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Fundamental Principles of Lagrangian Dynamics: Mechanical Systems with Non-ideal, Holonomic, and Nonholonomic Constraints

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This paper deals with the foundations of analytical dynamics. It obtains the explicit equations of motion for mechanical systems that are subjected to non-ideal holonomic and nonholonomic equality constraints. It provides an easy incorporation of such non-ideal constraints into the framework of Lagrangian dynamics. It bases its approach on a fundamental principle that includes non-ideal constraints and that reduces to D'Alembert's Principle in the special case when all the constraints become ideal. Based on this, the problem of determining the equations of motion for the constrained system is reformulated as a constrained minimization problem. This yields a new fundamental minimum principle of analytical dynamics that reduces to Gauss's Principle when the constraints become ideal. The solution of this minimization problem then yields the explicit equations of motion for systems with non-ideal constraints. An illustrative example showing the use of this general equation for a system with sliding Coulomb friction is given. © 2000 Academic Press

Key Words: non-ideal constraints; holonomic and nonholonomic equality constraints; fundamental principle of Lagrangian dynamics; new minimum principle of analytical dynamics; explicit equations of motion for non-ideal constraints; sliding friction.

1. INTRODUCTION

The equations of motion for constrained mechanical systems are based on a principle which was first enunciated by D'Alembert and later elaborated by Lagrange [7], and which today is commonly referred to as "D'Alembert's Principle." Though Johann Bernoulli, Euler, and Leibnitz

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made significant contributions to this line of thinking, it was Lagrange who provided a general theory of constrained motion and made this principle a centerpiece of analytical mechanics. The principle states that at each instant of time t, a constrained mechanical system evolves in such a manner that the total work done by all the forces of constraint under any set of virtual displacements is always zero. This principle, which in effect prescribes the *nature* of the forces of constraint which act upon a mechanical system, has been found to yield, in practice, adequate descriptions of the motion of large classes of mechanical systems, thereby making it an extremely useful and effective principle. As such, it has so far been regarded as the very foundation of Lagrangian dynamics, and constraints which satisfy D'Alembert's Principle are often called ideal constraints.

Realizing that Lagrange's multiplier approach to determining the equations of motion for constrained systems is suitable at best to problem-specific situations, the basic problem of constrained motion has since been worked on intensively by numerous scientists including Volterra, Boltzmann, Hamel, Novozhilov, Whittaker, and Synge, to name a few. About 100 years after Lagrange, Gibbs [4], and Appell [1] independently devised what is today known as the Gibbs–Appell method for obtaining the equations of motion for constrained mechanical systems with non-integrable equality constraints. The method relies on a felicitous choice of quasicoordinates and, like the Lagrange multiplier method, is amenable to problem-specific situations. The Gibbs–Appell approach relies on choosing certain quasicoordinates and eliminating others thereby falling under the general category of elimination methods [13]. The central idea behind these elimination methods was again first developed by Lagrange when he introduced the concept of generalized coordinates. Yet, despite their discovery more than a century ago, the Gibbs–Appell equations were considered by many up until very recently, to be at the pinnacle of our understanding of constrained motion; they have been referred to by Pars [8] in his opus on analytical dynamics as "probably the simplest and most comprehensive equations of motion so far discovered."

Dirac considered Hamiltonian systems with singular Lagrangians and constraints that were not explicitly dependent on time; he once more attacked the problem of determining the Lagrange multipliers of the Hamiltonian corresponding to the constrained dynamical system. By ingeniously extending the concept of Poisson brackets, he developed a method for determining these multipliers in a systematic manner through the repeated use of the consistency conditions [2, 11]. More recently, an explicit equation describing constrained motion of both conservative and nonconservative dynamical systems within the confines of classical mechanics was developed by Udwadia and Kalaba [12]. They used as their starting point Gauss's Principle [3] of Least Constraint and considered general equality constraints which could be both nonlinear in the generalized velocities and displacements, and explicitly dependent on time. Furthermore, their result does not require the constraints to be functionally independent.

