \tag{3.27}$$

Substituting (3.27) in Eq. (3.26), we get

$$\{(1 + c^2\alpha z^2)\ddot{x} + 2c^2\alpha\dot{x}\,z\,\dot{z} + c(1-\alpha)z\ddot{y}\}\delta x + \{\ddot{z} - c^2\alpha\dot{x}^2z\}\delta z = 0. \tag{3.28}$$

Since by (3.27) we see that the constraint is only between δy and δx, the virtual displacements δx and δz are independent of one another so that (3.28) yields the two relations

$$(1 + c^2\alpha z^2)\ddot{x} + 2c^2\alpha\dot{x}\,z\,\dot{z} + c(1-\alpha)z\ddot{y} = 0, \tag{3.29}$$

and

$$\ddot{z} - c^2\alpha\dot{x}^2z = 0. \tag{3.30}$$

Also from the constraint Eq. (3.18), we obtain

$$\ddot{y} = c\,z\,\ddot{x} + c\,\dot{x}\,\dot{z}, \tag{3.31}$$

and using Eq. (3.29) on the right hand side of (3.31) we obtain the differential equation for $\ddot{y}$, so that the constrained system's equation becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -\frac{c^2(\alpha+1)\dot{x}\,z\,\dot{z}}{1+c^2z^2} \\ \frac{c(1-c^2\alpha z^2)\dot{x}\,\dot{z}}{1+c^2z^2} \\ c^2\alpha\dot{x}^2z \end{bmatrix}. \tag{3.32}$$

Eqs. (3.25) and (3.32) are the same, and both of them confirm that both these ways of conceptualizing constrained motion are incorrect. Thus, once we conflate the unconstrained and the constrained systems, the resulting system can *no longer* be thought of, in general, as being *either* unconstrained *or* constrained. In other words, the virtual displacements cannot be taken to be either dependent (as dictated by the constraint equation) or independent (as when there are no constraints). Use of either of these conceptualizations eventually leads to the wrong equations of motion for the system.

The reader might have noticed that when $\alpha = 1$ the mass matrix in Eq. (3.23) – our presumed unconstrained description of the system – becomes singular. Strictly speaking then, since the matrix $[M|A^T]$ has full rank, one would have to use Eq. (2.7) (instead of (2.6)) to arrive at the constrained equation of motion as we did before in the previous example. Were we to do that, we would again obtain Eq. (3.32) with the right hand side evaluated at $\alpha = 1$; that is to say, the wrong equation of motion.

In the next example we go a step deeper into understanding the problem, and show that while the seemingly paradoxical situation exhibited by Hamel in Section 1 deals with conflating the descriptions of the unconstrained and the constrained system (as described in Section 2) by using the constraint equation to amend only the kinetic energy of the system from T to T^+, we could have come to such a situation, in general, by conflating the two descriptions in other ways.

3.3. Example 3

Consider a particle of unit mass moving in three-dimensional space subjected to no externally impressed forces. The position of the particle at time t is given by its coordinates (x, y, z) measured in an inertial frame of reference. Let the potential energy of the particle be $(1/2)x^2$, and let the particle be subjected to the nonholonomic constraint $x = c\,\dot{y}\,\dot{z}$, where $c \neq 0$ is a fixed constant.

3.3.1. The correct approach

Since the kinetic energy of the unconstrained system is given by (3.15) and the potential energy is $(1/2)x^2$, the Lagrangian of the unconstrained system is

$$L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{z}^2 - \frac{1}{2}x^2, \tag{3.33}$$

and the equations of motion for the unconstrained system are accordingly

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -x \\ 0 \\ 0 \end{bmatrix}. \tag{3.34}$$

Differentiating the constraint equation

$$x = c\,\dot{y}\,\dot{z}, \tag{3.35}$$

we get

$$c\,\dot{z}\,\ddot{y} + c\,\dot{y}\,\ddot{z} = \dot{x}, \tag{3.36}$$

so that $A = [0\ c\dot{z}\ c\dot{y}]$ and $b = \dot{x}$. Using (2.6) the correct equation of motion for the constrained system is then

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -x \\ 0 \\ 0 \end{bmatrix} + \frac{\dot{x}}{c^2(\dot{y}^2 + \dot{z}^2)} \begin{bmatrix} 0 \\ c\dot{z} \\ c\dot{y} \end{bmatrix} = \begin{bmatrix} -x \\ \frac{\dot{x}\dot{z}}{c(\dot{y}^2+\dot{z}^2)} \\ \frac{\dot{x}\dot{y}}{c(\dot{y}^2+\dot{z}^2)} \end{bmatrix}. \tag{3.37}$$

Again, we shall see what happens when we conflate the aforementioned distinction between the description of the unconstrained system and the constrained system.

