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Dynamics and Precision Control of Tumbling Multibody Systems

Prasanth B. Koganti* and Firdaus E. Udwadia†
University of Southern California, Los Angeles, California 90089

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Tumbling is an inherently nonlinear phenomenon, and this paper uses a generic model of a tumbling multibody system made up of a rigid body to which discrete masses are attached; it obtains the equations of motion of the system explicitly, exhibiting the highly nonlinear nature of the dynamics. Particularizations of the generic model used here are useful in applications such as liquid sloshing in rockets, biodynamics, and capture and refurbishing of space debris. It is assumed that the mathematical description of the system is accurately known. A uniform analytical dynamics-based approach is used to obtain both the equations of motion of the system as well as the requisite control torques and forces to satisfy control requirements. No linearizations or approximations of the nonlinear dynamics are made, and closed-form controls are obtained that precisely track user-specified time-dependent control requirements on the tumbling multibody system. The methodology is demonstrated using two examples of tumbling systems with internal degrees of freedom in a constant gravity field that possess significant nonlinear internal motions. Precision tumbling control of such tumbling-vibrating multibody systems is achieved with considerable ease, making the approach presented herein useful for real-time control.

I. Introduction

THE motion of a swimmer from a high-dive platform as he somersaults through the air or the motion of a mundane cube of nonhomogeneous material as it wobbles when thrown up in the air is always fascinating to watch. One of the reasons tumbling motions, especially in three dimensions, are almost magical to watch is that they are complex and often seem to defy intuition. It is the highly nonlinear nature of the equations that describe such motions that causes our intuition to falter, and hence the apparent strange behavior of such systems. This paper develops a generic model of a multi-degree-of-freedom tumbling system composed of a rigid body to which a set of discrete masses is attached that can move along a line fixed to the body. Each mass is connected to its nearest neighbor by a nonlinear (and/or linear) spring. As will be explained later, the discrete masses that form the "internal" degrees of freedom of the system can be large in comparison with the mass of the rigid body, and they can undergo large-amplitude nonlinear vibrations as the entire system tumbles. This generic model is useful for several applications such as liquid sloshing in rockets, biodynamics, and the capture and control of space debris.

There appear to be three main areas where the dynamics of tumbling and its control have been studied: 1) control of the orientation of a rigid body using the motion of internal parts [1–5], the study of tumbling dynamics of biological species and their internal control mechanisms through limbs or tails [6,7], and applications in bioinspired robots [8–10]; 2) work on attitude dynamics in the aerospace field (e.g., [11–30]); and 3) tumbling of biological species such as proteins, red blood cells, and flying insects (e.g., [31–34]). More recently, there has been activity in the capture of tumbling objects in space in which, for example, the use of optimal control methods for attitude orientation to capture tumbling objects was worked on [35]. In the following, we provide a review of the current literature in these three areas and the ways in which the present paper differs from the work that has been done to date.

References [1,2] explored the use of a spinning internal member to control the attitude of aerospace vehicles. Reference [3] used a spinning internal member for which the center of gravity was offset from the axis of symmetry of a projectile to stabilize the attitude of the projectile. Reference [4] used feedback linearization to control a kinetic warhead controlled using three moving mass actuators. Reference [5] used a vibrating internal mass mechanism to control the trajectory of a smart weapon. Full-state feedback control obtained using a feedback linearization technique was used. While the internal motion was leveraged to control the motion of a projectile, the control of both the internal masses and the tumbling body was not dealt with; it is the focus of the current work. In particular, this paper deals with situations where the internal masses can be a significant percent (more than 50%) of the total mass of the tumbling–vibrating body, thereby making the nonlinear interactions significant and placing them outside the scope of any linearizations. Also, the generic dynamical model used in this paper has a greater degree of generality because it deals with an arbitrary number of nonlinearly coupled internal masses moving along an (any) axis that may or may not pass through the center of mass of the tumbling body and is oriented along an arbitrary direction. The current paper also differs from [1–5] in the sense that no concepts borrowed from traditional control theory were used. The control approach relies on fundamental results from the closely allied field of analytical dynamics, and it provides exact control so that the control requirements become the integrals of motion of the controlled nonlinear system.

Reference [6] provided geometric insights into self-righting of falling cats. A feasibility analysis of maneuvers involving limb motions performed by a weightless astronaut to achieve self-orientation was provided in [7]. A three-dimensional analytical model was developed in [8] to investigate the use of inertial appendages to control the orientation of bioinspired robots. This study was inspired by self-righting maneuvers of lizards. A nonlinear feedback controller for a two-link robot with an active tail was proposed in [9]. The control strategy consisted of a simple proportional gain feedback. Reference [10] considered a general robot system that could be thought of as a connected chain of rigid bodies and developed control algorithms to achieve a pose that reduced the impact of landing. The methodology was inspired by falling cats. The current paper is different from the aforementioned work in the following ways:

1) Trajectory requirements are placed on internal masses as well as translation and rotation of the tumbling body, whereas [8–10] were only concerned with orientation control of a body with no control requirements on either the internal masses or the body.

2) The proposed control approach is inspired by recent advances in analytical dynamics, whereas the control approach presented in [8–10] was inspired by biological systems.

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*Department of Aerospace and Mechanical Engineering.
†Professor of Aerospace and Mechanical Engineering, Civil Engineering, Mathematics, and Information and Operations Management; fudwadia@gmail.com.

3) The proposed approach provides closed-form nonlinear control that exactly satisfies the control requirements cast in the form of control constraints.

4) The proposed control simultaneously minimizes a user-defined quadratic control cost at each instant of time.

The work done to date in the area of attitude control can be broadly classified into three groups. Considerable work has been done on the attitude control of rigid bodies (see, for example, [11–18]); this forms the first group of research activity. Here, different control methodologies such as feedback linearization, velocity feedback control, sliding-mode control, and State Dependent Riccati Equation (SDRE) control have been used to control the attitude of a rigid body, typically from a given rest position to another target rest position. As opposed to such standardly used methods developed by control theorists, [19] used an analytical dynamics-based method that unified the modeling and the control of rigid-body motions, resulting in a simpler closed-form exact control. The second focus of activity has been in developing optimal control methods for the attitude control of rigid bodies (see, for example, [20–22]); and a third focus has been on developing methods of attitude and position control of systems of rigid bodies, such as the controlled motion of one rigid-body spacecraft around another when the two move in uniform and nonuniform gravity fields (see, for example, [23–25]). Such work was recently also extended to include systems for which the properties may be only known imprecisely and/or systems on which the forces acting may be known with poor accuracy [26–28]. One of the areas where tumbling dynamics becomes important is when the fluid–structure interaction becomes significant; here, the main challenge is in the modeling of the unsteady nature of the complex fluid flow, which can often be turbulent. Given our imprecise knowledge of the proper fluid model, it is important that control methods that allow the inclusion of such imprecise descriptions are developed [29].

Work in the biodynamics field related to tumbling is more along experimental lines, with quantitative modeling being somewhat difficult at this time due to the lack of well-established dynamical models. References [30–33] show the importance of tumbling at the organism level, the subcomponent level, and at the molecular level (for example, in proteins).

As seen from the preceding discussion, though considerable work has been done in these three areas, precision control of a tumbling body that has numerous internal degrees of freedom (which one may also want to control) has not been dealt with so far in the manner described here. This paper develops a generic model of a general tumbling body with no axis of symmetry and an arbitrary number of internal masses that move along an arbitrary axis that may or may not pass through the center of mass, and that is oriented along an arbitrary direction. It focuses on the fundamental nonlinear behavior of a tumbling object that has several internal moving parts, with the internal parts also being coupled to one another through nonlinear elements. The explicit quaternion equations of motion that capture the ensuing large rotational dynamics are obtained. They expose the highly nonlinear nature of the dynamics wherein the vibratory motions of the internal degrees of freedom are nonlinearly coupled to the rigid-body dynamics. The general equations derived for the generic model are particularized to two pedagogical examples of practical importance. The first deals with the motion of a rigid parallelepiped, to which is attached a cylindrical rod along whose surface, discrete masses connected by nonlinear spring elements are allowed to slide and vibrate as the parallelepiped tumbles. This example shows the effect of internal vibrations on the tumbling of objects that do not have an axis of symmetry. The second deals with the dynamics of a cylinder, such as a spacecraft, along whose axis is a rod on which is mounted a chain of masses, with each connected to its nearest neighbor by nonlinear spring elements. This illustrates the nature of tumbling when the internal motions are along a line of axial symmetry of the body.

The paper develops a uniform approach based on results from analytical dynamics [36–39] for obtaining the equations of motion of the tumbling system and for controlling it to satisfy given control requirements. Both the tumbling of the system as well as its internal motions (the vibrating masses) are controlled with high precision in a user-prescribed time-dependent manner. The system and its parameters are assumed to be known, and the system is referred to as the nominal system. (Reference [29] deals with the dynamics and precision control of tumbling multibody uncertain systems.) The approach used here provides closed-form control of the nominal system so that it asymptotically fulfills all the control requirements. No linearization or approximations are made to the nonlinear dynamical model. The control is obtained in closed form, and can therefore be used in real time. In a sense, the control is "exact" because it causes the control requirements to become the integrals of motion of the controlled nonlinear system. In addition, the control cost (a quadratic function of the generalized control force) is minimized at each instant of time. The aforementioned two examples (the parallelepiped and the cylinder) are considered; and the simplicity, efficacy, and accuracy of the control approach developed herein is illustrated through numerical simulations.

II. Notation

This section introduces notation that will be used in the paper. In the following, u, v, and m are arbitrary n vectors (n-by-1 column vectors); A and B are m-by-m matrices; and w, w_1, and w_2 are m vectors. The ith element of n vectors u, v, and m are u_i, v_i, and m_i, respectively.

The Hadamard product (elementwise multiplication) of two n vectors, u and v, is denoted by $u \circ v$, so that

$$u \circ v = v \circ u := [u_1 v_1, u_2 v_2, \ldots, u_n v_n]^T \tag{1}$$

The Kronecker product of two matrices $C = [c_{pq}] \in R^{p\times q}$ and $D \in R^{r\times s}$ is defined as

$$C_{p\times q} \otimes D_{r\times s} := \begin{bmatrix} c_{11}D & c_{12}D & \cdots & c_{1q}D \\ c_{21}D & c_{22}D & \cdots & c_{2q}D \\ \vdots & \vdots & \ddots & \vdots \\ c_{p1}D & c_{p2}D & \cdots & c_{pq}D \end{bmatrix}_{pr\times qs} \tag{2}$$

The n-by-n diagonal matrix $\mathrm{diag}(u)$ has elements of n-vector u along its diagonal so that

$$\mathrm{diag}(u) := \begin{bmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{bmatrix} \tag{3}$$

and $(u \circ v) = \mathrm{diag}(u)v$.

The n-vector $\mathbf{1}_n$ is defined as a vector for which the elements are all unity as

$$\mathbf{1}_n := [1, 1, \ldots, 1]^T \tag{4}$$

so that one can write the n-by-n identity matrix $I_n = \mathrm{diag}(\mathbf{1}_n)$, and $u = (\mathbf{1}_n \circ u) = (u \circ \mathbf{1}_n)$. Furthermore,

$$\sum_{i=1}^{n} m_i = \mathbf{1}_n^T m = m^T \mathbf{1}_n, \quad (u \circ v)^T \mathbf{1}_n = u^T v, \quad \text{and}$$
$$\sum_{i=1}^{n} m_i u_i v_i = m^T (u \circ v) = (m \circ u)^T v \tag{5}$$

Similarly, the $r \times s$ matrix $0_{r\times s}$ has all its elements equal to zero.