All the above mentioned methods for obtaining the equations of motion for constrained mechanical systems deal with *ideal* constraints wherein the constraint forces *do no work* under virtual displacements. The motion of an unconstrained system is, in general, altered by the imposition of constraints; this alteration in the motion of the unconstrained system can be viewed as being caused by the creation of additional "forces of constraint" brought into play through the imposition of these constraints. One view of the main task of analytical dynamics is that it gives a prescription for (uniquely) determining the accelerations of point particles in a given mechanical system at any instant of time, given their masses, positions, and velocities, the nature of the constraints they need to satisfy, and the "given" (impressed) forces acting on them, at that instant.

The nature and properties of the constraint forces which are generated during the motion of a given mechanical system depend on the specific physical situation at hand; these properties need to be suitably prescribed —through careful inspection, experimentation, or otherwise—by the mechanician who is attempting to model the motion of the given mechanical system. D'Alembert's principle is *only one way of prescribing the nature* of the constraint forces that may actually exist; though useful from a practical standpoint, in most mechanical systems, it may still be oftentimes only an approximate description of the true nature of the constraint forces present.

In determining the equation of motion that models a given constrained mechanical system, the use of D'Alembert's Principle has four important consequences: (1) it relieves the mechanician who is modeling the particular mechanical system from deciphering the actual dynamical nature of the constraint forces that act, allowing him/her to get by with simply stating the kinematic nature (see Eqs. (2) and (3) below) of the constraints; (2) it brings about a simplification in the determination of the equation of motion, since the equation dealing with the work done by all the forces acting on the system under virtual displacements no longer contains any terms which involve the *unknown* constraint forces; (3) it provides just enough conditions so that the unknown accelerations and the unknown constraint forces at each instant of time can be *uniquely* determined—that is, the problem of finding all these unknowns at each instant of time is neither under-determined, nor over-determined [12], and (4) conceptually, the principle differentiates all the forces acting on a system of particles as falling into one of two disjoint classes: "given forces" and "constraint forces." Indeed, Lagrangian dynamics has come to accept, and even define, constraint forces as those forces for which the sum total of the work done under virtual displacements equals zero.

Despite the advantages that accrue from the use of D'Alembert's Principle in the modeling of the motion of a constrained mechanical system, there are many situations in which experiments show that the principle is not valid, and the forces of constraint indeed *do* work under virtual displacements. Perhaps the most significant such situation is where the constrained motion involves sliding Coulomb friction. Such constraints, which engender forces of constraint that do work under virtual displacements, are said to be *non-ideal*; they pose considerable difficulties in being included within the general framework of Lagrangian dynamics (see Ref. [10]). To date, their inclusion has not been accomplished. As stated by Goldstein [5, p. 17], "This [total work done by constraint forces equal to zero] is no longer true if sliding friction forces are present, and we must exclude such systems from our [Lagrangian] formulation." The equation dealing with work done by all the forces under virtual displacements now contains the unknown forces of constraint, and the simplications which accrue when the constraints are ideal disappear. More importantly, the character which such non-ideal constraints need to possess so that the consequent accelerations and the constraint forces (that describe the motion of the constrained system) can be *uniquely* determined, remains an open question in Lagrangian dynamics. Conceptually, the presence of non-ideal constraints leads to forces which cannot be categorized simply as being either "given" forces or "constraint forces." As stated by Pars [8] in his treatise on analytical dynamics, "There are in fact systems for which the principle enunciated—that the forces all belong either to the category of given forces or to the category of forces of constraint—does not hold. But such systems will not be considered in this book."

In this paper we base Lagrangian dynamics on a new fundamental principle—one might think of it as a *generalization* of D'Alembert's Principle—which encompasses *non-ideal* constraints where the constraint forces *do* work under virtual displacements. We consider both holonomic and nonholonomic equality constraints. The new principle reduces, as it must, to the standard D'Alembert's Principle in the special case when all the constraints become ideal. We use this principle to derive a new minimum principle of analytical dynamics that is now applicable to non-ideal constraints. Using this minimum principle, we obtain the explicit equations of motion for mechanical systems with non-ideal holonomic, and non-holonomic constraints, thereby introducing general, non-ideal, constraints in a simple and straightforward way into the fabric of Lagrangian dynamics.

The paper is organized as follows. Section 2 describes the problem of constrained motion and provides a fundamental principle on which we base Lagrangian dynamics. The principle explicitly includes the presence of non-ideal constraints. In Section 3 we present a fundamental minimum principle of analytical dynamics. In Section 4 we use this new principle to obtain the explicit equation of motion for constrained systems with non-ideal constraints. In Section 5 we present an example of sliding friction, and in Section 6 we give our conclusions.

