F. E. Udwadia
Professor of Civil Engineering, Aerospace and Mechanical Engineering, and Mathematics

R. E. Kalaba
Professor of Biomedical Engineering, Electrical Engineering, and Economics

University of Southern California,
Los Angeles, CA 90089-1453

Explicit Equations of Motion for Mechanical Systems With Nonideal Constraints

Since its inception about 200 years ago, Lagrangian mechanics has been based upon the Principle of D'Alembert. There are, however, many physical situations where this confining principle is not suitable, and the constraint forces do *work. To date, such situations are excluded from general Lagrangian formulations. This paper releases Lagrangian mechanics from this confinement, by generalizing D'Alembert's principle, and presents the explicit equations of motion for constrained mechanical systems in which the constraints are nonideal. These equations lead to a simple and new fundamental view of Lagrangian mechanics. They provide a geometrical understanding of constrained motion, and they highlight the simplicity with which Nature seems to operate.*
[DOI: 10.1115/1.1364492]

1 Introduction

One of the central problems in classical mechanics is the determination of the equations of motion for constrained systems. The importance of the problem stems from the fact that what makes a set of point masses and rigid bodies, a ''system,'' is the presence of constraints. When physical constraints are imposed on an unconstrained set of particles, forces of constraint are engendered which ensure the satisfaction of the constraints. The equations of motion developed to date for such constrained systems are based on a principle first enunciated by D'Alembert, and later elaborated by Lagrange [1] in his *Mechanique Analytique* which dates back to 1787. Today the principle is referred to as D'Alembert's principle, and it is the centerpiece of classical analytical dynamics. It states, simply, that the total work done by the forces of constraint under virtual displacements is always zero. Constraints for which D'Alembert's principle is applicable are referred to as *ideal* constraints.

Since its initial formulation by Lagrange more than 200 years ago, the problem of constrained motion has been vigorously and continuously worked on by numerous scientists including Volterra, Boltzmann, Hamel, Whittaker, and Synge, to name a few. In 1829, Gauss [2] provided a new general principle for the motion of constrained mechanical systems in what is today referred to as Gauss's Principle. About 100 years after Lagrange, Gibbs [3] and Appell [4] independently discovered what are known today as the Gibbs-Appell equations of motion ([3,4]). Pars ([5], p. 202) refers to the Gibbs-Appell equations as ([5]) ''... probably the most comprehensive equations of motion so far discovered.'' Dirac, because of his interest in constrained systems that arise in quantum mechanics, in a series of papers from 1951 to 1969 developed an approach for determining the Lagrange multipliers for constrained Hamiltonian systems ([6]). More recently, Udwadia and Kalaba [7] presented a simple, explicit, set of equations, applicable to general mechanical systems, with holonomic and nonholonomic constraints ([7,8]).

However, *all* these alternative descriptions of the motion of constrained systems discovered so far, as well as the numerous articles that have subsequently dealt with them, rely on D'Alembert's principle, and each of these mathematical formalisms is equivalent to the other. Despite the continuous and vigorous attention that this problem has received, the inclusion of situations where the physically generated forces of constraint in a mechanical system *do not* satisfy D'Alembert's principle has so far evaded Lagrangian dynamics. Yet, such forces of constraint are among those quite commonly found in nature. As stated by Goldstein ([9], p. 17), ''This [total work done by forces of constraint equal to zero] is no longer true if sliding friction is present, and we must exclude such systems from our [Lagrangian] formulation'' ([9]). And Pars in his treatise ([5]) on analytical dynamics (1979, p. 14) writes, ''There are in fact systems for which the principle enunciated [D'Alembert's principle] ... does not hold. But such systems will not be considered in this book.''

In this paper we obtain the equations of motion for constrained systems where the forces of constraint indeed do not satisfy D'Alembert's principle, and the sum total of the work done by them under virtual displacements no longer need be zero.

The outline of the paper is as follows. In Section 2.1 we generalize D'Alembert's Principle to include constraint forces that *do* work. This leads us to a deeper understanding of the specification of constraints in mechanical systems. This we discuss in Section 2.2. Section 3 deals with the mathematical statement of the problem of constrained motion. Section 4 states and verifies the explicit equation of motion for constrained systems with nonideal equality constraints. This equation leads to a new and fundamental principle of Lagrangian mechanics. The proof we give here is simpler than the one given in ([10]), and it yields an important geometrical interpretation that we discuss later. Section 5 gives an example of a nonholonomically constrained system for which the constraints are nonideal. We show here the ease of applicability of the explicit equation of motion obtained in the previous section and point out the insights it provides into understanding constrained motion where the constraint forces do work. Lastly, Section 6 deals with the geometry of constrained motion and exhibits the simplicity and aesthetics with which Nature seems to operate.