The Kronecker product has the following well-known properties, which will be used in the sequel:

$$[C_{p\times q} \otimes D_{r\times s}]^T = C_{p\times q}^T \otimes D_{r\times s}^T \tag{6}$$

$$(u^T \otimes A)(v \otimes w) = (u^T v)Aw \tag{7}$$

$$(u^T \otimes A)(\mathrm{diag}(v) \otimes B) = (u \circ v)^T \otimes AB \tag{8}$$

$$(\mathrm{diag}(u) \otimes w^T)(\mathrm{diag}(v) \otimes B) = \mathrm{diag}(u \circ v) \otimes w^T B \tag{9}$$

$$(\mathrm{diag}(u) \otimes w_1^T)(v \otimes w_2) = (u \circ v) w_1^T w_2 \tag{10}$$

III. Modeling a Multibody Tumbling System

In this section, we consider a general model of a composite multibody tumbling system undergoing three-dimensional tumbling as well as motion of its internal parts. The multibody system is assumed to comprise of the following two components (see Fig. 1a):

1) The first component includes a rigid body B of mass m_B and a rigid rod R of mass m_R that is fixed to the body B. The center of mass of the composite rigid body (denoted hereafter as BR) is located at C (see Fig. 1a). The coordinate axes of the body-fixed coordinate frame xyz, for which the origin is chosen to coincide with C, lie along the principal axes of inertia of BR. The direction of the axis of the rod is specified by the unit vector a, for which the components in the body-fixed coordinate frame are the constants a_1, a_2, and a_3, respectively. The coordinates of C in the inertial coordinate frame XYZ are denoted by (X_c, Y_c, Z_c). The vector from C to a suitable point O' on the rod R is the three-vector d, for which the components in the xyz frame are the constants d_1, d_2, and d_3, respectively (see Fig. 1a).

2) The second component includes a set of discrete masses $m_i, i = 1, 2, \ldots, n$, that slide along the rod R, with each mass connected to its nearest neighbor by linear and/or nonlinear spring elements (see Fig. 1b). The position of the masses measured from O' along the rod R are denoted by $p^i, i = 1, 2, \ldots, n$. As shown in Fig. 1b, the spring element k_{i+1} connecting mass m_i to mass m_{i+1} is assumed, for simplicity, to consist of a linear elastic spring element with stiffness k_{i+1}^l in parallel with a cubically nonlinear elastic spring element with stiffness k_{i+1}^n. The equilibrium positions of the discrete masses are given by $p_e^i, i = 1, 2, \ldots, n$, as measured along the rod R from O'.

A uniform analytical dynamics approach to the determination of the explicit equations of motion of this composite body and its precision control is developed. Explicit controls are found so that 1) the angular velocity vector follows a user-prescribed time-dependent trajectory, and 2) each of the discrete masses follows user-prescribed time-dependent internal motions along the rod R. These trajectory requirements need to be satisfied as the composite system tumbles, vibrates, and translates under gravity. No linearizations and/or approximations of the nonlinear dynamics of the system are made in finding the closed-form controls. In [29], the system is assumed to be only imprecisely known, and an approach to controlling this uncertain system is developed so that it satisfies the aforementioned set of user-prescribed trajectory requirements.

The rigid body BR has six degrees of freedom, and each point mass has one degree of freedom. Thus, a total of $n+6$ coordinates are required to describe the configuration of the system. However, in what follows, an additional coordinate will be used to describe the composite system's configuration, and its rotational motion will be described by four quaternions. The mass of BR (bodies B and R) is denoted by m_{BR}, and its mass moment of inertia matrix with respect to the body-fixed frame located at C (for which the directions are along the principal axes of inertia of BR) is denoted by $J = \mathrm{diag}(J_x, J_y, J_z)$.

IV. Equations of Motion of the Uncontrolled System

The rotational displacement of the system is described by the unit quaternion $u = [u_0, u_1, u_2, u_3]^T$. The active rotation matrix $S(u)$ used to transform from body coordinates to inertial coordinates is given by

$$S(u) := [S_1 \quad S_2 \quad S_3] = \begin{bmatrix} 2u_0^2 - 1 + 2u_1^2 & 2u_1u_2 - 2u_0u_3 & 2u_1u_3 + 2u_0u_2 \\ 2u_1u_2 + 2u_0u_3 & 2u_0^2 - 1 + 2u_2^2 & 2u_2u_3 - 2u_0u_1 \\ 2u_1u_3 - 2u_0u_2 & 2u_2u_3 + 2u_0u_1 & 2u_0^2 - 1 + 2u_3^2 \end{bmatrix} \tag{11}$$

The angular velocity of the composite body BR is given by

$$\omega = H\dot{u} := \begin{bmatrix} -2u_1 & 2u_0 & 2u_3 & -2u_2 \\ -2u_2 & -2u_3 & 2u_0 & 2u_1 \\ -2u_3 & 2u_2 & -2u_1 & 2u_0 \end{bmatrix} \begin{bmatrix} \dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} \tag{12}$$

where the matrix H is defined as

$$H := \begin{bmatrix} -2u_1 & 2u_0 & 2u_3 & -2u_2 \\ -2u_2 & -2u_3 & 2u_0 & 2u_1 \\ -2u_3 & 2u_2 & -2u_1 & 2u_0 \end{bmatrix} \tag{13}$$

In the preceding and throughout the rest of the paper, one dot on top of a variable denotes a derivative with respect to time and two dots represent a second derivative with respect to time t.

The rotational kinetic energy of the body-rod BR system is given by

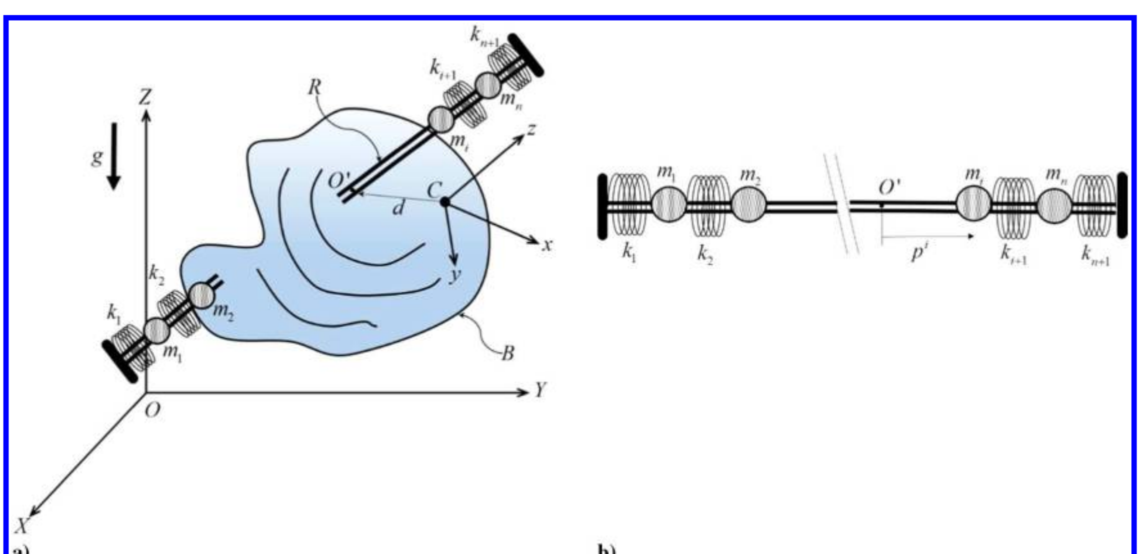

Fig. 1 Representations of a) generic multibody tumbling system and b) schematic of the rod R.

$$KE_R = \frac{1}{2}\omega^T J\omega = \frac{1}{2}\dot{u}^T H^T J H \dot{u} \tag{14}$$

where $J = J_{BR}$ is the moment of inertia of BR, and ω is the angular velocity of BR expressed in the xyz body-fixed frame that is attached to BR. It is useful to note the following partial derivatives of ω:

$$\frac{\partial\omega}{\partial\dot{u}} = H \quad \text{and} \quad \frac{\partial\omega}{\partial u} = -\dot{H} \tag{15}$$

It can also be easily verified that the matrix H satisfies the relation

$$\dot{H}\,\dot{u} = 0 \tag{16}$$

Denoting by vector R, the position of the center of mass C of the body BR

$$R = [X_c, Y_c, Z_c]^T \tag{17}$$

the translational kinetic energy (KE) of BR is given by

$$KE_T = \frac{1}{2} m_{BR}\dot{R}^T\dot{R} \tag{18}$$

Similarly, the potential energy (PE) of BR is

$$PE = m_{BR} g Z_c = m_{BR} g e_3^T R \tag{19}$$

where g is the (constant) acceleration due to gravity, and e_3 denotes the basis vector;

$$e_3 = [0, 0, 1]^T \tag{20}$$

The coordinates of the ith discrete mass in the inertial coordinate system are denoted by $r^i = (r_X^i, r_Y^i, r_Z^i)^T$. These are computed, using the active rotation matrix $S(u)$, as

$$\begin{aligned}(r_X^i, r_Y^i, r_Z^i)^T &= (X_c, Y_c, Z_c)^T + p^i S(u)a + Sd \\ &= R + p^i(S_1 a_1 + S_2 a_2 + S_3 a_3) + Sd\end{aligned} \tag{21}$$

where S_i is the ith column of the matrix $S(u)$; d is the position vector of O', which is a suitable reference point on the rod R in the body-fixed coordinate frame (see Fig. 1a); and a is the unit vector (for which the components are given in the body frame) in the direction of the axis of the rod.