2. FUNDAMENTAL PRINCIPLES OF LAGRANGIAN DYNAMICS WITH NON-IDEAL CONSTRAINTS

Consider an "unconstrained system" of point particles, each particle having a constant, but possibly different, mass. We can write down the equations of motion for such a system, using either Lagrange's equations or Newtonian mechanics, in the form

$$M(q,t)\ddot{q} = Q(q,\dot{q},t), \qquad q(0) = q_0, \qquad \dot{q}(0) = \dot{q}_0, \tag{1}$$

where $q(t)$ is an n-vector (i.e., n by 1 vector) of generalized coordinates, M is an n by n symmetric, positive-definite matrix, Q is the "known" n-vector of impressed (or "given") forces, and the dots refer to differentiation with respect to time. By "unconstrained system" we mean that the components of the n-vector $\dot{q}_0$ can be independently specified. By "known" we mean a known function of the arguments. We note from Eq. (1) that the quantity $a(t) = M^{-1}Q$ gives the acceleration of the unconstrained system at time t.

Next, let this system be subjected to a set of $m = h + s$ consistent constraints of the form

$$\varphi(q,t) = 0 \tag{2}$$

and

$$\psi(q,\dot{q},t) = 0, \tag{3}$$

where φ is an h-vector and ψ an s-vector. Furthermore, we shall assume that the initial conditions q_0 and $\dot{q}_0$ satisfy these constraint equations at time $t = 0$.

Assuming that Eqs. (2) and (3) are sufficiently smooth, we differentiate Eq. (2) twice with respect to time, and Eq. (3) once with respect to time, to obtain the equation

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t), \tag{4}$$

where the matrix A is m by n, and b is the m-vector which results from carrying out the differentiations. It is important to note that Eq. (4) is equivalent to Eqs. (2) and (3). This set of constraint equations include among others, the usual holonomic, nonholonomic, scleronomic, rheonomic, catastatic, and acatastatic varieties of constraints.

Because of the constraints imposed on the system, the motion of the constrained mechanical system at any time t, now deviates, in general, from what it might have been were there no constraints acting. This deviation of the constrained motion from that of the unconstrained system may be thought of as being brought about by an additional force at time t, an n-vector Q^c called the force of constraint. The *properties and nature* of this force of constraint are situation-specific and the mechanician who is modeling the motion of the system needs to prescribe them in order to obtain the requisite equation of motion for the specific system under consideration. The motion of the constrained mechanical system is thus described by the equation

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q^c(q,\dot{q},t), \tag{5}$$

where the n-vector Q^c denotes the engendered force of constraint.

Equation (1) thus pertains to the description of the motion of the unconstrained system at a given time t; Eqs. (2) and (3) pertain to the further (kinematic) constraints imposed on the system. *Along with a characterization of the nature and properties of the force of constraint that is prescribed by the mechanician* and is situation-specific, Eqs. (1), (2), and (3) then specify the constrained mechanical system. The task of analytical dynamics can now be viewed as determining the acceleration n-vector, $\ddot{q}(t)$, of the constrained system from our knowledge at time t of: the n-vectors $q(t)$, and $\dot{q}(t)$; the constraint relations (2) and (3) (or alternatively, relation (4)); and, the prescribed additional information about the nature and properties of the constraint force n-vector Q^c.

For future use, we define a virtual displacement n-vector $v(t)$ at time t as any vector that satisfies the relation (see, for example, [13])

$$A(q,\dot{q},t)v = 0. \tag{6}$$

We are now ready to state the following Fundamental Principle on which we shall base Lagrangian dynamics.

The constrained mechanical system described by Eqs. (1), (2), *and* (3) *evolves in time in such a manner that the total work done at any time*, t, *by the constraint force n-vector* Q^c *under virtual displacements at time t is given by*

$$v^T Q^c = v^T C(q,\dot{q},t), \tag{7}$$

where $C(q,\dot{q},t)$ *is a known*, *prescribed*, *sufficiently smooth n-vector* (*it needs only to be* $\mathscr{C}^1$), *and* v *is any virtual displacement n-vector at time t*. *The vector C is pertinent to the specific mechanical system under consideration and needs to be prescribed by the mechanician who is modeling its motion*.