2 Generalization of D'Alembert's Principle, Constraint Forces That *Do* Work, and Their Specification

2.1 Generalized D'Alembert's Principle. Consider an unconstrained system of particles, each particle having a constant mass. By ''unconstrained'' we mean that the number of generalized coordinates, n, used to describe the configuration of the sys-

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, April 2, 2000; final revision, Oct. 9, 2000. Associate Editor: N. C. Perkins. Discussion on the paper should be addressed to the Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

tem at any time, t, equals the number of degrees-of-freedom of the system. The Lagrangian equation of motion for such a system can be written in the form

$$M(q,t)\ddot{q}=Q(q,\dot{q},t), \quad q(0)=q_0, \;\dot{q}(0)=\dot{q}_0 \tag{1}$$

where $q(t)$ is the n-vector (i.e., n by 1 vector) of generalized coordinates, M is an n by n symmetric, positive-definite matrix, Q is the ''known'' n-vector of impressed forces, and the dots refer to differentiation with respect to time. By ''known,'' we shall mean that Q is a known function of its arguments. The acceleration, a, of the unconstrained system at any time t is then given by the relation $a(q,\dot{q},t)=M^{-1}(q,t)Q(q,\dot{q},t)$.

We shall assume that this system is subjected to a set of $m=h+s$ consistent equality constraints of the form

$$\varphi(q,t)=0 \tag{2}$$

and

$$\psi(q,\dot{q},t)=0, \tag{3}$$

where φ is an h-vector and ψ an s-vector. Furthermore, we shall assume that the initial conditions q_0 and $\dot{q}_0$ satisfy these constraint equations at time $t=0$. Assuming that Eqs. (2) and (3) are sufficiently smooth, we differentiate Eq. (2) twice with respect to time, and Eq. (3) once with respect to time, to obtain the equation

$$A(q,\dot{q},t)\ddot{q}=b(q,\dot{q},t), \tag{4}$$

where the matrix A is m by n, and b is a suitably defined m-vector that results from carrying out the differentiations.

This set of constraint equations includes among others, the usual holonomic, nonholonomic, scleronomic, rheonomic, catastatic, and acatastatic varieties of constraints; combinations of such constraints may also be permitted in Eq. (4). It is important to note that Eq. (4), together with the initial conditions, is equivalent to Eqs. (2) and (3).

Consider now any instant of time t. When the equality constraints (Eqs. (2) and (3)) are imposed at that instant of time on the unconstrained system, the motion of the unconstrained system is, in general, altered from what it would have been (at that instant of time) in the absence of these constraints. We view this alteration in the motion of the unconstrained system as being caused by an additional set of forces, called the ''forces of constraint,'' acting on the system at that instant of time. The equation of motion of the constrained system can then be expressed as

$$M(q,t)\ddot{q}=Q(q,\dot{q},t)+Q^c(q,\dot{q},t), \quad q(0)=q_0, \dot{q}(0)=\dot{q}_0 \tag{5}$$

where the additional ''constraint force'' n-vector, $Q^c(q,\dot{q},t)$, arises by virtue of the constraints (2) and (3) imposed on the unconstrained system, which is described by Eq. (1). Our aim is to determine Q^c explicitly at time t in terms of the known quantities M, Q, A, b, ***and*** information about the nonideal nature of the constraint force, at time t. The latter comes from looking at the physics of the system.

A virtual displacement ([8]) at time t is any nonzero n-vector v such that $A(q,\dot{q},t)v=0$. When the constraint force n-vector does *no* work under virtual displacements v, we have $v^TQ^c=0$. This is also referred to as D'Alembert's principle, and it is the basis that underlies *all* the different formalisms ([1–8]) hereto developed of the equations of motion for mechanical systems subjected to the constraints described by Eqs. (2) and (3).

As demonstrated elsewhere ([7,8]), one formalism that yields an *explicit* equation describing the motion of such a constrained system that abides by D'Alembert's principle is given by $M\ddot{q}=Q+M^{1/2}B^{+}(b-Aa)=Q+Q^c$, where the m by n matrix $B=AM^{-1/2}$, and B^{+} stands for the Moore-Penrose generalized inverse ([11]) of the matrix B.