Consider now the first column S_1 given by

$$S_1 = [2u_0^2 + 2u_1^2 - 1, 2u_1u_2 + 2u_0u_3, 2u_1u_3 - 2u_0u_2]^T \tag{22}$$

The time derivative of the vector S_1 can be computed as

$$\begin{aligned}\dot{S}_1 &= \begin{bmatrix} 4u_0\dot{u}_0 + 4u_1\dot{u}_1 \\ 2\dot{u}_1u_2 + 2u_1\dot{u}_2 + 2\dot{u}_0u_3 + 2u_0\dot{u}_3 \\ 2\dot{u}_1u_3 + 2u_1\dot{u}_3 - 2\dot{u}_0u_2 - 2u_0\dot{u}_2 \end{bmatrix} \\ &= \begin{bmatrix} 4u_0 & 4u_1 & 0 & 0 \\ 2u_3 & 2u_2 & 2u_1 & 2u_0 \\ -2u_2 & 2u_3 & -2u_0 & 2u_1 \end{bmatrix}\begin{bmatrix}\dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3\end{bmatrix} := L_1\dot{u}\end{aligned} \tag{23}$$

Furthermore, it can be verified from Eq. (23) that

$$\frac{\partial S_1}{\partial u} = L_1 \quad \text{and} \quad \frac{\partial \dot{S}_1}{\partial u} = \dot{L}_1 \tag{24}$$

In a similar manner, one obtains relations related to the second and third columns, S_2 and S_3, of the matrix $S(u)$ so that one gets

$$\frac{\partial S_i}{\partial u} = L_i, \quad \frac{\partial \dot{S}_i}{\partial u} = \dot{L}_i, \quad \text{and} \quad \dot{S}_i = L_i\dot{u}, \quad i = 1, 2, 3 \tag{25}$$

where

$$L_2 = \begin{bmatrix} -2u_3 & 2u_2 & 2u_1 & -2u_0 \\ 4u_0 & 0 & 4u_2 & 0 \\ 2u_1 & 2u_0 & 2u_3 & 2u_2 \end{bmatrix}$$

and

$$L_3 = \begin{bmatrix} 2u_2 & 2u_3 & 2u_0 & 2u_1 \\ -2u_1 & -2u_0 & 2u_3 & 2u_2 \\ 4u_0 & 0 & 0 & 4u_3 \end{bmatrix} \tag{26}$$

Hence, the following relations are obtained:

$$\dot{S}(u)a = \sum_{i=1}^{3} a_i\dot{S}_i = \left(\sum_{i=1}^{3} a_i L_i\right)\dot{u} := L_a\dot{u} \tag{27}$$

$$\frac{\partial(Sa)}{\partial u} = L_a, \quad \text{and} \quad \frac{\partial(\dot{S}a)}{\partial u} = \dot{L}_a \tag{28}$$

where we have denoted

$$L_a := \sum_{i=1}^{3} a_i L_i \tag{29}$$

Furthermore, Eq. (27) yields

$$\ddot{S}(u)a = \dot{L}_a\dot{u} + L_a\ddot{u} \tag{30}$$

To simplify the analysis, we define m as the mass vector consisting of the values of the point masses, p as the vector of their positions along the rod R with respect to point O', and r as the vector consisting of their respective coordinates in an inertial frame of reference so that

$$m := [m_1, m_2, \ldots, m_n]^T, \qquad p := [p^1, p^2, \ldots, p^n]^T \tag{31}$$

and

$$r := [r_X^1, r_Y^1, r_Z^1, r_X^2, r_Y^2, r_Z^2, \cdots, r_X^n, r_Y^n, r_Z^n]^T = 1_n \otimes R + p \otimes Sa + 1_n \otimes Sd \tag{32}$$

Assuming the components of the vector d are $d = [d_1, d_2, d_3]^T$, the partial derivatives of r are

$$\frac{\partial r}{\partial R} = 1_n \otimes I_3 \tag{33}$$

$$\frac{\partial r}{\partial u} = p \otimes \frac{\partial(Sa)}{\partial u} + 1_n \otimes \frac{\partial(Sd)}{\partial u} = p \otimes L_a + 1_n \otimes L_d$$

and

$$\frac{\partial r}{\partial p} = I_n \otimes Sa \tag{34}$$

where

$$L_d = \sum_{i=1}^{3} d_i L_i$$

Defining

$$M_D := \mathrm{diag}(m) \otimes I_3 \tag{35}$$

the kinetic energy of all the discrete point masses is compactly written as

$$T_D = \frac{1}{2}\dot{r}^T M_D \dot{r} \tag{36}$$

and the potential energy is

$$U_D = g r^T (m \otimes e_3) + U_s. \tag{37}$$

U_s is the potential energy in the spring elements given by

$$U_s = \sum_{i=1}^{n+1} \frac{1}{2} k_i^l \left(p^i - p^{i-1} - p_e^i + p_e^{i-1}\right)^2 + \frac{1}{4} k_i^n \left(p^i - p^{i-1} - p_e^i + p_e^{i-1}\right)^4 \tag{38}$$

where k_i^l and k_i^n are the linear and nonlinear stiffnesses of the ith spring element, and $p^0 = p^{n+1} = p_e^0 = p_e^{n+1} = 0$.

Differentiating Eq. (32) with respect to time yields

$$\begin{aligned}\dot{r} &= 1_n \otimes \dot{R} + \dot{p} \otimes Sa + p \otimes \dot{S}a + 1_n \otimes \dot{S}d \\ &= 1_n \otimes \dot{R} + \dot{p} \otimes Sa + p \otimes L_a \dot{u} + 1_n \otimes L_d \dot{u}\end{aligned} \tag{39}$$

and

$$\begin{aligned}\ddot{r} &= 1_n \otimes \ddot{R} + \ddot{p} \otimes Sa + 2\dot{p} \otimes \dot{S}a + p \otimes \ddot{S}a + 1_n \otimes \ddot{S}d \\ &= 1_n \otimes \ddot{R} + \ddot{p} \otimes Sa + 2\dot{p} \otimes L_a \dot{u} + p \otimes (\dot{L}_a \dot{u} + L_a \ddot{u}) \\ &\quad + 1_n \otimes (\dot{L}_d \dot{u} + L_d \ddot{u})\end{aligned} \tag{40}$$

The partial derivatives of $\dot{r}$ that will be needed for obtaining the Lagrange equations are given by

$$\frac{\partial \dot{r}}{\partial \dot{u}} = p \otimes L_a + 1_n \otimes L_d \tag{41}$$

$$\begin{aligned}\frac{\partial \dot{r}}{\partial u} &= \dot{p} \otimes \frac{\partial Sa}{\partial u} + p \otimes \frac{\partial \dot{S}a}{\partial u} + 1_n \otimes \frac{\partial \dot{S}d}{\partial u} \\ &= \dot{p} \otimes L_a + p \otimes \dot{L}_a + 1_n \otimes \dot{L}_d\end{aligned} \tag{42}$$

$$\frac{\partial \dot{r}}{\partial \dot{p}} = I_n \otimes Sa$$

and

$$\frac{\partial \dot{r}}{\partial p} = I_n \otimes \dot{S}a = I_n \otimes L_a \dot{u} \tag{43}$$

The configuration space vector

$$q := [R^T, u^T, p^T]^T = \Big[\underbrace{X_c, Y_c, Z_c}_{R^T}, \underbrace{u_0, u_1, u_2, u_3}_{u^T}, \underbrace{p_1, p_2, \ldots, p_n}_{p^T}\Big]^T \tag{44}$$

is used to describe the configuration of the system at any time t. A total of $(n+7)$ coordinates are used, all of which are not independent because the rotation is overparametrized by the quaternion four-vector u, which is subject to the constraint $u^T u = 1$.

The total kinetic energy of the system is

$$T = T_{BR} + T_D = \frac{1}{2} m_{BR} \dot{R}^T \dot{R} + \frac{1}{2}\omega^T J \omega + \frac{1}{2}\dot{r}^T M_D \dot{r} \tag{45}$$

and the potential energy is

$$U = m_{BR} g R^T e_3 + g r^T (m \otimes e_3) + U_s \tag{46}$$

The Lagrangian for the composite system is

$$\mathcal{L} = T - U \tag{47}$$

Because the configuration coordinates are not all independent of one another [see Eq. (44)], the explicit equations of motion of the uncontrolled system are now obtained using the following three-step approach [36–39]:

1) Obtain the equation of motion of the unconstrained system in the form $M(q)\ddot{q} = Q(q, \dot{q})$; here, all the components of the quaternion are taken to be independent.

2) Express the quaternion constraint (namely, that its norm is unity) in the form $A_m(q, \dot{q}, t)\ddot{q} = b_m(q, \dot{q}, t)$.

3) Use the fundamental equation of mechanics (we omit for clarity the arguments of the various quantities):

$$M\ddot{q} = Q + A_m^T (A_m M^{-1} A_m^T)^+ (b_m - A_m M^{-1} Q) \tag{48}$$

The matrix X^+ denotes the Moore–Penrose inverse of the matrix X.

Step 1: We begin by finding the unconstrained equations of motion of the system, which are given by Lagrange's equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, n+7 \tag{49}$$

Result 1: The equation of the motion of the unconstrained multibody system given by Eq. (49) is

$$M(q)\ddot{q} = Q(q, \dot{q}) \tag{50}$$

where M is the generalized mass matrix, and Q is the generalized force vector given by

$$M = \begin{bmatrix} \{m_{BR} + m^T 1_n\} I_3 & \{m^T p\} L_a + \{m^T 1_n\} L_d & (Sa) m^T \\ \{m^T p\} L_a^T + \{m^T 1_n\} L_d^T & H^T J H + \{(p \circ m)^T p\} L_a^T L_a + A_d & L_a^T Sa (p \circ m)^T + L_d^T Sa m^T \\ m (Sa)^T & (p \circ m)(Sa)^T L_a + m (Sa)^T L_d & \{(Sa)^T Sa\} \mathrm{diag}(m) \end{bmatrix} \tag{51}$$

where

$$A_d = \{m^T p\}(L_a^T L_d + L_d^T L_a) + \{m^T 1_n\} L_d^T L_d$$

and

$$Q = \begin{bmatrix} -2\{m^T \dot{p}\} L_a \dot{u} - (\{m^T p\} \dot{L}_a + \{m^T 1_n\} \dot{L}_d)\dot{u} - g(m_{BR} + \{m^T 1_n\}) e_3 \\ -2\dot{H}^T J H \dot{u} - 2\{(p \circ m)^T \dot{p}\} L_a^T L_a \dot{u} - \{(p \circ m)^T p\} L_a^T \dot{L}_a \dot{u} - g\{p^T m\} L_a^T e_3 + B_d \\ -\{(Sa)^T \dot{L}_a \dot{u}\}(p \circ m) - 2\{(Sa)^T L_a \dot{u}\}(m \circ \dot{p}) - \{(Sa)^T \dot{L}_d \dot{u}\} m + F^S - g\{(Sa)^T e_3\} m \end{bmatrix} \tag{52}$$

where

$$B_d = -(2\{m^T \dot{p}\}L_d^T L_a + \{m^T p\}L_a^T \dot{L}_d + \{m^T p\}L_d^T \dot{L}_a + \{m^T \mathbb{1}_n\}L_d^T \dot{L}_d)\dot{u} - g\{m^T \mathbb{1}_n\}L_d^T e_3$$

In the preceding, F^S is the force (column) vector due to the spring elements, for which the ith component is

$$F_i^S = -k_i^l(p^i - p^{i-1} - p_e^i + p_e^{i-1}) - k_i^n(p^i - p^{i-1} - p_e^i + p_e^{i-1})^3 + k_{i+1}^l(p^{i+1} - p^i - p_e^{i+1} + p_e^{i-1}) + k_{i+1}^n(p^{i+1} - p^i - p_e^{i+1} + p_e^{i-1})^3 \tag{53}$$

For convenience, scalars are shown in curly brackets.

Proof: See the Appendix A.