We note that when $C \equiv 0$, the total work done by the forces of constraint under virtual displacements becomes zero, and so this principle reduces to the usual D'Alembert's Principle in the special case when all the constraints are ideal. In the following section we shall use this principle to reduce Lagrangian dynamics to a quadratic minimization problem.

3. THE FUNDAMENTAL QUADRATIC MINIMIZATION PRINCIPLE OF ANALYTICAL DYNAMICS

Let us denote by a *possible* acceleration of the system *any* acceleration n-vector which satisfies the equations of constraint. Since at time t the values of the n-vectors $q(t)$ and $\dot{q}(t)$ are assumed to be known, a *possible* acceleration, $\hat{\ddot{q}}(t)$, is then any n-vector which satisfies the constraint Eq. (4) so that

$$A(q, \dot{q}, t)\hat{\ddot{q}} = b(q, \dot{q}, t). \tag{8}$$

Subtracting Eqs. (4) and (8) we obtain

$$A(q, \dot{q}, t)\left(\hat{\ddot{q}} - \ddot{q}\right) = 0. \tag{9}$$

The n-vector $d(t) = (\hat{\ddot{q}} - \ddot{q})$ represents the *deviation* at time t of a *possible* acceleration, $\hat{\ddot{q}}(t)$, from the true acceleration, $\ddot{q}(t)$, of the constrained system.

Since any n-vector $v(t)$ which satisfies, at time t, the relation $A(q, \dot{q}, t)v = 0$ constitutes a virtual displacement at time t, the n-vector $d(t)$ represents a virtual displacement. In what follows we shall drop the arguments of the various quantities, unless their presence is necessary for clarification.

Using Eq. (5), the fundamental principle then states that at each instant of time t,

$$v^T Q^c = v^T [M\ddot{q} - Q] = v^T C. \tag{10}$$

The last equality above then requires that at each instant of time t,

$$v^T [M\ddot{q} - Q - C] = 0, \tag{11}$$

which becomes

$$d^T [M\ddot{q} - Q - C] = 0 \tag{12}$$

in view of the fact that the n-vector $d(t) = (\hat{\ddot{q}} - \ddot{q})$ qualifies as a virtual displacement vector for any *possible* acceleration vector $\hat{\ddot{q}}(t)$.

We now present the following lemmas.

LEMMA 1. *For any n by n symmetric matrix Y, and any set of n-vectors e, f, and g, we have*

$$(e-g, e-g)_Y - (e-f, e-f)_Y$$
$$= (g-f, g-f)_Y - 2(e-f, g-f)_Y, \tag{13}$$

where we define $(a, b)_Y \equiv a^T Y b$ *for any two n-vectors a and b.*

Proof. This identity can be verified directly. ∎

When Y is a positive definite matrix, result (13) may be geometrically thought of as a generalization of the "cosine rule" in a triangle using the metric given by Y.

LEMMA 2. *Any n-vector* $d(t) = (\hat{\ddot{q}} - \ddot{q})$ *satisfies the relation*

$$\left(M\hat{\ddot{q}} - (Q+C), M\hat{\ddot{q}} - (Q+C)\right)_{M^{-1}}$$
$$-\left(M\ddot{q} - (Q+C), M\ddot{q} - (Q+C)\right)_{M^{-1}}$$
$$= (d, d)_M + 2\left(M\ddot{q} - (Q+C), d\right), \tag{14}$$

where M is any symmetric, positive-definite matrix.

Proof. Set $Y = M$, $e = M^{-1}(Q+C)$, $f = \ddot{q}$, and $g = \hat{\ddot{q}}$ in Eq. (13). The result follows. ∎

We are now ready to state the Fundamental Minimum Principle of analytical dynamics.