The central question that arises now is how to incorporate into the equation of motion, *constraints* that *do* do work under virtual displacements, thereby bringing such constraints within the Lagrangian framework. Such nonideal constraint forces (for example, sliding frictional forces) are in fact commonplace, and have to date defied ([5,9]) inclusion in a simple way within the general framework of analytical mechanics. The main reason for this difficulty is that three obstacles need to be simultaneously surmounted. Firstly, we require the specification of such constraint forces to be general enough so that they encompass problems of practical utility. Secondly, this specification must, in order to comply with physical observations, yield the accelerations of the constrained system *uniquely* when using the accepted mathware of analytical dynamics that has been developed over the last 250 years. And lastly, when the constraint forces do no work, we must obtain the usual formalisms/equations that have thus far been obtained (e.g., by Lagrange, Gibbs, Appell, and Gauss), and are known to be of practical value.

Clearly, the work done by such a constraint force under virtual displacements v at each instant of time needs to be known, and must therefore be specified using some known n-vector $C(q,\dot{q},t)$, as v^TC. Such an additional specification calls for a generalization of D'Alembert's principle. We make this generalization in the following manner:

For any virtual displacement v at time t, the constraint force n-vector Q^c at time t does a prescribed amount of work given by

$$v^TQ^c(t)=v^TC(q,\dot{q},t). \tag{6}$$

Here $C(q,\dot{q},t)$ is a known n-vector (i.e., a known function of q, $\dot{q}$, and t) that needs to be specified and depends on the physics of the situation, as discussed in the example below. The work done by the constraint force in a virtual displacement may thus be *positive, negative, or zero*.

Relation (6) constitutes a new principle. This principle requires a description of the nature of the nonideal constraint force at time t through a specification of the work it does during a virtual displacement at that time. It generalizes D'Alembert's principle, and when $C\equiv 0$, it reduces to it. In what follows we shall often refer to the constraint force n-vector, Q^c, as the constraint force.

2.2 Specification of Constraints. The equations of motion provide a mathematical model for describing the motion of any given physical mechanical system. The constraints specify the conditions that the generalized displacements and/or velocities must satisfy at each instant of time as the motion of the system ensues under the action of the impressed forces. However, the equations that state these conditions (Eqs. (2) and (3)) *do not completely specify the influence of these constraints on the motion of the mechanical system*. For short, we shall say that Eqs. (2) and (3) do not completely specify the constraints on the mechanical system. This is what the generalized D'Alembert's principle tells us.

There is a second part to the specification of the constraints, and this deals with the *nature of the forces that are created by virtue of the presence of the constraints*. For this, the mechanician who is modeling a specific mechanical system needs to study the system, possibly through experimentation, or otherwise. It is this information regarding the nonideal *nature* of the force of constraint that is encapsulated in the vector $C(q,\dot{q},t)$.

For example, consider a rigid block that is confined to move on a horizontal surface $z=0$. The specification of this relation (i.e., $z=0$) does *not* constitute a complete specification of the constraint. For, the presence of this constraint creates a constraint force, and this force influences the motion of the block. So to adequately model the motion of the block on the surface, one needs to prescribe the *nature* of this constraint force. Such a prescription is situation-specific and must be specified by the mechanician either by experimentation with the system, by observation, by analogy with other systems (s)he has experience with, or by some other means. For example, if the mechanician finds that the surfaces in contact are rough (s)he may want to perform some experiments to understand the nature of the forces created by the presence of this constraint. For a specific setup, (s)he may find that the work done by the constraint force under virtual displacements is proportional to the speed of the block, or perhaps to the square of its speed. Thus, depending on the situation at hand, C would then be specified as $-a_0[\begin{smallmatrix}\dot{q}_x\\ \dot{q}_y\end{smallmatrix}]$ or $-a_0|\dot{q}|[\begin{smallmatrix}\dot{q}_x\\ \dot{q}_y\end{smallmatrix}]$ respectively, where a_0 may be a suitable constant whose value would also need to be prescribed (perhaps by performing more experiments). If, further, the roughness of the surface changes from location to location, additional experimentation may be warranted, and a further refinement may be required in specifying the vector C. Or, in some other situation, C may perhaps be modeled as $-a_0 Q_i^c(q,\dot{q},t)$ (see Eq. (11) below).

The invocation of D'Alembert's principle when modeling a mechanical system is then clear. D'Alembert's principle specifies the nature of the constraint forces by simply setting $C\equiv 0$. It points to the genius of Lagrange, for this specification accomplishes the following three things simultaneously.

1 It provides a condition that enables the accelerations of the constrained system to be *uniquely* determined, something desirable when dealing with mechanical systems.