Remark 1: It is assumed in obtaining Eqs. (50–52) that the masses $m_i, i = 1, \ldots, n$, are point masses. If they are not assumed to be point masses and possess substantial rotational kinetic energy, these equations can be modified in a simple and straightforward manner. Referring to $r^i(p^i)$ now as the locations in the inertial frame (along R from O') of the center of mass C_i of m_i, we can denote the moment of inertia of mass m_i about its principal axes going through C_i by the diagonal 3-by-3 matrix J_i. The contribution to the total rotational kinetic energy (of the composite system) made by the rotational motion of mass m_i is then $(1/2)\omega_p^T J_i \omega_p$, where ω_p is the angular velocity three vector for which the components are the angular velocities along the principal axes of m_i. But, because the angular velocity of BR about its body-fixed frame xyz is ω, we have $\omega_p = P_i\omega$, where P_i is a rotation matrix. Hence, the contribution of mass m_i becomes $(1/2)\omega^T P_i^T J_i P_i \omega$. Thus, the rotational energy that is given in Eq. (14) gets modified to

$$KE_R = \frac{1}{2}\omega^T J_{BR}\omega + \frac{1}{2}\sum_{i=1}^{n}\omega^T P_i^T J_i P_i \omega = \frac{1}{2}\omega^T\left[J_{BR} + \sum_{i=1}^{n} P_i^T J_i P_i\right]\omega \tag{54}$$

where we have explicitly denoted the moment of inertia of body BR about the body-fixed axes xyz by J_{BR}. Consequently, all that is needed to include the rotational kinetic energy of masses m_i, $i = 1, \ldots, n$, is to replace J in Eqs. (50–52) by

$$J_{BR} + \sum_{i=1}^{n} P_i^T J_i P_i$$

Step 2: The modeling constraint is the requirement that the norm of the quaternion is unity:

$$\phi_m := u^T u - 1 = 0 \tag{55}$$

By differentiating Eq. (55) twice with respect to time t, we obtain the modeling constraint equation in the form

$$A_m(q, \dot{q}, t)\ddot{q} = b_m(q, \dot{q}, t) \tag{56}$$

where

$$A_m := [0_{1\times 3}, \quad u^T, \quad 0_{1\times n}]$$

and

$$b_m := -\dot{u}^T \dot{u} \tag{57}$$

One could also use the dynamical system

$$\ddot{\varphi}_m + \delta_1\dot{\varphi}_m + \delta_2\varphi_m = 0, \qquad \delta_1, \delta_2 > 0 \tag{58}$$

as a modeling constraint on the unconstrained system of Eq. (50). Note that $\varphi_m(0) = 0$ and this constraint ensure that $\varphi_m \to 0$ as $t \to \infty$, thus attempting to keep $\varphi_m(t)$ close to zero. Use of Eq. (58) leads to

$$A_m = [0_{1\times 3}, u^T, 0_{1\times n}]$$

and

$$b_m = -\delta_1 u^T \dot{u} - \frac{\delta_2}{2}(u^T u - 1) - \dot{u}^T \dot{u} \tag{59}$$

Because A_m is a row vector, the matrix $A_m M^{-1} A_m^T$ in Eq. (48) is therefore a scalar for which the generalized inverse is simply its reciprocal.

Step 3: The equation of motion of the multibody system is then given explicitly by Eq. (48), where the matrix M and the vector Q are explicitly given in Eqs. (51) and (52), respectively; and A_m and b_m are given in Eq. (57). This yields the equation of motion of the multibody system as

$$M\ddot{q} = Q - \begin{bmatrix} 0_{3\times 1} \\ u \\ 0_{n\times 1} \end{bmatrix}\frac{1}{(u^T[M^{-1}]_{22}u)} \times \left[\delta_1 u^T\dot{u} + \frac{\delta_2}{2}(u^T u - 1) + \dot{u}^T\dot{u} + A_m M^{-1} Q\right] \tag{60}$$

where we have denoted by $[M^{-1}]_{22}$ the (2,2) block matrix element of M^{-1} [see Eq. (51)]. We note that the quantity in the denominator $(u^T[M^{-1}]_{22}u)$ is a positive scalar.

Remark 2: If external "given" forces and/or torques are applied to the multibody system, then the vector Q in Eq. (52) needs to be augmented by the generalized forces that are engendered. The generalized force four-vector Q_τ corresponding to the quaternion u for a given set of physically applied torques τ about the xyz body-fixed axes is given by the relation [19]

$$Q_\tau = H^T \tau \tag{61}$$

Remark 3: As seen from Eqs. (51), (52), and (60), the dynamics is highly nonlinear. The translation of the body BR, its tumbling, and the vibration of the discrete masses that slide along the rod R are all highly nonlinearly coupled.

V. Simulation of Dynamics of an Uncontrolled Tumbling System

In this section, we consider two examples of the generic tumbling multibody as described in the previous section. The first is a rectangular parallelepiped (body B); at its corner stands a cylindrical rod (rod R) perpendicular to one surface (see Fig. 2). Two discrete masses slide along R, and each mass is connected to its nearest neighbor (and the two ends of R) by a linear spring element in parallel with a nonlinear cubic one. The second example deals with a cylindrical body B; along its axis is a rod R. Five discrete masses move along the rod, with each mass being connected to its nearest neighbor (and the ends of R) with linear and nonlinear spring elements as before. These same examples will be used in Sec. VII, which deals with the precision control of this tumbling, vibrating system as it falls under gravity. Reference [29] deals with the control of the system when the parameters describing it and the forces acting on it are assumed to be known only imprecisely.

A. Example 1

1. Description of Body BR

A rigid 16-m-long aluminum (density of 2.7×10^3 kg/m^3) cylindrical rod R of radius $r_0 = 1$ m is mounted at one corner on the surface of a rigid rectangular aluminum block B of dimensions $10 \times 6 \times 2$ m, as shown in Fig. 2, so that $L = 5$, $b = 3$, and

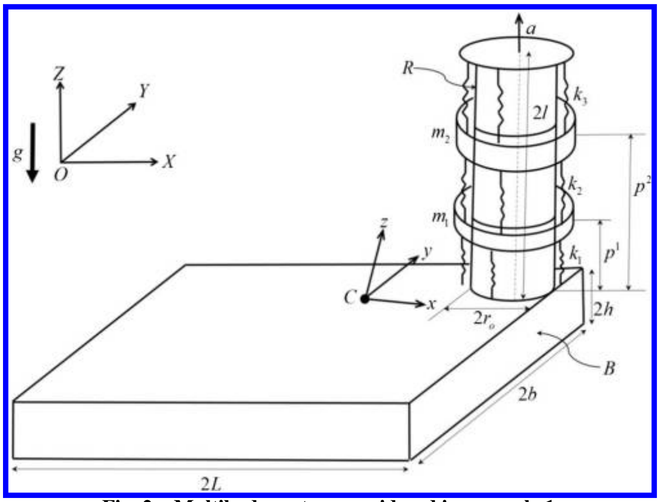

Fig. 2 Multibody system considered in example 1.

$h = 1$ m. An inertial XYZ frame is chosen so that the Z axis points vertically upward. The X axis of the frame is parallel to the edge of body B of dimension $2L$, and the Y axis is parallel to the edge of dimension $2b$. The center of mass of BR (bodies B and R) is located at C. A body-fixed xyz coordinate system for which the origin is located at C, and for which the axes are along the principal axes of inertia of BR, is used. The moment of inertia matrix of BR about these body-fixed axes is given by the matrix $J = 10^7 \times \mathrm{diag}(1.3626, 1.5333, 0.3848)\ \mathrm{kg/m^2}$. The mass of the body BR is 4.597×10^5 kg. The reference point O' on the rod R is located at the center of the circle where the rod R meets the parallelepiped B. The vector from point C to the reference point O' is $d = [3.2616, 1.4196, -0.1623]^T$ m. The unit vector a in the body-fixed xyz coordinate frame (with origin at C) is $a = [-0.3985, -0.1481, 0.9051]^T$. The equilibrium positions m of the discrete masses measured along R from O' (see Fig. 2) are taken to be $p_e = [4, 12]^T$. The parameters for the modeling constraint are $\delta_1 = 0.5$ and $\delta_2 = 8$ [see Eq. (58)]. The configuration of the system is described by the 9-vector $q = [r^T, u^T, p^1, p^2]^T$. At time $t = 0$, the origin of the inertial XYZ frame and point C coincide.

2. Description of the Discrete Masses and Spring Elements

The discrete masses that slide along the rod R have values of $m_1 = 5 \times 10^5$ kg and $m_2 = 4 \times 10^5$ kg (see Fig. 2). The linear spring elements connecting the discrete masses m_1 and m_2 have stiffnesses $k_1^l = 6$ MN/m, $k_2^l = 7.5$ MN/m, and $k_3^l = 3$ MN/m; and the stiffnesses of the nonlinear cubic spring elements are $k_1^n = 0.55\ \mathrm{MN/m^3}$, $k_2^n = 0.3\ \mathrm{MN/m^3}$, and $k_3^n = 0.2\ \mathrm{MN/m^3}$.

The rotational kinetic energy of masses m_1 and m_2 is also included. As shown in Remark 1, this entails adding to the moment of inertia matrix J_{BR} the expression

$$\sum_{i=1}^{n} P_i^T J_i P_i$$

The moments of inertia of masses m_i, $i = 1, 2$, about the direction of the unit vector a (along the axis of the cylinder R) are taken to be $m_i r_0^2$. Thus,

$$J_i = \mathrm{diag}(m_i r_0^2/2, m_i r_0^2/2, m_i r_0^2)$$

yielding

$$J_1 = 10^5 \times \mathrm{diag}(2.5, 2.5, 5), \quad J_2 = 10^5 \times \mathrm{diag}(2, 2, 4) \tag{62}$$

And, the orthogonal matrix $P = P_1 = P_2$ in Eq. (54) is

$$P = \begin{bmatrix} 0.9171 & -0.0765 & 0.3912 \\ 0.0113 & 0.9860 & 0.1664 \\ -0.3985 & -0.1481 & 0.9051 \end{bmatrix} \tag{63}$$

3. Description of Initial Conditions

The initial position of the center of mass C of the body BR in the inertial frame XYZ (for which the Z axis points upward) is taken to be the origin O so that $R(0) = [0, 0, 0]^T$, and its initial velocity (in meters per second) is taken to be the vector $\dot{R}(0) = [1, 2, 20]^T$.

The initial orientation of the body BR at time $t = 0$ is obtained by rotating it about the unit vector $\nu = 1/\sqrt{3}[1, 1, 1]^T$ in the inertial XYZ frame through an angle $\theta = \pi/3$ so that the initial quaternion that describes the orientation of the xyz body-fixed axes is

$$u(0) = [0.8034, 0.1600, 0.4272, 0.3828]^T$$

The initial angular velocity (in radians per second) about this body-fixed frame (see Fig. 2) is taken to be the three-vector $[\omega_x(0), \omega_y(0), \omega_z(0)]^T = [1, -1, 0.5]^T$, so that

$$\dot{u}(0) = [0.0379, 0.6999, -0.2503, -0.0927]^T$$

Finally, the initial positions (in meters) of the two masses are taken as $p(0) = [5, 11]^T$, and their initial velocities (in meters per second) are taken as $\dot{p}(0) = [-0.4, 0.3]^T$.