Result 1. A constrained mechanical system subjected to *non-ideal* holonomic and/or nonholonomic constraints evolves in time in such a way that its acceleration n-vector $\ddot{q}$ at each instant of time t (given q and $\ddot{q}$ at time t) minimizes the quadratic form

$$G_{ni}(\hat{\ddot{q}}) := \left(M\hat{\ddot{q}} - (Q+C), M\hat{\ddot{q}} - (Q+C)\right)_{M^{-1}} \tag{15}$$

over all *possible* accelerations $\hat{\ddot{q}}$ at that instant of time t. Noting Eq. (5), the mechanical system evolves *as though* it minimizes the measure of constraint given by

$$G_{ni} = (Q^c - C, Q^c - C)_{M^{-1}}, \tag{16}$$

where the work done at time t by the constraint force Q^c under virtual displacements v is given by $v^T C$.

Proof. For the constrained mechanical system described by Eqs. (1), (2), and (3), the n-vector $d(t)$ satisfies relation (12). Hence the last member on the right hand side of Eq. (14) becomes zero, and Eq. (14) can be rewritten as

$$\left(M\hat{\ddot{q}} - (Q + C), M\hat{\ddot{q}} - (Q + C)\right)_{M^{-1}} \\ -\left(M\ddot{q} - (Q + C), M\ddot{q} - (Q + C)\right)_{M^{-1}} = (d, d)_{M}. \quad (17)$$

But M is a positive definite matrix, and hence the scalar on the right hand side of Eq. (17) must be positive for any nonzero vector d. The minimum of (15) must therefore occur when $\hat{\ddot{q}}(t) = \ddot{q}(t)$. Furthermore, the actual motion of the constrained mechanical system with non-ideal constraints is such as to minimize the measure of constraint force given by (16). ▮

Remark 1. The above result is a generalization of Gauss's principle of least constraint which is now applicable to "non-ideal" constraints. The subscript *ni* on G in Eqs. (15) and (16) is meant to indicate this. It can be considered to be an alternative starting point for analytical dynamics involving non-ideal constraints. The quantity $Q^c - C$ is that part of the constraint force that does *no* work under virtual displacements.

Remark 2. The Fundamental Principle of analytical dynamics stated in Result 1 appears to be the only true "minimum principle" in analytical dynamics; the others (like Hamilton's Principle) deal, in general, with extremization of functionals. In addition, it should be pointed out that unlike other extremization principles that involve integrals over time, this minimum principle is valid at *each instant* of time.

Remark 3. We observe that when the constraints are ideal, then $C = 0$, and the minimum principle stated above in (15) and (16) becomes Gauss's Principle of Least Constraint (Gauss [3]) which requires the minimization of the constraint measure

$$G_i = (Q^c, Q^c)_{M^{-1}} \quad (18)$$

over all *possible* accelerations of the system at each instant of time t.

Remark 4. We note from the proof that the minimum in (15) and (16) is *global* since the *possible* accelerations are not restricted in magnitude, as long as they satisfy Eq. (4).

Remark 5. Result 1 states the following: Given that we know the state (i.e., q and $\dot{q}$) of a mechanical system at time t, of all the possible accelerations at that time consistent with the constraint equations and with the prescription of C provided by the mechanician, the actual acceleration that the constrained mechanical system "chooses" is the one that minimizes G_{ni}.

Remark 6. At each instant of time t, the determination of the acceleration vector, $\ddot{q}$, of the constrained system with non-ideal constraints leads to the following constrained quadratic minimization problem: At each instant of time t,

$$\underset{\{\hat{\ddot{q}} \mid A\hat{\ddot{q}} = b\}}{\text{Min}} \left[G_{ni}(\hat{\ddot{q}}) \right]. \tag{19}$$

The $\hat{\ddot{q}}$ that minimizes this quadratic form (19) at the instant of time t is then the acceleration of the mechanical system, $\ddot{q}(t)$.

4. EXPLICIT EQUATIONS OF MOTION FOR SYSTEMS WITH NON-IDEAL CONSTRAINTS

In this section we obtain the explicit equations of motion. Noting that the acceleration of the unconstrained system is given by $a = M^{-1}Q$, and denoting $c = M^{-1}C$, the expression for G_{ni} can be expressed as

$$G_{ni}(\ddot{q}) = (\ddot{q} - a - c)^T M(\ddot{q} - a - c). \tag{20}$$

From all those $\ddot{q}$'s which satisfy the relation $A\ddot{q} = b$ at time t, we need to find that $\ddot{q}$ which minimizes (20). Let $r = M^{1/2}(\ddot{q} - a - c)$, so that