2 It specifies the nature of the constraint force through the ad hoc specification of $C\equiv 0$. This allows the mechanician to model a given mechanical system *without* having to explicitly provide further information (beyond that contained in the constraint Eqs. (2) and/or (3)) on the *nature* of the constraint forces that are created by the presence of the constraints. Most importantly, it therefore obviates the need for situation-specific experimentation, observation, etc., that would have been otherwise necessary to specify C when modeling a specific mechanical system.

3 This specification of $C\equiv 0$ works well (or at least sufficiently well) in many practical situations. This is perhaps the most remarkable attribute of D'Alembert's principle, and it points to the genius of Lagrange.

All this becomes quite obvious, especially when modeling the problem of sliding friction where we immediately recognize that the equation that describes the motion of the block on a horizontal surface must depend not only of the constraint equation, $z=0$, but indeed also on the *nature* of the constraint force engendered by this constraint. And the latter depends on the physics of the specific situation—the materials in contact, the surface roughnesses, etc., and, of course, the intended use that the mechanician wants to put the model to.

But in analytical dynamics, we may have got so used to invoking D'Alembert's principle, which obviates the explicit need to specify the *nature* of the constraint force for any given mechanical system by implicitly taking $C\equiv 0$, that it is tempting to think that such a specification may be wholly unnecessary, even in general. One perhaps may then get the impression that the equations that specify the constraints (Eqs. (2) and/or (3)) are all that is necessary for properly posing the problem of constrained motion. *This indeed is not so*. Specification of the nature of the constraint forces is *always* necessary. The generalized D'Alembert's principle stated in Section 2.1 reminds us that, D'Alembert's principle provides, in fact, *one particular* specification for the nature of the constraint force. As in the case of sliding friction, C may not be zero, and its explicit specification is necessary, in general. Such a specification, as mentioned before, is situation-specific and relies on the discernment and discretion of the mechanician who is modeling the system.

Having explained what we mean by ''specification of constraints'' for a given, constrained mechanical system at hand, we now need to explicitly determine its equation of motion. We start by providing a statement of the problem of constrained motion.

3 General Statement of the Problem of Constrained Motion With Constraints That Do Work

In the notation that we have thus far developed, the problem of constrained motion can now be mathematically stated as follows.

We require to find the n-vector $Q^c(q,\dot{q},t)$ such that

1 $M(q,t)\ddot{q}=Q(q,\dot{q},t)+Q^c(q,\dot{q},t)$, with $q(0)=q_0$, $\dot{q}(0)=\dot{q}_0$, and Q a known function of q, $\dot{q}$, and t; (S1)

2 $\varphi(q,t)=0$, $\psi(q,\dot{q},t)=0$, with $\varphi(q_0,0)=0$; $\dot{\phi}(q_0,t)=0$, and $\psi(q_0,\dot{q}_0,0)=0$; and, (S2)

3 for all vectors v such that $A(q,\dot{q},t)v=0$, we require $v^T Q^c(t)=v^T C(q,\dot{q},t)$, where the n-vector $C(q,\dot{q},t)$ is a known function of its arguments. It specifies the *nature* of the constraint forces. (S3)

We remind the reader that item (S2) above is equivalent to Eq. (4), and item (S3) is our generalized D'Alembert's principle as stated in Section 2.

Next we shall provide the explicit equation of motion that emerges from the above mathematical statement, and furthermore show that the accelerations provided by it are unique. From here on, for clarity, we shall suppress the arguments of the various quantities.

4 Equation of Motion for Constrained System With Nonideal Constraints

Result 1. An equation of motion of the constrained mechanical system that satisfies conditions (S1)–(S3) given in the previous section is explicitly given by

$$M\ddot{q}=Q+Q^c=Q+M^{1/2}B^+(b-Aa)+M^{1/2}\{I-B^+B\}M^{-1/2}C. \tag{7}$$

Proof. We shall prove that the constraint force n-vector, Q^c, given by Eq. (7) satisfies (S1)–(S3).

(S1) The form of Eq. (7) shows that (S1) is satisfied.

(S2) Using $\ddot{q}$ from Eq. (7) in Eq. (4) gives

$$\begin{aligned}A\ddot{q}&=Aa+BB^+(b-Aa)+B(I-B^+B)M^{-1/2}C\\ &=Aa+BB^+b-BB^+BM^{1/2}a\\ &=Aa+BB^+b-BM^{1/2}a=BB^+b,\end{aligned} \tag{8}$$

where we have used the relations $a=M^{-1}Q$, $BB^+B=B$, and $B=AM^{-1/2}$. Equation (4) can be expressed as $B(M^{1/2}\ddot{q})=b$, and being consistent, implies ([8]) that $BB^+b=b$. Using this in the right-hand side in (11) proves that the acceleration $\ddot{q}$ satisfies Eq. (4). Hence (S2) is satisfied.