3.3.2. The incorrect approach

Were we to substitute for x from the constraint Eq. (3.35) into the Lagrangian L in (3.33), we would get

$$L^+ = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{z}^2 - \frac{1}{2}(c\,\dot{y}\,\dot{z})^2, \tag{3.38}$$

and the equations of motion of our unconstrained system would become

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (1 - c^2\dot{z}^2) & -2c^2\dot{y}\dot{z} \\ 0 & -2c^2\dot{y}\dot{z} & (1 - c^2\dot{y}^2) \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.39}$$

Imposition of the constraint (3.35) on this presumably unconstrained system, (assuming that $1 - c^2\dot{z}^2$ and $1 - c^2(3\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ are both non-zero), yields on using Eq. (2.6),

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\dot{x}\dot{z}}{c}\left(\frac{1+c^2\dot{y}^2}{2\dot{x}^2+\dot{y}^2+\dot{z}^2}\right) \\ \frac{\dot{x}\dot{y}}{c}\left(\frac{1+c^2\dot{z}^2}{2\dot{x}^2+\dot{y}^2+\dot{z}^2}\right) \end{bmatrix}. \tag{3.40}$$

From (3.40) and (3.37) we see that we have clearly obtained the wrong equations of motion because we have conceptually mixed up our unconstrained and constrained descriptions of our mechanical system.

As in the previous example, since we have already used the constraint equation in describing the Lagrangian of the system so as to obtain L^+, we might think that the virtual displacements should no longer be considered independent. Working out the equations from the Lagrangian mechanics point of view, we would, as before, get the sum of three terms adding up to zero, and would obtain

$$\{\ddot{x}\}\delta x + \left\{(1 - c^2\dot{z}^2)\ddot{y} - 2c^2\dot{y}\,\dot{z}\,\ddot{z}\right\}\delta y + \left\{-2c^2\dot{y}\,\dot{z}\,\ddot{y} + (1 - c^2\dot{y}^2)\ddot{z}\right\}\delta z = 0. \tag{3.41}$$

The virtual displacements being no longer independent (since we have now conceptualized the system with the Lagrangian L^+ as being constrained), they must satisfy the relation

$$\dot{z}\delta y = -\dot{y}\delta z. \tag{3.42}$$

Substituting (3.42) in Eq. (3.41), we get

$$\{\ddot{x}\}\delta x + \left\{-(1 - c^2\dot{z}^2)\frac{\dot{y}}{\dot{z}}\ddot{y} + 2c^2\dot{y}^2\ddot{z} - 2c^2\dot{y}\dot{z}\ddot{y} + (1 - c^2\dot{y}^2)\ddot{z}\right\}\delta z = 0. \tag{3.43}$$

The constraint (3.42) causes the virtual displacements δy and δz to be related; but δx and δz are independent of one another so that (3.43) yields the two relations

$$\ddot{x} = 0, \tag{3.44}$$

and

$$-(1 - c^2\dot{z}^2)\frac{\dot{y}}{\dot{z}}\ddot{y} + 2c^2\dot{y}^2\ddot{z} - 2c^2\dot{y}\,\dot{z}\,\ddot{y} + (1 - c^2\dot{y}^2)\ddot{z} = 0. \tag{3.45}$$

Also from the constraint Eq. (3.36), we obtain

$$\ddot{z} = \frac{\dot{x} - c\,\dot{z}\,\ddot{y}}{c\,\dot{y}}, \tag{3.46}$$

and substituting Eq. (3.46) in the left hand side of (3.45) we obtain the differential equation for $\ddot{y}$. The constrained system's equation then become

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\dot{x}\dot{z}}{c}\left(\frac{1+c^2\dot{y}^2}{2\dot{x}^2+\dot{y}^2+\dot{z}^2}\right) \\ \frac{\dot{x}\dot{y}}{c}\left(\frac{1+c^2\dot{z}^2}{2\dot{x}^2+\dot{y}^2+\dot{z}^2}\right) \end{bmatrix}. \tag{3.47}$$

Eqs. (3.40) and (3.47) are the same, and both of them confirm that both these ways of conceptualizing constrained motion are incorrect. Thus, once we conflate the unconstrained and the constrained descriptions, the resulting system, in general, can *no longer* be thought of as being *either* unconstrained *or* constrained!