$$\ddot{q} = M^{-1/2}r + a + c, \tag{21}$$

and the relation $A\ddot{q} = b$ transforms to

$$AM^{-1/2}r = b - Aa - Ac. \tag{22}$$

The quadratic minimization problem reduces to the determination of the vector r which satisfies Eq. (22) and minimizes $\|r\|_2^2$. But the solution to this problem is simply [6]

$$r = (AM^{-1/2})^{\{1,4\}}(b - Aa - Ac), \tag{23}$$

where the superscript $\{1, 4\}$ denotes *any* $\{1, 4\}$ generalized inverse of the matrix $AM^{-1/2}$ [13]. Noting Eq. (21), and the definitions of a and c, we have our next result.

Result 2. The explicit equation of motion for a mechanical system with non-ideal constraints is given by

$$\begin{aligned} \ddot{q} &= a + M^{-1/2}(AM^{-1/2})^{\{1,4\}}(b - Aa) \\ &\quad + M^{-1/2}\left[I - (AM^{-1/2})^{\{1,4\}}AM^{-1/2} \right] M^{-1/2}C \end{aligned} \tag{24}$$

or

$$M\ddot{q} = Q + M^{1/2}(AM^{-1/2})^{\{1,4\}}(b - AM^{-1}Q) + M^{1/2}\Big[I - (AM^{-1/2})^{\{1,4\}}(AM^{-1/2})\Big]M^{-1/2}C. \qquad (25)$$

When all the constraints are ideal, $C = 0$, and the third member on the right-hand side of each of Eqs. (24) and (25) disappears. Comparing Eq. (25) with Eq. (5) we observe that the total constraint force n-vector Q^c can be written as the sum of two n-vectors as

$$Q^c = Q_i^c + Q_{ni}^c, \qquad (26)$$

where

$$Q_i^c = M^{1/2}(AM^{-1/2})^{\{1,4\}}(b - AM^{-1}Q), \qquad (27)$$

and

$$Q_{ni}^c = M^{1/2}\Big[I - (AM^{-1/2})^{\{1,4\}}(AM^{-1/2})\Big]M^{-1/2}C. \qquad (28)$$

We thus see that the total constraint force is made up of two contributions: (1) the force Q_i^c which is the force of constraint *had all the constraints been ideal*; and (ii) the force Q_{ni}^c which is the contribution to the total constraint force from the non-ideal nature of the constraints. For a *given* mechanical system the vector $C(q, \dot{q}, t)$ (or its equivalent) in Eq. (28) needs to be prescribed by the mechanician at each instant of time. When the mechanician specifies that $C \equiv 0$ for all time, i.e., the constraints are ideal, then the second contribution Q_{ni}^c equals zero; however, the first contribution Q_i^c is *ever-present* whether or not the constraints are ideal.

Equation (25) may be considered to be a generalization of the equation of motion obtained by Udwadia and Kalaba [12], which now includes the possible presence of non-ideal holonomic and nonholonomic constraints. Equations (24) and (25) show the simple and straightforward way in which non-ideal constraints are hereby included in Lagrangian dynamics.

Remark 7. It should be noted that though the expression for the solution n-vector r obtained in Eq. (23) utilizes *any* $\{1,4\}$ generalized inverse of the matrix $AM^{-1/2}$, the solution vector r is uniquely determined [13]. Consequently, the acceleration n-vector $\ddot{q}$ is *uniquely* determined from the right hand side of Eq. (24).

Remark 8. A special $\{1,4\}$ inverse of the matrix $AM^{-1/2}$ is the usual pseudo-inverse [9], also called the $\{1,2,3,4\}$ inverse, and denoted by $(AM^{-1/2})^+$. Using this inverse in Eqs. (24) and (25) we get the explicit equation of motion of the mechanical system to be

$$\ddot{q} = a + M^{-1/2}(AM^{-1/2})^{+}(b - Aa) + M^{-1/2}\left[I - (AM^{-1/2})^{+}AM^{-1/2}\right]M^{-1/2}C, \tag{29}$$

and

$$M\ddot{q} = Q + M^{1/2}(AM^{-1/2})^{+}(b - AM^{-1}Q) + M^{1/2}\left[I - (AM^{-1/2})^{+}(AM^{-1/2})\right]M^{-1/2}C. \tag{30}$$

In Eqs. (27) and (28) we could similarly replace $(AM^{-1/2})^{\{1,4\}}$ by $(AM^{-1/2})^{+}$ to obtain the corresponding expressions for Q_i^c and Q_{ni}^c, respectively.