(S3) As seen from (7), the constraint force, Q^c, is given by

$$Q^c=Q_i^c+Q_{ni}^c=M^{1/2}B^+(b-Aa)+M^{1/2}\{I-B^+B\}M^{-1/2}C. \tag{9}$$

Since $B=AM^{-1/2}$, after setting $v=M^{-1/2}\mu$, (S3) is equivalent to proving that

$$\{\mu|B\mu=0,\mu\neq 0\}\}\Rightarrow \mu^T M^{-1/2}Q^c=\mu^T M^{-1/2}C. \tag{10}$$

But $B\mu=0$ implies $\mu^+B^+=0$, and this ([8]) implies $\mu^T B^+=0$. By Eq. (9) we then have $\mu^T M^{-1/2}Q^c=\mu^T B^+(b-Aa)+\mu^T\{I-B^+B\}M^{-1/2}C=\mu^T M^{-1/2}C$, which is the required result (S3).□

Result 2. The equation of motion for the constrained system given by (7) is unique.

Proof. Assume there exists another set of solution vectors $\ddot{q}+\ddot{e}$ and Q^c+R such that (S1)–(S3) are also satisfied. We must then have $M(\ddot{q}+\ddot{e})=Q+Q^c+R$, and by (5), $M\ddot{e}=R$. Similarly, $A(\ddot{q}+\ddot{e})=b$, and by Eq. (4), $A\ddot{e}=0$. So the n-vector $\ddot{e}$ qualifies as a virtual displacement. Also, for all virtual displacements v, we must have $v^T(Q^c+R)=v^T C$, so that $v^T R=0$. Thus $\ddot{e}^T R=\ddot{e}^T M\ddot{e}=0$, and hence $\ddot{e}=0$ because M is positive definite. Since $R=M\ddot{e}=0$, uniqueness follows. □

Thus Eq. (7) gives the *unique* equation of motion describing the acceleration of a constrained mechanical system where the constraints are nonideal and the constraint forces do an amount of work (under the virtual displacement, v) given by $v^T C(q,\dot{q},t)$, with the n-vector C being known. We explain the salient features of Eqs. (7) and (9) in the following series of remarks.

Remark 1. The equation of motion, (7), for the constrained system does not contain any ‘‘multipliers’’ that need to be solved for, as found in Lagrange’s equations that describe constrained motion with ideal constraints. □

Remark 2. No elimination of coordinates (or velocities) is done; therefore, no set of coordinates (or velocities) is singled out for special treatment, as in the Gibbs-Appell approach that is applicable for ideal constraints. The equation of motion is stated in the *same* coordinates as those describing the *unconstrained* system. This makes it simple to directly assess the influence that the presence of the constraints have on the accelerations of the unconstrained system. The next remarks deal with this. □

Remark 3. The total constraint force n-vector, Q^c, is given by $Q^c = Q_i^c + Q_{ni}^c$, and it is seen to be made up of *two additive* contributions. The first member on the right-hand side of Eq. (9) given by

$$Q_i^c = M^{1/2}B^{+}(b - Aa) \tag{11}$$

is the constraint force *that would have been engendered were all the constraints ideal*, and $C \equiv 0$. This contribution is ever present, no matter whether the constraints are ideal or not.

The second member on the right-hand side of Eq. (9) given by

$$Q_{ni}^c = M^{1/2}\{I - B^{+}B\}M^{-1/2}C \tag{12}$$

gives the *additional* contribution to the constraint force due to the presence of nonideal constraints where the constraint forces *do* work under virtual displacements. This breakdown of the total constraint force n-vector explicitly shows the way in which knowledge of the virtual work done by nonideal constraints enters the equation of motion of the constrained system. □

Remark 4. The contribution, Q_i^c, to total force of constraint, Q^c, does no work under virtual displacements. For, as in the proof of Result 1, $v^T Q_i^c = v^T M^{1/2} B^{+}(b - Aa) = \mu^T B^{+}(b - Aa) = 0$, for all μ such that $B\mu = 0$. Hence, at each instant of time $v^T Q^c = v^T Q_{ni}^c = v^T C$. □

Remark 5. The force $C(q,\dot{q},t)$ provides a mathematical specification of the nonideal nature of the constraints by informing us of the work done by the constraint force n-vector, Q^c, under virtual displacements, v. Its specification depends on the physics of any given particular situation. It engenders a contribution, Q_{ni}^c, to the total constraint force, Q^c, but in general, this contribution is such that, $Q_{ni}^c \neq C$. As seen from Eq. (12), only at those instants of time when $M^{-1/2}C$ lies in the null space of the matrix B, does $Q_{ni}^c = C$.