So far we have been careful to choose constraints that have all been nonholonomic. This leaves open the following question: If we had only holonomic constraints acting on a mechanical system and we were to use the constraint equation(s) to amend the Lagrangian L to L^+ as before, and then use the usual approach for obtaining the constrained equations of motion, would we obtain the correct equations? In the next section we show that in this special case we indeed will, and we explain the reason for that.

3.4. Example 4

A particle of unit mass is moving in three-dimensional Cartesian space. The impressed force on the particle is $F = [F_x, F_y, F_z]^T$. The position of the particle in a rectangular inertial coordinate frame is described by its coordinates (x,y,z). The particle is subjected to the holonomic constraint $y = \frac{1}{2}z^2$.

3.4.1. The correct approach

Using the three-steps outlined in Section 2 we have the following results for each step.

1. The equation describing the motion of the unconstrained system, assuming that all the coordinates (and the corresponding virtual displacements) are independent is

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}. \tag{3.48}$$

2. To impose the constraint

$$y = \frac{1}{2}z^2, \tag{3.49}$$

on this unconstrained description of the system, we differentiate Eq. (3.49) twice with respect to time in order to put the constraint equation in the form of Eq. (2.3), and get

$$\ddot{y} = z\ddot{z} + \dot{z}^2. \tag{3.50}$$

Thus, $A = [0\ 1\ \ -z]$ and $b = \dot{z}^2$.

3. Using the fundamental Eq. (2.6), we then obtain

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -z \end{bmatrix} \frac{\dot{z}^2 - F_y + zF_z}{1+z^2} = \begin{bmatrix} F_x \\ \frac{\dot{z}^2+z^2F_y+zF_z}{1+z^2} \\ \frac{-z\dot{z}^2+zF_y+F_z}{1+z^2} \end{bmatrix}, \tag{3.51}$$

as the equation of motion of our constrained system.

Let us contemplate a suitable change of coordinates to convert the constrained holonomic system to an unconstrained system (Pars [6]). This can be done because the constraint has usable information about both the generalized coordinates and their derivatives. Let us employ a new coordinate $q = y - \frac{1}{2}z^2$ instead of the coordinate 'y' so that we describe the configuration of the system by the coordinates (x, q, z) instead of (x, y, z). The mapping from (x, y, z) to (x, q, z) must be 1–1, and since

$$\det\left(\frac{\partial(x,q,z)}{\partial(x,y,z)}\right) = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -z & 1 \end{bmatrix}\right) = 1 \neq 0, \tag{3.52}$$

so it is. Note that since from the constraint Eq. (3.49), we must have $q(t) = y - \frac{1}{2}z^2 = 0$ for *all* time, which in turn implies that $\dot{q}(t) \equiv 0$ and $\ddot{q}(t) \equiv 0$ for all time. In addition, we can differentiate $q = y - \frac{1}{2}z^2$, with respect to time, to obtain

$$\dot{y} = \dot{q} + z\,\dot{z}. \tag{3.53}$$

Then, inserting Eq. (3.53) into the expression for the kinetic energy (we are conflating the unconstrained and constrained descriptions here),

$$T = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{1}{2}\dot{z}^2, \tag{3.54}$$

of the system, yields

$$T^{+} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(\dot{q} + z\,\dot{z})^2 + \frac{1}{2}\dot{z}^2. \tag{3.55}$$

But since $\dot{q}(t) \equiv 0$, we get

$$T^{+} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(z\,\dot{z})^2 + \frac{1}{2}\dot{z}^2 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(1 + z^2)\dot{z}^2. \tag{3.56}$$