5. ILLUSTRATIVE EXAMPLE

We show now the use of the general Eq. (25) which describes the motion of a mechanical system with non-ideal constraints when applied to a problem involving sliding Coulomb friction.

Consider a particle of unit mass constrained to move in a circle in the vertical plane on a circular ring of radius R under the action of gravity. We use Cartesian coordinates with the origin at the center of the ring. The unconstrained motion of the particle is given by

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} \tag{31}$$

and the constraint is represented by $x^2 + y^2 = R^2$, which upon two differentiations with respect to time becomes

$$[x \quad y]\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = -(\dot{x}^2 + \dot{y}^2), \tag{32}$$

so that $A = [x \quad y]$, and $A^{+} = (1/R^2)\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. Were this constraint to be ideal, the force of constraint Q_i^c would be given by Eq. (27) so that

$$Q_i^c = -\begin{bmatrix} x/R \\ y/R \end{bmatrix}\frac{(\dot{x}^2 + \dot{y}^2 - gy)}{R}, \tag{33}$$

and the equation of motion of the constrained system becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} - \begin{bmatrix} x/R \\ y/R \end{bmatrix} \frac{\left(\dot{x}^2 + \dot{y}^2 - gy\right)}{R}. \tag{34}$$

The magnitude of this constraint force, had the constraint been ideal, is given by

$$|Q_i^c| = \left| \frac{\left(\dot{x}^2 + \dot{y}^2 - gy\right)}{R} \right|. \tag{35}$$

Let the nature of the non-ideal constraint generated by sliding friction between the ring and the mass be described by

$$v^T Q^c = v^T C \equiv -v^T \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \frac{\mu |Q_i^c|}{\sqrt{\left(\dot{x}^2 + \dot{y}^2\right)}}, \tag{36}$$

where μ is the coefficient of friction between the ring and the mass.

Along the circular trajectory of the particle, $x\dot{x} = -y\dot{y}$, and we get

$$C = -\frac{\mu |Q_i^c|}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = -\frac{\mu |Q_i^c|}{R} \begin{bmatrix} -y \operatorname{sgn}(x) \\ |x| \end{bmatrix} \operatorname{sgn}(\dot{y}). \tag{37}$$

The contribution to the constraint force provided by this non-ideal constraint is then given by Eq. (28). Replacing $A^{\{1,4\}}$ by A^+ (see Remark 8) in Eq. (28) we get (note $M = I_2$),

$$Q_{ni}^c = \{I - A^+ A\} C = -\frac{\mu |Q_i^c|}{R} \left\{ I - \frac{1}{R^2} \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} \right\} \begin{bmatrix} -y \operatorname{sgn}(x) \\ |x| \end{bmatrix} \operatorname{sgn}(\dot{y})$$

$$= -\mu |Q_i^c| \begin{bmatrix} -y \operatorname{sgn}(x)/R \\ x \operatorname{sgn}(x)/R \end{bmatrix} \operatorname{sgn}(\dot{y}). \tag{38}$$

The explicit equation of motion for the particle then becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} - \begin{bmatrix} x/R \\ y/R \end{bmatrix} \frac{\left(\dot{x}^2 + \dot{y}^2 - gy\right)}{R} - \mu |Q_i^c| \begin{bmatrix} -y \operatorname{sgn}(x)/R \\ x \operatorname{sgn}(x)/R \end{bmatrix} \operatorname{sgn}(\dot{y}). \tag{39}$$

We could have also used the relation for C (given by the first equality in Eq. (37)) directly in Eq. (30) to obtain the equation of motion of the non-ideally constrained system. For brevity, we have not shown examples of nonholonomic, non-ideal constraints. They may be found elsewhere (see Ref. [14]).