Furthermore, at those instants of time when $M^{-1/2}C$ is such that it lies in the range space of B^T, then $Q_{ni}^c = 0$. For then, $M^{-1/2}C$ can be expressed as $B^T w$ for some suitable vector w, and by Eq. (12) we have, $M^{-1/2}Q_{ni}^c = (I - B^{+}B)B^T w = [B^T - (B^{+}B)^T B^T]w = [B^T - B^T(B^T)^{+}B^T]w = 0$. Here, in the second and third equalities we use the properties of the Moore-Penrose inverse ([8]). □

Remark 6. When the constraints are ideal, $C \equiv 0$, and the equation of motion given by Eq. (7) reverts to one that is well known ([8]), and has been shown to be equivalent to the usual Lagrange equations with multipliers, and to the Gibbs-Appell equations, each of which is valid only for ideal constraints. □

5 Example

We illustrate the power of our result by considering a particle of unit mass moving in an inertial Cartesian frame subjected to a set of impressed forces $f_x(x,y,z,t)$, $f_y(x,y,z,t)$, $f_z(x,y,z,t)$ acting in the x, y, and z-directions, respectively. The particle is subjected to the nonholonomic, constraint $\dot{y} = z^2\dot{x}$. *The presence of this nonideal constraint creates a force of constraint*. For the specific system at hand, we assume that this force of constraint does work under virtual displacements given by $v^T Q^c = -v^T(a_0 u^T u)(u/|u|)$, where u is the velocity of the particle and $|u| = +\sqrt{u^T u}$. Such a specification of the *nature* of the constraint force is left to the discretion of the mechanician who is modeling the system, and it would depend on the physics of any particular situation (see Section 2.2). What is the equation of motion of this nonholonomically constrained system in which the constraints create nonideal forces of constraint?

Using Eq. (7) we can write down an explicit equation for the motion of the particle as follows.

Differentiating the constraint equation $\dot{y} = z^2\dot{x}$, we get

$$A = [-z^2 \quad 1 \quad 0], \tag{13}$$

with

$$b = 2\dot{x}\dot{z}z. \tag{14}$$

We note that it is the *existence* of the constraint $\dot{y} = z^2\dot{x}$ that *creates* the force of constraint. This force of constraint is nonideal. It does work under virtual displacements; its magnitude is proportional to the square of the speed of the particle, and it opposes the particle’s motion. It is *not* an ‘‘impressed force’’ on the particle. It would disappear in the absence of the constraint.

Since $M = I_3$, $B = A$. By Eq. (11) we then obtain

$$Q_i^c = [-z^2 \quad 1 \quad 0]^T \frac{(2z\dot{x}\dot{z} + z^2 f_x - f_y)}{(1 + z^4)}, \tag{15}$$

and, by Eq. (12),

$$Q_{ni}^c = -a_0 \begin{bmatrix} \dot{x} + z^2\dot{y} \\ z^2\dot{x} + z^4\dot{y} \\ \dot{z}(1 + z^4) \end{bmatrix} \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}}{(1 + z^4)}. \tag{16}$$

The equation of motion of the nonholonomically constrained system with nonideal constraints then becomes

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = Q + Q_i^c + Q_{ni}^c = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} + \frac{(2z\dot{x}\dot{z} + z^2 f_x - f_y)}{(1 + z^4)} \begin{bmatrix} -z^2 \\ 1 \\ 0 \end{bmatrix} - a_0 \begin{bmatrix} \dot{x} + z^2\dot{y} \\ z^2\dot{x} + z^4\dot{y} \\ \dot{z}(1 + z^4) \end{bmatrix} \frac{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}}{(1 + z^4)}. \tag{17}$$

The last member on the right-hand side of Eq. (17) exposes explicitly the contribution that the nonideal character of this nonholonomic constraint provides to the total constraint force, Q^c. The second member on the right informs us of the constraint force the particle *would* be subjected to, were the nonholonomic constraint $\dot{y} = z^2\dot{x}$ ideal. As stated in Remark 5, in this example $Q_{ni}^c \neq C$.