In summary, we note that the ratio $(m_1 + m_2)/m_{BR} \approx 2$ so that the vibrating masses, which are connected by nonlinear spring elements and which slide along the rod R, account for approximately two-thirds the total mass of the composite body. The body BR is given an initial velocity of 1 m/s in the X direction, 2 m/s in the Y direction, and 20 m/s in the Z direction. Its initial orientation is obtained by rotating it about the vector υ through an angle of 60 deg in the inertial frame. It is given an angular velocity about the three principal body-fixed axes of 1, −1, and 0.5 rad/s, respectively. The displacement of m_1 relative to its equilibrium position is 1 m, and the displacement of m_2 relative to its equilibrium position is −1 m. The spring elements connecting them to one another (and to the ends of the rod R) have a cubic nonlinearity. The initial velocities of m_1 and m_2 are −0.4 and 0.3 m/s, respectively. The positions p^i and velocities $\dot{p}^i$ of the discrete masses are measured from O' along the rod R. The values chosen for δ_1 and δ_2 in Eq. (58) are $\delta_1 = 0.5$ and $\delta_2 = 8$.

With the initial conditions described previously, the body is allowed to fall freely under gravity ($g = 9.81\ \mathrm{m/s^2}$). Using the equations of motion given by Eq. (60), where M and Q are given in Eqs. (51) and (52), the motion of this tumbling, vibrating body is simulated. The computations are done using ode113 in the MATLAB environment using a relative error tolerance of 10^{-12} and an absolute error tolerance of 10^{-13}.

Figure 3 shows the positions $p^1(t)$ and $p^2(t)$ from O' (measured along the rod R; see Fig. 2) of the two discrete masses m_1 and m_2, as well as the angular velocity of body BR as it descends under gravity over a time duration of 20 s.

Figure 4 shows the computed quaternion 4-vector $u(t)$ and the motion of point C, the center of mass of the body BR. The error $e_u = u_0^2 + u_1^2 + u_2^2 + u_3^2 - 1$ in the norm of the quaternion 4-vector is $O(10^{-15})$ throughout the interval of integration.

The dynamics of the system show that it translates, vibrates, and tumbles; the discrete masses vibrate along the rod R during the tumbling. As indicated by the complex nonlinear equations that describe the motion, the translation, the tumbling, and the vibration of the discrete masses are all nonlinearly coupled to one another.

B. Example 2

As a second example, we consider a rigid closed hollow cylinder B with a rigid rod R down its axis, along which a set of five discrete masses m_i, $i = 1, 2, \ldots, 5$, slide (see Fig. 5). The masses, as before, are each connected to their nearest neighbors by a linear spring in parallel with a nonlinear (cubic) spring. The discrete masses at either

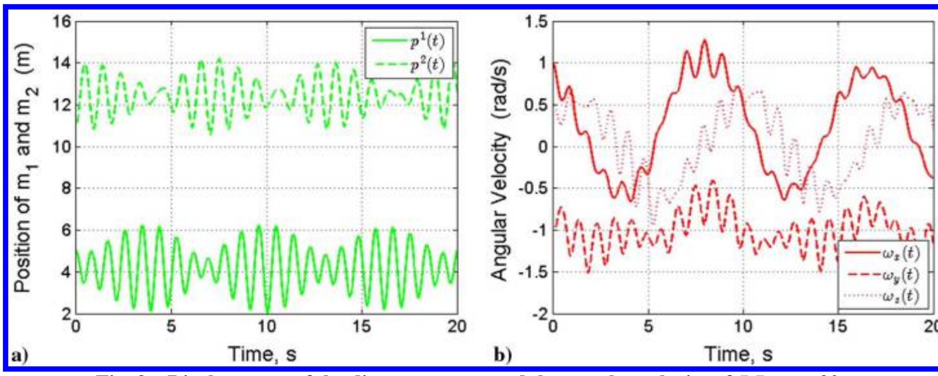

Fig. 3 Displacement of the discrete masses and the angular velocity of BR over 20 s.

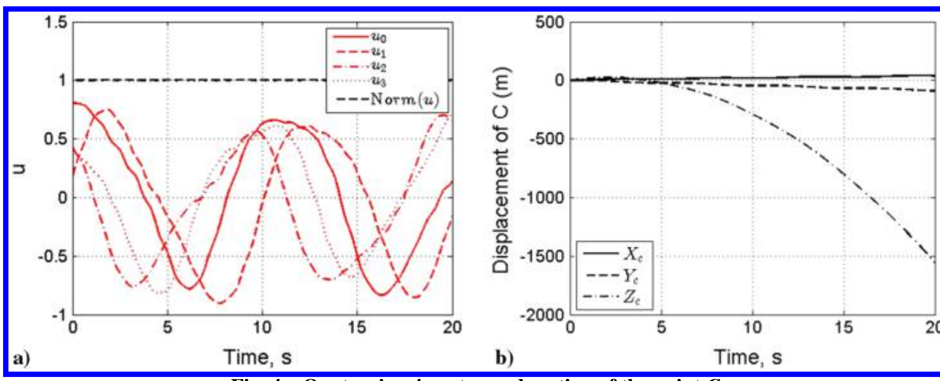

Fig. 4 Quaternion 4-vector and motion of the point C.

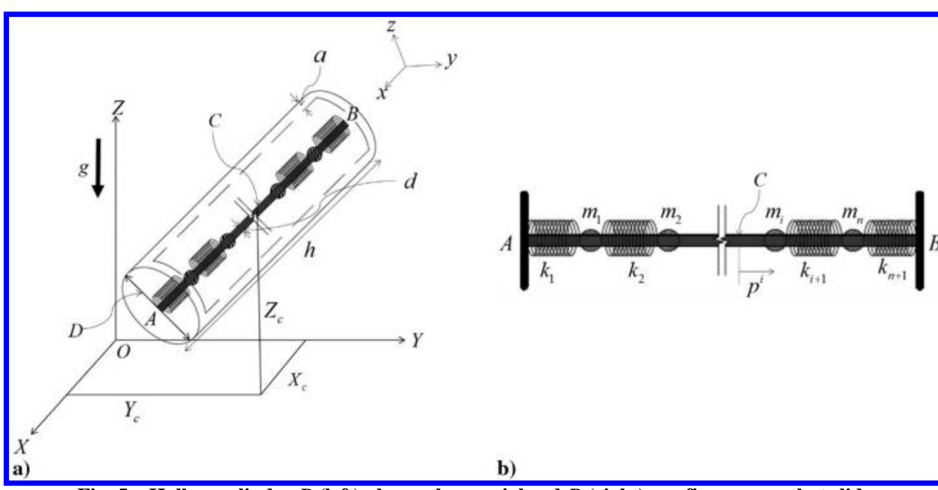

Fig. 5 Hollow cylinder B (left) along whose axial rod R (right) are five masses that slide.

ends of the rod are connected to the end plates that cap the cylinder. The body, denoted BR, consists of the cylinder B and the rod R.

1. Description of the Body BR

The outer diameter of the cylinder B is 8 m, its outer length is $h = 12$ m, and its wall thickness is $a = 2$ cm; the diameter of the rod R is 20 cm. The cylinder's ends are capped with circular disks, also of thicknesses of 2 cm. The cylinder and the rod are made of steel, with a density of 7×10^3 kg/m^3. The center of mass C of the body BR is located along the axis of the cylinder at the center of the rod R, and one principal axis (body-fixed x-axis) lies along the cylinder axis, with the other two perpendicular to the cylinder axis (in Fig. 5, the xyz body-fixed frame is shown outside the figure to avoid clutter, but it is fixed at the center of mass C of BR). The moment of inertia matrix (kg/m^2) of BR about these principal axes is $J = 10^5 \times \mathrm{diag}(7.809, 7.535, 7.535)$; its mass is 2.934×10^4 kg. Positions of the discrete masses that slide along R are measured (both in the positive and negative directions) from $C \equiv O'$ because the vector $d = 0_{3\times1}$ (see Fig. 1). The unit vector a that shows the direction of the rod R in the body-fixed frame is $a = [1, 0, 0]^T$. The origin of the XYZ inertial frame at $t = 0$ coincides with C. The configuration vector is $q = [R^T, u^T, p^T]^T$, where p is now a 5-vector of the positions of each of the discrete masses from C along the rod R. The values chosen for δ_1 and δ_2 in Eq. (58) are $\delta_1 = 0.5$ and $\delta_2 = 8$.

2. Description of the Discrete Masses and Spring Elements

Along the axis of the cylinder are five discrete masses ($m_i, i = 1, 2, \ldots, 5$) for which the values (in kilograms) are given in our notation by $m = 10^4 \times [1, 2, 2, 1, 1]^T$. The stiffnesses of the linear springs (in meganewtons per meter) connecting the five masses are

$$k^l = [70, 50, 70, 100, 70, 50]^T \tag{64}$$

and those of the nonlinear springs (in kilonewtons per cubic meter) that are in parallel are

$$k^{nl} = [30, 70, 40, 60, 50, 80]^T \tag{65}$$

We assume here that the contribution of these masses to the rotational kinetic energy of the composite system is negligible.

3. Description of Initial Conditions

The initial position of the center of mass C of body BR in an inertial frame for which the Z axis points upward is taken to be the origin so that $R(0) = [0, 0, 0]^T$, and its initial velocity (in meters per second) is taken to be the vector $\dot{R}(0) = [1.2, 20]^T$.

The inertial Z axis points vertically upward. Starting with the body-fixed x, y, and z axes of BR along the inertial X, Y, and Z axes, respectively, the initial orientation (at time $t = 0$) of the body is obtained by rotating it about the unit vector $\nu = 1/\sqrt{3}[1, 1, 1]^T$ in the inertial frame through an angle of $\theta = \pi/3$ so that the initial quaternion $u(0) = [\cos(\theta/2), \sin(\theta/2)v]^T$. The initial angular velocity (in radians per second) about the principal axes is taken to be the three-vector $[\omega_x(0), \omega_y(0), \omega_z(0)]^T = [1, -1, 2]^T$, so that $\dot{u}(0) = [-0.2887, 0.8660, -0.5477, 0.5774]^T$. Finally, the equilibrium positions (in meters) of the discrete masses measured along R are uniformly spaced along its length of $h - 2a$. The initial positions (in meters) measured from C along R of the five masses and their initial velocities (in meters per second) are

$$p(0) = [-4.4, \quad -1.8, \quad 1.4, \quad 2.5, \quad 4.2]^T \tag{66}$$

and

$$\dot{p}(0) = [0.2, \quad -0.3, \quad -0.2, \quad 0.1, \quad -0.2]^T \tag{67}$$

In summary, we note that the ratio

$$\left(\sum m_i\right)/m_{BR} \approx 2.4$$

so that the vibrating masses that slide along the rod R account for approximately 70% of the total mass of the composite body. As in example 1, at $t = 0$, the point C coincides with the inertial frame and the body BR is given an initial velocity of 1 m/s in the X direction, 2 m/s in the Y direction, and 20 m/s in the Z direction. Its orientation at $t = 0$ is obtained by rotating it about the vector v through 60 deg. It is given an angular velocity about the three (rotated) principal axes of 1, −1, and 2 rad/s, respectively.

With these initial conditions, the body is allowed to fall freely under gravity ($g = 9.81\ \mathrm{m/s^2}$). Using again the equations of motion given by Eq. (60), where M and Q are given in Eqs. (51) and (52), the motion of this tumbling, vibrating body is simulated. Error tolerances for the numerical integration using ode113 are the same as those in example 1. The simulation is carried out for 20 s.