6. CONCLUSIONS

Since its inception about 200 years ago, Lagrangian mechanics has been built upon the underlying principle of D'Alembert. This principle makes the confining assumption that all constraints are ideal constraints for which the sum total of the work done by the forces of constraint under virtual displacements is zero. Though often applicable, experiments show that this assumption may be invalid in many practical situations, such as when sliding Coulomb friction is important. This paper releases Lagrangian mechanics from this confinement and obtains the explicit equations of motion allowing for holonomic and/or nonholonomic constraints which are non-ideal. The explicit equations of motion obtained here are accordingly based on a more general principle, which then includes D'Alembert's Principle as a special case when the constraints are ideal. From this new principle is derived a Fundamental Minimum Principle of analytical dynamics that reduces to Gauss's Principle of Least Constraint in the special case when the constraints are all ideal. The solution of the constrained minimization problem gives the general, explicit equations of motion for mechanical systems with non-ideal constraints.

We list below the main contributions of this paper.

1. We have stated a fundamental principle of mechanics which encompasses non-ideal constraints. The principle reduces to D'Alembert's Principle when all the constraints become ideal.

2. On the basis of this principle we have reformulated Lagrangian dynamics as a constrained quadratic minimization problem. The resulting formulation results in a new fundamental minimum principle of analytical dynamics that is now applicable to non-ideal constraints; it reduces to Gauss's Principle of Least Constraint in the special situation when all the constraints are ideal.

3. By solving this quadratic minimization problem we have obtained the general, explicit equations of motion pertinent to mechanical systems subjected to non-ideal holonomic and nonholonomic constraints.

4. The total constraint force exerted on the system by virtue of the constraints is shown to be made up of two additive contributions. The first contribution, Q_i^c, comes from the constraints *as though they were* ideal; the second, Q_{ni}^c, comes from the non-ideal character of the constraints. For a

given mechanical system, Q^c_{ni} depends on the vector function $C(q, \dot{q}, t)$ that needs to be prescribed by the mechanician.

5. We have shown that the accelerations under general, non-ideal, constraints can be uniquely determined when C is known (i.e., a known function of its arguments) at each instant of time.

6. No elimination of coordinates or quasi-coordinates (as required by the Gibbs–Appell approach) is undertaken [1, 4]. Consequently, the equations of motion pertinent to the constrained system with non-ideal holonomic and/or nonholonomic constraints are obtained in the *same set of coordinates* which are used to describe the unconstrained system. This facilitates a direct comparison between the equations of motion for the constrained and the unconstrained mechanical system, thereby showing simply and explicitly the effects that the addition of constraints, whether they be ideal or not, have on the equations of motion of the unconstrained system.

REFERENCES

1. P. Appell, Sur une forme generale des equations de la dynamique, *C. R. Acad. Sci. Paris* **129** (1899), 459–460.
2. P. A. M. Dirac, Lectures in quantum mechanics, Yashiva University, New York, 1964.
3. C. Gauss, Uber ein neues allgemeines grundgesetz der mechanik, *J. Reine Angew. Math.* **4** (1829), 232–235.
4. W. Gibbs, On the fundamental formulae of dynamics, *Amer. J. Math.* **2** (1879), 49–64.
5. H. Goldstein, “Classical Mechanics,” 2nd ed., Addison–Wesley, Reading, MA, 1981.
6. F. Graybill, “Matrices and Applications to Statistics,” Wadsworth, Belmont, CA, 1983.
7. J. L. Lagrange, “Mecanique Analytique,” Mme Ve Couricer, Paris, 1787.
8. L. A. Pars, “A Treatise on Analytical Dynamics,” Oxbow Press, Woodridge, CT, 1979.
9. R. Penrose, A generalized inverse of matrices, *Math. Proc. Cambridge Philos. Soc.* **51** (1955), 406–413.
10. R. Rosenberg, “Analytical Dynamics of Discrete Systems,” Plenum, New York, 1972.
11. E. C. G. Sudarshan and N. Mukunda, “Classical Dynamics: A Modern Perspective,” Wiley, New York, 1974.
12. F. E. Udwadia and R. E. Kalaba, A new perspective on constrained motion, *Proc. Roy. Soc. London* **439** (1992), 407–410.
13. F. E. Udwadia and R. E. Kalaba, “Analytical Dynamics: A New Approach,” Cambridge Univ. Press, New York, 1996.
14. F. E. Udwadia and R. E. Kalaba, Nonideal constraints and Lagrangian dynamics, *J. Aerospace Engrg.* **13** (2000), 17–22.