Note that when $a_0 = 0$, the third member on the right of Eq. (17) disappears, and we get the correct equation of motion that is valid for ideal constraints. Then, our equation becomes equivalent to Lagrange’s equation with multipliers and the Gibbs-Appell equation, both of which are valid only for ideal constraints.

In Ref. ([10]) we handle the sliding friction problem of a bead running down a wire. As expected, Eq. (7) indeed yields the proper equations of motion, which in this case are easy to verify using Newtonian mechanics.

Holonomically constrained systems where the constraint forces are nonideal, as in sliding friction, may at times be handled by the Newtonian approach. However, to the best of our knowledge there is no way to date to obtain the equations of motion for nonholonomically constrained systems where the constraint forces are nonideal. Thus, seemingly simple problems like the one considered in this section have so far been beyond the compass of the Lagrangian formulation (see Refs. [5] and [9] for a more extensive discussion).

6 The Geometry of Constrained Motion

The geometrical simplicity of the equation of motion (7) developed herein can perhaps be best captured by using the ''scaled'' accelerations $\ddot{q}_s = M^{1/2}\ddot{q}$, $a_s = M^{1/2}a = M^{-1/2}Q$, $\ddot{q}_s^c = M^{-1/2}Q^c$ and $c_s = M^{1/2}(M^{-1}C) = M^{-1/2}C$. The equation of motion (5) of the constrained system can then be written in terms of these scaled accelerations as

$$\ddot{q}_s(t) = a_s(t) + \ddot{q}_s^c(t), \tag{18}$$

and the problem of finding the equation of motion of the constrained system then reduces, as pointed out by Gauss [2], to finding the *deviation* $\Delta\ddot{q}_s \equiv \ddot{q}_s^c(t) = \ddot{q}_s(t) - a_s(t)$ of the scaled acceleration of the constrained system, $\ddot{q}_s(t)$, from its known, unconstrained, scaled acceleration, $a_s(t)$. Equation (7), then takes on the simple form

$$\ddot{q}_s = (I - B^+B)(a_s + c_s) + B^+b, \tag{19}$$

from which we can explicitly obtain the deviation, $\Delta\ddot{q}_s$, as

$$\Delta\ddot{q}_s = B^+(b - Ba_s) + (I - B^+B)c_s. \tag{20}$$

Let us denote $N = (I - B^+B)$, and $T = B^+B$. To understand the first member on the right-hand side of Eq. (20), we note that the extent to which the acceleration a of the unconstrained system does *not* satisfy the constraint Eq. (4) is given by

$$e = b - Aa = b - Ba_s. \tag{21}$$

Equations (19) and (20) can now be rewritten as

$$\ddot{q}_s = N(a_s + c_s) + B^+b \tag{22}$$

and

$$\Delta\ddot{q}_s = B^+b - Ta_s + Nc_s = B^+e + Nc_s. \tag{23}$$

Noting the definition of $\Delta\ddot{q}_s$, Eq. (23) can be expressed alternatively as

$$\ddot{q} - a = (M^{-1/2}B^+)e + (M^{-1/2}NM^{-1/2})C. \tag{24}$$

This form of our result leads to the following new fundamental principle of Lagrangian mechanics:

> *The motion of a discrete mechanical system subjected to constraints that are nonideal evolves, at each instant of time, in such a way that the deviation of its accelerations from those it would have at that instant if there were no constraints on it, is made up of two components. The first component is proportional to the extent to which the accelerations corresponding to the unconstrained motion, at that instant, do not satisfy the constraints; the matrix of proportionality is $M^{-1/2}B^+$, and the measure of the dissatisfaction of the constraints is provided by the vector e. The second component is proportional to the vector C that specifies the work done by the constraint forces under virtual displacements, at that instant, and the matrix of proportionality is $(M^{-1/2}NM^{-1/2})$.*

Now the operator N, being symmetric and idempotent, is an orthogonal projection operator on the null space of B, and the vector B^+b belongs to the range space of B^T. Furthermore, the two right-hand members of Eq. (22) constitute two n-vectors that are orthogonal to each other, because

$$N^TB^+ = (I - B^+B)^TB^+ = (I - B^+B)B^+ = B^+ - B^+BB^+ = 0, \tag{23}$$

since $B^+BB^+ = B^+$. Equation (22) thus informs us that the scaled acceleration of the constrained system is simply the sum of two *orthogonal* vectors, one belonging to the null space of B—denoted $\mathcal{N}(B)$, and the other belonging to the range space of B^T—denoted $\mathcal{R}(B^T)$. Figure 1 depicts relations (22) and (23) pictorially, and reveals the geometrical elegance with which Nature appears to operate. It generalizes the results obtained in Ref. ([12]) to include systems in which nonideal forces of constraint exist.