Note that this is tantamount to simply using the constraint (3.53) to replace the $(1/2)\dot{y}^2$ term in the kinetic energy – a key observation. From (3.53), after setting $\dot{q}(t) \equiv 0$, we get

$$\delta y = z\delta z. \tag{3.57}$$

The generalized forces in the new coordinates (x,q,z) are obtained by equating the virtual work done by the forces, given by

$$\delta W = F_x\delta x + F_y z\delta z + F_z\delta z = \widehat{F}_x\delta x + \widehat{F}_q\delta q + \widehat{F}_z\delta z, \tag{3.58}$$

which gives

$$F_x\delta x + (zF_y + F_z)\delta z = \widehat{F}_x\delta x + \widehat{F}_q\delta q + \widehat{F}_z\delta z, \tag{3.59}$$

in view of (3.57). Noting now that $q(t) \equiv 0$ so that $\delta q(t) \equiv 0$, we get

$$\widehat{F}_x = F_x \quad \text{and} \quad \widehat{F}_z = (zF_y + F_z). \tag{3.60}$$

Since by (3.57) the constraint is only between δy and δz, the virtual displacements δx and δz are independent of one another. Applying Lagrange's equations to the coordinates (x,z), we obtain

$$\ddot{x} = F_x, \tag{3.61}$$

$$(1 + z^2)\ddot{z} = -z\,\dot{z}^2 + zF_y + F_z. \tag{3.62}$$

The time evolution of the last coordinate q of the triple (x,q,z) that we are using to describe the configuration of the system is simply given by $\ddot{q}(t) \equiv 0$, as we saw before.

We see that the dynamical equation describing the evolution of q is trivial (i.e., $q(t) \equiv 0$) and it is uncoupled from (3.61) and (3.62). We can interpret, from a physical viewpoint, what has been done in arriving at Eqs. (3.61) and (3.62) as an elimination of one of the coordinates of our triplet (x,y,z) to obtain the dynamical equations of the system in only the coordinates x and z. These coordinates may be considered to be independent, since the corresponding virtual displacements δx and δz in them can be specified independently. Thus, (3.61) and (3.62) represent, technically speaking, the equations of motion of an unconstrained system (see Section 2). That the equation describing the evolution of the coordinate $q(t)$ is uncoupled from these two equations is an important observation, which we shall make use of later on.

To get back to our original configuration space coordinates (x, y, z), we use (3.50) along with (3.61) and (3.62) to get

$$\ddot{y} = \frac{\dot{z}^2 + z^2F_y + zF_z}{1 + z^2}. \tag{3.63}$$

Notice that in order to transform the coordinates back from (x, q, z) to(x, y, z), we have to use the Eq. (3.50) which causes the virtual displacements δy and δz to be related, and so the Eqs. (3.61)–(3.63) now correspond to the constrained system. They are indeed identical to (3.51), as they should.

We then see that the substitution of the holonomic constraint in the kinetic energy (or Lagrangian) of the unconstrained system, namely,

$$T(x,y,z,\dot{x},\dot{y},\dot{z}) \to T\left(x,\frac{1}{2}z^2,z,\dot{x},z\dot{z},\dot{z}\right), \tag{3.64}$$

is actually just an appropriate change in coordinates from the original coordinates (x,y,z) to (x,q,z), namely,

$$T(x,y,z,\dot{x},\dot{y},\dot{z}) \to T\left(x,q+\frac{1}{2}z^2,z,\dot{x},\dot{q}+z\dot{z},\dot{z}\right), \tag{3.65}$$

in which the coordinate q is so chosen that $q(t) = \dot{q}(t) \equiv 0$, thus making the right hand sides of the expressions in (3.64) and (3.65) identical. Were we to consider the system described by $T^+ = T(x,\frac{1}{2}z^2,z,\dot{x},z\dot{z},\dot{z})$ and the impressed force F as constituting an unconstrained system (since x and z are independent coordinates now), and use Lagrange's equations we would get the singular mass system described by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (1+z^2) \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ -z\dot{z}^2 + F_z \end{bmatrix}. \tag{3.66}$$

According to Section 2, we would then need to impose the constraint (3.49) on this unconstrained system, and obtain the description of the constrained system by further using Eq. (2.7). Since the matrix $[M|A^T]$ has full rank, we indeed can use Eq. (2.7) and if we did so, we would get the same (correct) set of equations of motion (3.61)–(3.63) as we did before.