Figure 6a shows a small segment from 12 to 15 s of the positions of the various discrete masses as the cylinder tumbles with the masses vibrating along the rod R. Figure 6b shows the variation in the components of the angular velocity of BR over the duration of the simulation. The quaternion 4-vector u and the motion of the center of mass C of body BR are shown in Fig. 7. The dashed line in Fig. 7a represents the time history of Norm(u). The error in the modeling constraint $e_u = u^T u - 1$ is of $O(10^{-15})$ throughout the integration interval.

Because the only external force acting on the falling multibody parallelepiped (example 1) and the multibody cylinder (example 2) is gravity, the angular momentum in the inertial frame of reference about the center of mass of each of these tumbling, vibrating multibody systems must be conserved. The two examples considered are fairly complex, and so this serves as a check on the computational results shown. The constancy of the components of the angular momentum vector (about the center of mass in the inertial frame) for the two examples is shown in Fig. 8. As seen, the components of the angular momentum are constant for each example.

VI. Equations of Motion of Controlled System: Explicit Determination of Generalized Control Force

We now place control requirements on the tumbling, vibrating, generic multibody system. The determination of the explicit

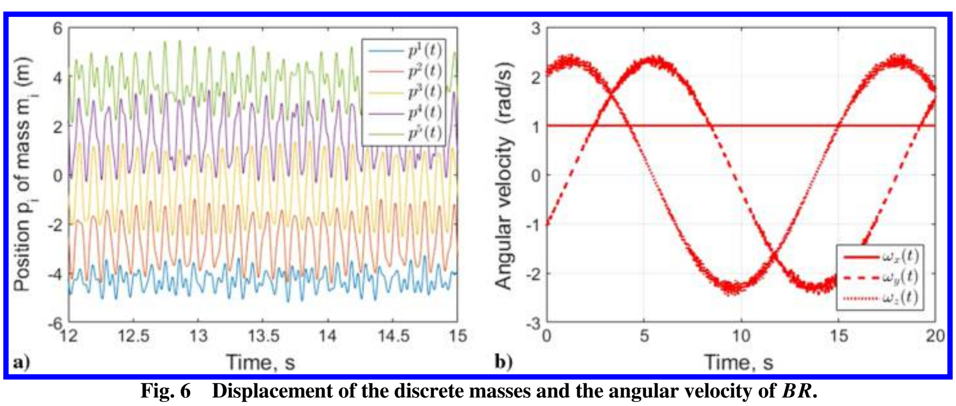

Fig. 6 Displacement of the discrete masses and the angular velocity of BR.

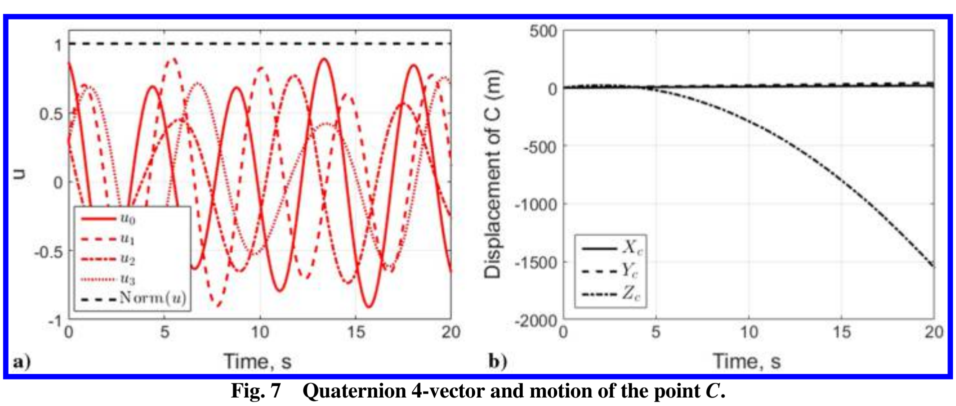

Fig. 7 Quaternion 4-vector and motion of the point C.

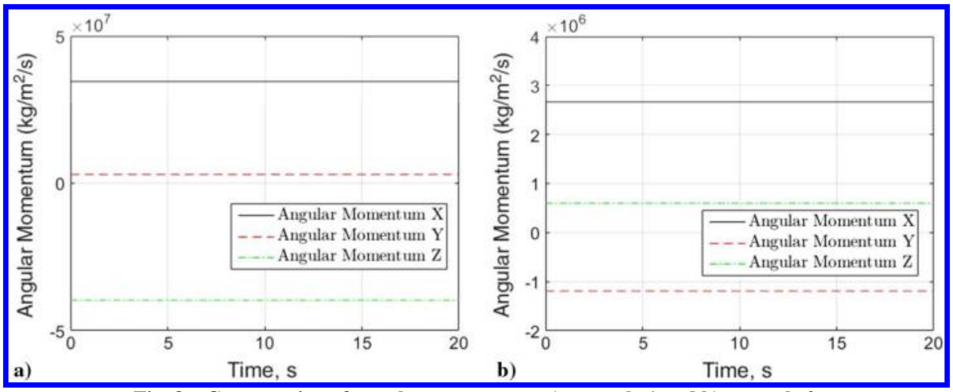

Fig. 8 Conservation of angular momentum: a) example 1 and b) example 2.

generalized control force to obtain precision control without making any linearizations or approximations in the dynamics of the system is one of the main contributions of this paper. Because the control forces are computed using a simple explicit expression, the computational costs are negligibly small and the method can be used for real-time control.

The approach is based on recent (exact) results from analytical dynamics [36–41]. First, the control requirements are framed as constraints on the nonlinear dynamical system. Two types of constraints generally arise, as we shall soon see. They are of the form

$$\varphi_c(q,t)=0 \tag{68}$$

where φ_c is an h vector, and

$$\psi_c(\dot{q},q,t)=0 \tag{69}$$

where ψ_c is a j vector.

We differentiate each equation in the set of Eq. (68) twice with respect to time t and each equation in the set of Eq. (69) once with respect to time. We then form the dynamical equations

$$\ddot{\varphi}_c+\mathrm{diag}(\alpha_1,\alpha_2,\ldots\alpha_h)\dot{\varphi}_c+\mathrm{diag}(\beta_1,\beta_2,\ldots,\beta_h)\varphi_c=0 \tag{70}$$

and

$$\dot{\psi}_c+\mathrm{diag}(\gamma_1,\gamma_2,\ldots,\gamma_j)\psi_c=0 \tag{71}$$

in which the parameters in the diagonal matrices are chosen to be positive. Matrices that are more general than the diagonal matrices shown can also be used; but, for simplicity, we will use these here. This choice of parameters ensures that $\varphi_c,\psi_c\to 0$ asymptotically as $t\to\infty$.

Next, Eqs. (70) and (71) are used as the constraints on the unconstrained system for which the equation is given by $M\ddot{q}=Q$ [see Eq. (50)]. Furthermore, Eqs. (70) and (71) can always be expressed in the form

$$A_c(q,\dot{q},t)\ddot{q}=b_c(q,\dot{q},t) \tag{72}$$

On including the modeling constraints, if any, which are of the form $A_m(q,\dot{q},t)\ddot{q}=b_m(q,\dot{q},t)$, we obtain the set of all the constraints on the unconstrained dynamical system [described by Eq. (50)] as

$$A\ddot{q}:=\begin{bmatrix}A_m\\A_c\end{bmatrix}\quad\ddot{q}=\begin{bmatrix}b_m\\b_c\end{bmatrix}:=b \tag{73}$$

We now obtain the equation of motion of the controlled system that satisfies the control constraints [Eqs. (68) and (69)] asymptotically by using the fundamental equation of mechanics

$$M\ddot{q}=Q+A^T(AM^{-1}A^T)^+(b-AM^{-1}Q)=Q+Q^C \tag{74}$$

in which the vector Q^C explicitly gives the generalized control force needed. It should be pointed out that this explicit generalized control force minimizes at each instant of time the control cost $J(t)=[Q^C]^TM^{-1}Q^C$. (For general weighting matrices, other than M^{-1} in the cost function $J(t)$, and a more general discussion of the exponential stability of the control methodology, see Appendix B).

VII. Simulation of Dynamics with Controlled Tumbling and Controlled Vibrations: Example 1 (Continued)

We continue the examples from Sec. V with the following control objectives:

1) The vibrating discrete masses m_i are required to oscillate harmonically about their equilibrium positions p_e^i with amplitude l^i and frequency λ_i so that

$$\bar{p}^i(t)=p_e^i+l^i\cos(\lambda_i t),\qquad i=1,\ldots,n \tag{75}$$

2) The angular velocity component $\omega_i(t)$ of the body BR (about its principal axes) is required to oscillate sinusoidally with amplitude b_i and frequency σ_i such that

$$\bar{\omega}_i(t)=b_i\cos(\sigma_i t),\qquad i=1,2,3 \tag{76}$$

where the parameters $l=[l^1,\ldots,l^n]^T$, $\lambda=[\lambda_1,\ldots,\lambda_n]^T$, $b=[b_1,b_2,b_3]^T$, and $\sigma=[\sigma_1,\sigma_2,\sigma_3]^T$ are user-specified constants. The bars above the quantities denote that these are the user-prescribed control objectives. Our control requirements are $p\to\bar{p}$ and $\omega\to\bar{\omega}$ as $t\to\infty$.

Thus it is required that 1) the two discrete masses execute sinusoidal motion along the rod R, each with a different (given) amplitude and a different (given) frequency; 2) whereas the body tumbles with its angular velocity sinusoidally changing (about each principal axis of BR) with a given amplitude and a given frequency. We note that the control requirements are time dependent. Our aim is to obtain closed-form precision control of the tumbling and the vibrational behavior of the system.

As stated in Sec. VI, we first cast the control requirements given by Eqs. (75) and (76) as constraints on the nonlinear dynamical system. The control requirement regarding the positions of the masses m_i can be expressed as the constraint

$$\varphi_p(t):=p(t)-\bar{p}(t)=0 \tag{77}$$

on the dynamical system, where $\bar{p}(t)=[\bar{p}^1,\ldots,\bar{p}^n]^T$ is given in Eq. (75). Because the system may not start on the constraint manifold, one can think of the quantity $\varphi_p=p-\bar{p}$ as the trajectory error in $p(t)$ and use the dynamical equation

$$\ddot{\varphi}_p+\mathrm{diag}(\alpha_1,\alpha_2)\dot{\varphi}_p+\mathrm{diag}(\beta_1,\beta_2)\varphi_p=0,\ \alpha_i,\beta_i>0,\ i=1,\ldots,n \tag{78}$$

as a constraint on the dynamical system. For suitable positive values of α_i,β_i, $i=1,\ldots,n$, this constraint will ensure that $\varphi_p\to 0$

asymptotically as $t \to \infty$. Different choices of the parameters α_i and β_i will, of course, cause different rates at which asymptotic convergence occurs. Noting Eq. (77), the last equation can be rewritten as

$$\ddot{p} = -\mathrm{diag}(\alpha_1, \ldots, \alpha_n)(\dot{p} - \dot{\bar{p}}) - \mathrm{diag}(\beta_1, \ldots, \beta_n)(p - \bar{p}) + \ddot{\bar{p}} \tag{79}$$

and, by using the configuration vector $q = [R, u, p]^T$, it can be compactly expressed as

$$A_p \ddot{q} = b_p(q, \dot{q}, t) \tag{80}$$

where

$$A_p = [0_{n\times 7}, I_n]$$

and

$$b_p = -\mathrm{diag}(\alpha_1, \ldots, \alpha_n)(\dot{p} - \dot{\bar{p}}) - \mathrm{diag}(\beta_1, \ldots, \beta_n)(p - \bar{p}) + \ddot{\bar{p}} \tag{81}$$