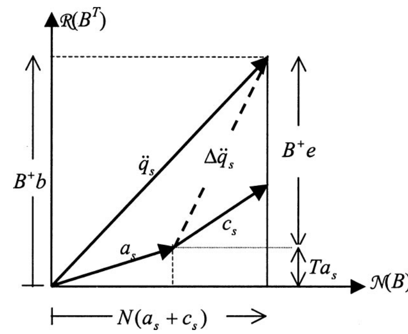

Fig. 1 The geometry of constrained motion is depicted using projections on $\mathcal{N}(B)$ and $\mathcal{R}(B^T)$. The projection of $\ddot{q}_s$ on $\mathcal{N}(B)$ is the same as that of $(a_s + c_s)$ because $N\ddot{q}_s = N(a_s + c_s)$. The vector B^+b is orthogonal to this projection.

It should come as no surprise that the vectors a_s and c_s enter Eq. (19) in the same way. Though their genesis is vastly different, they come, after all, from forces that act on the system. Notice, however, that the sum $(a_s + c_s)$ does not enter directly. The matrix $N = (I - B^+B)$ is a projection on the null space of B, and hence it is the sum's projection on this space that enters the equation of motion.

Conclusions

We summarize the contribution in this paper as follows.

1 To date, Largangian mechanics has been built upon the Principle of D'Alembert. This principle restricts Lagrangian mechanics to situations where the work done by the forces of constraint under virtual displacements is zero. In this paper we relax this restriction and thereby release Lagrangian mechanics from this confinement.

2 We have generalized D'Alembert's principle to include situations in which the constraints are not ideal, and the forces of constraint may do positive, negative, or zero work under virtual displacements. The generalized principle reduces to the usual D'Alembert's principle when the constraints are ideal.

3 The generalized D'Alembert's principle highlights the fact that the description of the motion of a constrained mechanical system requires more than just a statement of the equations of constraint, i.e., Eqs. (2) and/or (3). It *always* also requires a specification of the *nature* of the forces of constraint that the constraints engender. This is done in terms of the work done by the forces of constraint under virtual displacements, through a prescription of the n-vector $C(q,\dot{q},t)$. D'Alembert's principle is thus seen as *one particular way* of specifying the nature of the forces of constraint, for it prescribes the vector $C(q,\dot{q},t)$ to be identically zero. In general, one has to rely on the discretion of the mechanician to specify the vector $C(q,\dot{q},t)$ upon examination of the specific system whose motion needs to be modeled. When D'Alembert's principle is invoked while dealing with a given constrained mechanical system—and this is most often the case in analytical dynamics, to date—the burden of this specification ''seems'' lifted from the shoulders of the mechanician, for the principle simply sets $C(q,\dot{q},t)$ to the zero vector. However, the conscientious mechanism needs to examine if, and how well, the forces of constraint (in the given physical system being modeled) exhibit the behavior subsumed by this principle.

4 The framework of Lagragian mechanics is used to show that this generalized D'Alembert's principle provides just the right extent of information to yield the accelerations of the constrained system *uniquely*, as demanded by practical observation. In the situation that the constraints are ideal, these accelerations agree

with those determined using formalisms developed by Lagrange, Gibbs, and Appell, each of these being applicable only to the case of ideal constraints.

5 We have presented here the general, explicit, equations of motion for mechanical systems with nonideal, equality, constraints. They lead to a new and fundamental understanding of constrained motion. To the best of our knowledge, these equations are arguably the simplest and most comprehensive so far discovered. They will aid in understanding the dynamics of mechanical systems in various fields such as biomechanics, robotics, and multibody dynamics, where such nonideal constraints abound.

6 Our equations show that the constraint force *n*-vector is made up of two additive contributions: $Q^c = Q_i^c + Q_{ni}^c$. Explicit expressions for each of these contributions are given in this paper. The contribution Q_i^c *always* exists whether or not the constraints are ideal, and it is dictated by the kinematic nature of the constraints. The contribution Q_{ni}^c arises from a *specification* by the mechanician of the nonideal nature of the constraints that may be involved in any particular situation; it prevails when the constraint forces do work under virtual displacements.

7 We have provided an insight into the geometry of constrained motion revealing the simplicity and elegance with which Nature seems to operate.

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