Lastly, we consider what might happen if we were to partially conflate the kinetic energy T of the system by expressing the term $\frac{1}{2}\dot{y}^2$ as $\frac{1}{2}\alpha\dot{y}^2 + \frac{1}{2}(1-\alpha)\dot{y}^2$, and replacing $\frac{1}{2}\alpha\dot{y}^2$ by $\frac{1}{2}\alpha(\dot{q}+z\dot{z})^2$, while retaining the term $\frac{1}{2}(1-\alpha)\dot{y}^2$ in our amended kinetic energy T^+, analogous to what we did in arriving at (3.22). With this, the kinetic energy

$$T^+ = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha(\dot{q}+z\dot{z})^2 + \frac{1}{2}(1-\alpha)\dot{y}^2 + \frac{1}{2}\dot{z}^2 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha(z\dot{z})^2 + \frac{1}{2}(1-\alpha)\dot{y}^2 + \frac{1}{2}\dot{z}^2. \tag{3.67}$$

Notice that we have now made a transformation $(x,y,z) \to (x,y,z,q)$ in going from T to T^+; but since $q(t) \equiv 0$, the equation of motion $\ddot{q}(t) = 0$ is *uncoupled* from that for x, y, and z. This amounts to a transformation from $(x,y,z) \to (x,y,z,q \equiv 0)$, or simply from $(x,y,z) \to (x,y,z)$. One can then use the right-most expression for T^+ in (3.67) (instead of T) as the kinetic energy of the system along with the constraint (3.49) to get the correct equations of motion of the system. This happens because the use of the constraint Eq. (3.53) in T to get T^+ as we did in (3.67), is tantamount to the coordinate transformation from $(x,y,z) \to (x,y,z)$, i.e. the same coordinate system, so that T^+ is also the correct kinetic energy of the unconstrained system.

Thus, only in the special case when holonomic constraints are present, we can use T^+ and the impressed forces to comprise a constrained system whose virtual displacements are no longer independent as we did in (3.56)–(3.63), or we can consider it as an unconstrained system on which the constraints need to be further applied as we did in using (3.66)and later (2.7). Either of these conceptualizations will yield the correct equations of motion. The reason for this is that such substitutions of the constraints in the kinetic energy (Lagrangian) are tantamount to an appropriate special change of coordinates. But this will not work for more general systems with nonholonomic (or a combination of holonomic and nonholonomic) constraints since in that case, one cannot demonstrate such a coordinate transformation.

4. Conclusions

The main contributions of this paper are as follows.

(i) Hamel's paradox is not limited to the specific skate problem considered by him in his text. It brings out deeper issues in analytical dynamics – specifically, the way in which one needs to conceptualize a constrained mechanical system so that the equations of motion so obtained are consistent with physical observation. In doing this we are led to the view first proposed by Gauss which provides a uniform three-step method of conceptualizing such systems. This three-step conceptualization of constrained motion involves:
   1. description of the unconstrained system in which the coordinates are all independent of each other.
   2. description of the constraints, and
   3. description of the constrained system using the previous two descriptions.
      It provides a systematic way to obtain the correct equations of motion – those that are consistent with physical observation – of a constrained mechanical system.

(ii) In general, substitution of the constraint equation(s) in the kinetic energy (or in the Lagrangian) of a general constrained mechanical system conflates the unconstrained and the constrained descriptions of such a system. The system with such an amended kinetic energy, T^+ (or Lagrangian, L^+), cannot, in general, be considered either as (i) an appropriate description of the given constrained system, or (ii) an appropriate description of the given unconstrained system on which the constraints then need to be further imposed.

(iii) In the special situation in which we have only holonomic constraints such substitutions represent simply an appropriate change in coordinates, and the use of the amended kinetic energy (Lagrangian) will then lead to the correct equations of motion.
(iv) We note that to have a rigorous explanation of Hamel's paradox, we are led deeper to the foundations of analytical dynamics and to the use of newly developed concepts that deal with singular mass matrices.
(v) Though Lagrange's equations are over 200 years old, and the problem of constrained motion has been worked on near-continuously by numerous researchers since the time it was first conceived by Lagrange, analytical dynamics still has many interesting aspects that seem to need considerable care, and still call for improved understanding.

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