Similarly, the control requirement on the angular velocity can be expressed as a constraint on the dynamical system given by [see Eq. (12)]

$$\varphi_\omega(t) := \omega(t) - \bar{\omega}(t) = H\dot{u}(t) - \bar{\omega}(t) = 0 \tag{82}$$

Again, because one may not start on the constraint manifold, one can conveniently use the dynamical equation

$$\ddot{\varphi}_\omega(t) + \gamma\varphi_\omega = 0, \qquad \gamma > 0 \tag{83}$$

as a constraint on the dynamical system. Equation (83) ensures, for $\gamma > 0$, that $\varphi_\omega \to 0$ exponentially as $t \to \infty$, with the rate of convergence depending on the value of γ that is chosen. In view of Eq. (82), the last equation can be rewritten as (recall $\dot{H}\,\dot{u} = 0$)

$$H\ddot{u}(t) = -\gamma(H\dot{u} - \bar{\omega}) + \dot{\bar{\omega}}(t) \tag{84}$$

As before, one can compactly write this relation in terms of the configuration vector $q(t)$ as

$$A_\omega \ddot{q} = b_\omega(q, \dot{q}, t) \tag{85}$$

where

$$A_\omega = [0_{3\times 3}, H, 0_{3\times n}] \quad \text{and} \quad b_\omega = -\gamma(H\dot{u} - \bar{\omega}) + \dot{\bar{\omega}}(t) \tag{86}$$

The equation of motion of the controlled dynamical system can now be obtained following the same three-step procedure that was used earlier

The first step (Step 1) is the unconstrained system given by

$$M(q)\ddot{q} = Q(q, \dot{q}, t) \tag{87}$$

where M and Q are given in Eqs. (51) and (52).

The second step (Step 2) is the statement of all the constraints in the form $A(q, \dot{q}, t)\ddot{q} = b(q, \dot{q}, t)$.

1) We have the modeling constraint given by Eq. (56) that ensures that the norm of the quaternion is unity.

2) In addition, to control the system so as to satisfy the control requirements, we have two sets of constraints given by Eqs. (80) and (85). We call them control constraints.

The three constraints can be written as

$$\begin{aligned} A\ddot{q} &:= \begin{bmatrix} 0_{(n+4)\times 3} & \hat{A}_{(n+4)\times(n+4)} \end{bmatrix} := \begin{bmatrix} 0_{1\times 3} u^T 0_{1\times n} \\ 0_{3\times 3} H 0_{3\times n} \\ 0_{n\times 3} 0_{n\times 4} I_n \end{bmatrix} \begin{bmatrix} \ddot{R} \\ \ddot{u} \\ \ddot{p} \end{bmatrix} \\ &= \begin{bmatrix} -\delta_1 u^T \dot{u} - \frac{\delta_2}{2}(u^T u - 1) - \dot{u}^T \dot{u} \\ -\gamma(H\dot{u} - \bar{\omega}) + \dot{\bar{\omega}}(t) \\ -\mathrm{diag}(\alpha_1, \ldots, \alpha_n)(\dot{p} - \dot{\bar{p}}) - \mathrm{diag}(\beta_1, \ldots, \beta_n)(p - \bar{p}) + \ddot{\bar{p}} \end{bmatrix} \\ &:= b \end{aligned} \tag{88}$$

Note that the rank of matrix A is $(n+4)$, and it cannot change because the matrix $[u\,\frac{1}{2}H^T]$ is orthogonal (with determinant unity).

Remark 4: In the preceding, we assume that all the masses m_i are to be controlled, as well as each of the components of the angular velocity of BR. If a control requirement is to be placed only on, say, m_2, then only the second row of A_p in Eq. (81) will be considered as a constraint so that A_p will then be a row vector; correspondingly, only the second element of the column vector b_p in Eq. (81) will be considered. Similar remarks apply if we want to control just one (or two) of the three components of the angular velocity of BR.

For the third step (Step 3), the equation of motion of the controlled system is now given by

$$M\ddot{q} = Q + A^T(AM^{-1}A^T)^+(b - AM^{-1}Q) = Q + Q^C \tag{89}$$

where matrix A and column vector b are shown in Eq. (88). Noting the structure of matrix A given in Eq. (88), it is easy to see that

$$(AM^{-1}A^T)^+ = (\hat{A}[\hat{M}]^{-1}\hat{A}^T)^{-1} \tag{90}$$

where $[\hat{M}]^{-1}$ stands for the lower corner $(n+4)\times(n+4)$ block of matrix M^{-1}, which is positive definite.

Remark 5: The requisite generalized control force needed to control the system so that it asymptotically satisfies the control requirements given in Eqs. (75) and (76) is explicitly given by vector Q^C. Observe that no approximations of the nonlinear equations describing the dynamical system are made. The control force needed to satisfy the control requirements, theoretically speaking, is exact in the sense that these requirements are made to be the integrals of motion of the controlled system. The (generalized) control force is provided in closed form through the simple multiplication of matrices; therefore, the computational cost for obtaining it is negligible. This makes the control approach useful for real-time applications. Furthermore, it minimizes, at each instant of time, the control cost $J(t) = [Q^C]^T M^{-1} Q^C$.

To illustrate the ease and accuracy with which precision tumbling control can be achieved along with precision vibrational control of the nonlinear motion of the discrete masses that oscillate by sliding along the rod R, we consider the two examples discussed in Sec. V.

A. Example 1 (Continued)

We consider the dynamical system that was described in Sec. V (example 1), keeping the description of the body BR, the values of the discrete masses and spring elements, and the initial conditions the same.

For control requirements, the parameters that specify our control requirements given in Eqs. (75) and (76) are chosen as follows:

$$l = [-1, 2]^T, \ \lambda = [2\pi, \pi/2]^T, \ b = [-10, 8, 15]^T, \ \text{and} \ \sigma = [0, \pi, 2\pi]^T \tag{91}$$

and those in Eqs. (79) and (84) are taken to be

$$\alpha_1 = \alpha_2 = 2, \quad \beta_1 = \beta_2 = 12, \quad \delta_1 = 0.5, \quad \delta_2 = 8, \quad \text{and } \gamma = 0.6 \tag{92}$$

Equation (91) states that the control requirement on the positions of the discrete masses m_1 and m_2 is that the two masses oscillate sinusoidally about their (respective) equilibrium positions with amplitudes $-\cos(2\pi t)$ and $2\cos(\pi t/2)$, respectively. The control requirement on the body BR is that its angular velocities be $\omega_x = -10$ rad/s, $\omega_y = 8\cos(\pi t)$ rad/s, and $\omega_z = 15\cos(2\pi t)$ rad/s. Note that, because $\omega_x(0) = 1$, the rotation about the x-principal axis is required to reverse direction, whereas the angular velocities about the other two axes are required to vary periodically.

The simulation is again carried out using ode113 in the MATLAB environment, with the same error tolerances for the numerical integration as in Sec. V. Figure 9 shows the displacement of the discrete masses m_1 and m_2 and the error in tracking the required control objective expressed in Eq. (75). At the end of 20 s, the tracking error $e_p(t) = p(t) - \bar{p}(t)$ is $O(10^{-8})$ m. Upon continuing the integration to 60 s, the error $e_p(t = 60)$ reduces and becomes $O(10^{-12})$ m. Precision control is thus stably obtained.

The angular velocity of the body BR is shown in Fig. 10 along with the error in tracking the angular velocity $e_\omega(t) = \omega(t) - \bar{\omega}(t)$. At the end of 20 s, the error $e_\omega(t = 20)$ is of $O(10^{-4})$ rad/s. When the computations are continued to 60 s, the error $e_\omega(t = 60)$ for the controlled system reduces and is of $O(10^{-11})$ rad/s. Figure 10a can be compared with Fig. 3b for the uncontrolled system. Precision control is again obtained.

Figure 11 shows the generalized control forces that need to be applied to control the tumbling, vibrating body so that it satisfies all the control requirements imposed. The control force exerted at the point C is zero, and only the forces on the discrete masses and torques are therefore shown. The precision control obtained here is effected by the application of forces F_1 and F_2 along the rod R to masses m_1 and m_2, as well as by the application of torques about the principal axes that meet at C. For simplicity, we have not considered the auxiliary problem of including the control mechanisms for generating these forces and the torques in the dynamical equations.

Lastly, the motion of point C in the inertial frame and the quaternion 4-vector $u(t)$ for the controlled motion are shown in Fig. 12. The error in the norm of the quaternion $e_u = u_0^2 + u_1^2 + u_2^2 + u_3^2 - 1$ is of $O(10^{-13})$ throughout the interval of integration.

Remark 6: The reasons why the tracking errors become very small with increasing time are as follows:

1) The complete nonlinear dynamical system is handled when finding the generalized control forces Q^C with no approximation and/or linearizations.

2) Equation (89) provides the control force Q^C that, theoretically speaking, causes the constraint equations [Eqs. (58), (78), and (83)] to be exactly satisfied. And, each of these equations shows that the quantities φ_m, φ_p, and φ_ω exponentially go to zero in time. But, as seen from Eqs. (55), (77), and (82), the quantities φ_m, φ_p, and φ_ω are nothing but the tracking errors: a) between Norm(u) and unity, which is a modeling requirement; b) between $p(t)$ and the desired $\bar{p}(t)$; and 3) between $\omega(t)$ and the desired $\bar{\omega}(t)$. The latter two are our control requirements.

3) The rate of convergence to the desired control objectives can be altered by a proper choice of the parameters used in Eq. (92). This will, of course, affect the control forces needed to satisfy the objectives.

4) As mentioned before, the control force Q^C, while ensuring that the tracking error (theoretically speaking) goes to zero, also ensures simultaneously that the control cost $J(t) = [Q^C]^T M^{-1} Q^C$ is minimized at each instant of time.

B. Example 2 (Continued)

We continue the example of the cylinder shown in Fig. 5 that has five internally vibrating discrete masses. The description of the body BR, the values of the discrete masses and spring elements, and the

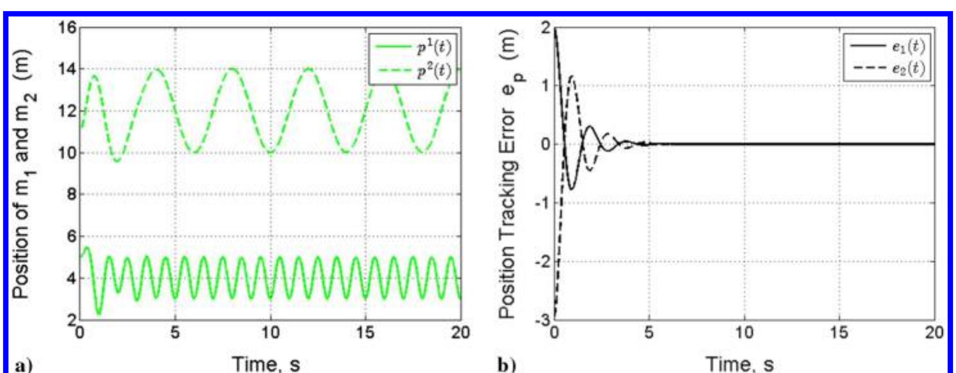

Fig. 9 Displacement of discrete masses and tracking error $e_p(t) := [e_1(t) e_2(t)]^T$.

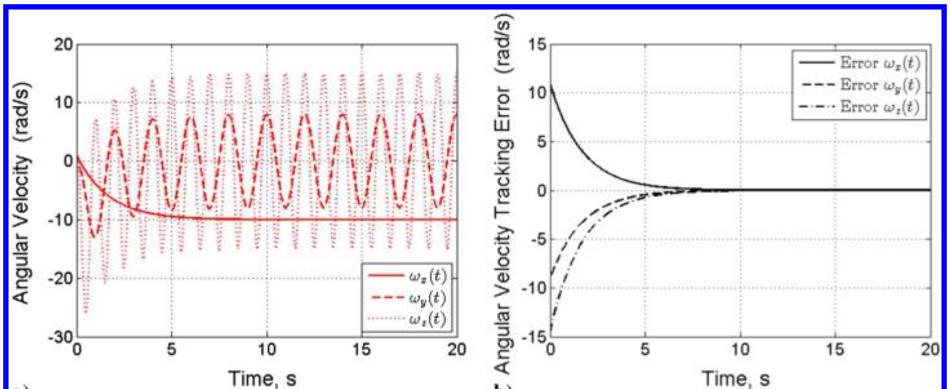

Fig. 10 Angular velocity and tracking error e_ω.

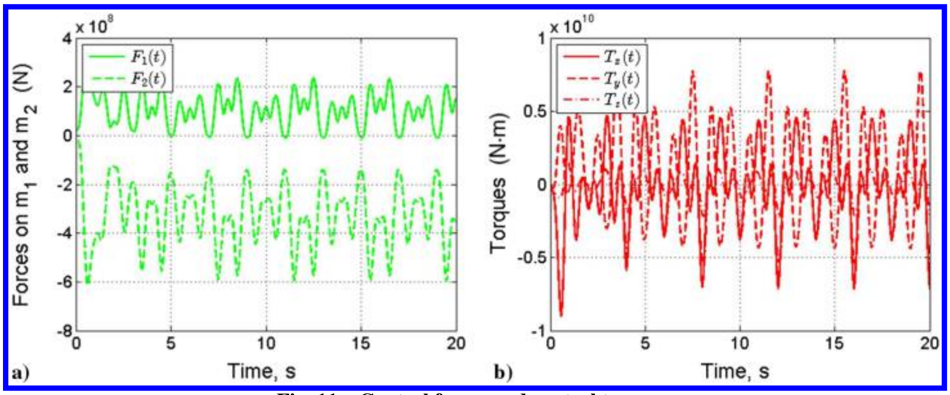

Fig. 11 Control forces and control torques.

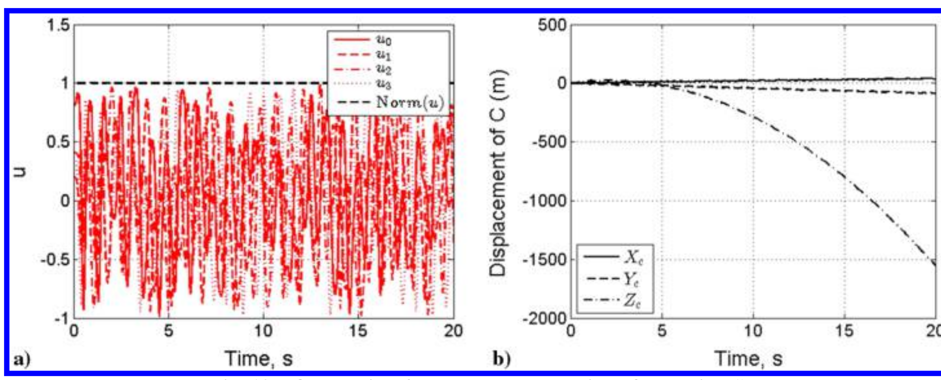

Fig. 12 Quaternion 4-vector and the motion of the point *C*.

initial conditions are the same as those used in Sec. V. Recall that the sum of the discrete masses is about 2.4 times the mass of the cylinder–rod system *BR*.

For the control requirements, of the five discrete masses, we take as our control objective the control of the positions of *only* masses m_1, m_2, and m_4, allowing the motion of masses m_3 and m_5 to remain uncontrolled. Hence, our control requirements on the positions of the masses are as follows:

$$\bar{p}^i(t) = p_e^i + l^i \cos(\lambda_i t), \qquad i = 1, 2, 4 \tag{93}$$

As before, the constraints imposed on the dynamical system are then [see Eq. (79)]

$$\ddot{p}^i = -\alpha_i(\dot{p}^i - \dot{\bar{p}}^i) - \beta_i(p^i - \bar{p}^i) + \ddot{\bar{p}}^i \coloneqq \hat{g}_i, \quad i = 1, 2, 4 \tag{94}$$

One can express these three control requirements in the form $A_p\ddot{q} = b_p(q, \dot{q}, t)$, where matrix $A_p = [0_{3\times7}, V_{3\times5}]$ and matrix V have all elements as zero, except for (1,1), (2,2), and (3,4) elements, which are each equal to unity. Similarly, the three-vector $b_p \coloneqq \hat{g} = [\hat{g}_1, \hat{g}_2, \hat{g}_4]^T$ has the right-hand sides of Eq. (94) as its three elements.

Control requirements on the components of the angular velocities (about the principal axes of the cylinder) are imposed so that, as before,

$$\bar{\omega}_i(t) = b_i \cos(\sigma_i t), \qquad i = 1, 2, 3 \tag{95}$$

This leads to

$$A_\omega = [0_{3\times3}, H, 0_{3\times5}]$$

and

$$b_\omega = -\gamma(H\dot{u} - \bar{\omega}) + \dot{\bar{\omega}}(t) \tag{96}$$

in a manner similar to Eq. (86), making allowance for the fact that the vector p is five-by-one now because we have five discrete masses. We assemble the control constraints now so that

$$A\ddot{q} \coloneqq \begin{bmatrix} 0_{1\times3} & u^T & 0_{1\times5} \\ 0_{3\times3} & H_{3\times4} & 0_{3\times5} \\ 0_{3\times3} & 0_{3\times4} & V_{3\times5} \end{bmatrix} \begin{bmatrix} \ddot{R} \\ \ddot{u} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} -\delta_1 u^T\dot{u} - \frac{\delta_2}{2}(u^T u - 1) - \dot{u}^T\dot{u} \\ -\gamma(H\dot{u} - \bar{\omega}) + \dot{\bar{\omega}}(t) \\ \hat{g} \end{bmatrix} \coloneqq b \tag{97}$$

The rank of matrix A is seven now, and it does not change because, again, the matrix $[u\frac{1}{2}H^T]$ is orthogonal. We next show some simulation results.

The parameters that specify our control requirements given in Eqs. (93) and (95) are chosen as follows:

$$l^1 = 0.8, \ \ l^2 = -1, \ \ l^4 = 0.5; \ \ \lambda_1 = \pi/2, \ \ \lambda_2 = \pi, \ \ \lambda_4 = \pi/4 \tag{98}$$

and

$$b = [8, \quad -10, \quad 15]^T$$

and

$$\sigma = [\pi, \quad 0, \quad 2\pi]^T \tag{99}$$

Furthermore, we choose the same parameters in Eq. (97) as in example 1 so that

$$\alpha_i = 2, \ \ \beta_i = 12, \quad i = 1, 2, 3 \tag{100}$$

$$\delta_1 = 0.5, \quad \delta_2 = 8, \quad \text{and} \quad \gamma = 0.4 \tag{101}$$

We consider the cylinder–rod system that was described in Sec. V (example 2), keeping the description of the body BR, the description and values of the masses and springs, and the initial conditions the same. The simulations are again done over an interval of 20 s.

Figure 13 shows the controlled motion of the five discrete masses that vibrate along the rod R inside the cylinder. The motions of masses m_1, m_2, and m_4 are controlled, with the control requirement being that they oscillate sinusoidally with amplitudes of 0.8, −1, and 0.5 m, respectively, and with periods of 4, 2, and 8 s, respectively. The error in tracking the control requirements of the three discrete masses is given by $e_i(t) = p^i(t) - \bar{p}^i(t)$, $i = 1, 2, 4$. The 3-vector $e_p(t) \coloneqq [e_1(t), e_2(t), e_4(t)]^T$ is also shown. These errors at $t = 20$ s are of $O(10^{-8})$ m. When the computations are continued to 60 s, the errors at the end of $e_p(60)$ are of $O(10^{-11})$ m. It is seen that masses m_3 and m_5, which are not controlled, show (relatively) high-frequency oscillations.

Figure 14 shows the three components of the angular velocity $e_\omega(t) = \omega(t) - \bar{\omega}(t)$. At the end of 20 s, the error $e_\omega(t = 20)$ is of $O(10^{-3})$ rad/s. When the computations are continued to 60 s, $e_\omega(t = 60)$ reduces and is of $O(10^{-9})$ rad/s, showing the accuracy attainable with which the control objectives are met. The effect of the control is illustrated by comparing Figs. 14a and 6b.

The necessary forces on the discrete masses along the rod R and the required control torques about the principal axes of BR passing through C [so that the control requirements expressed by Eqs. (93) and (95) are met] are shown in Fig. 15. As seen from the figure, because only masses m_1, m_2, and m_4 are controlled, no forces (along the rod R) are needed to be applied to masses m_3 and m_5.

Lastly, we show the quaternion 4-vector $u(t)$ and the motion of point C (center of mass of body BR) in Fig. 16. The error in the norm of the quaternion 4-vector $e_u = u_0^2 + u_1^2 + u_2^2 + u_3^2 - 1$ is of $O(10^{-12})$ throughout the interval of integration.

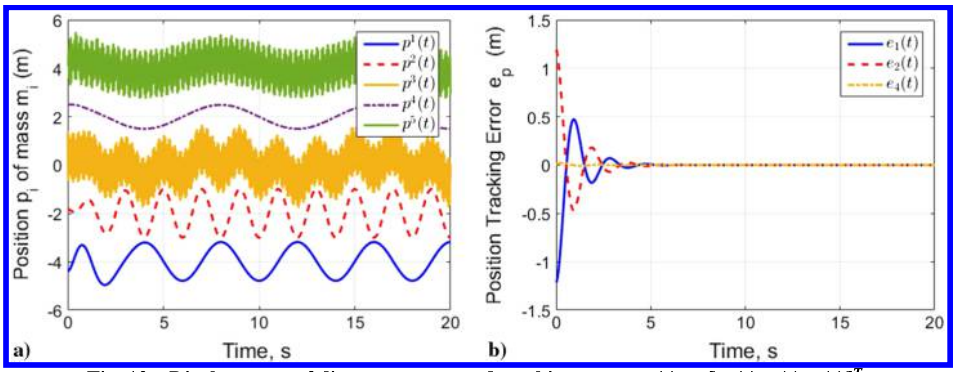

Fig. 13 **Displacement of discrete masses and tracking error $e_p(t) \coloneqq [e_1(t)e_2(t)e_4(t)]^T$.**

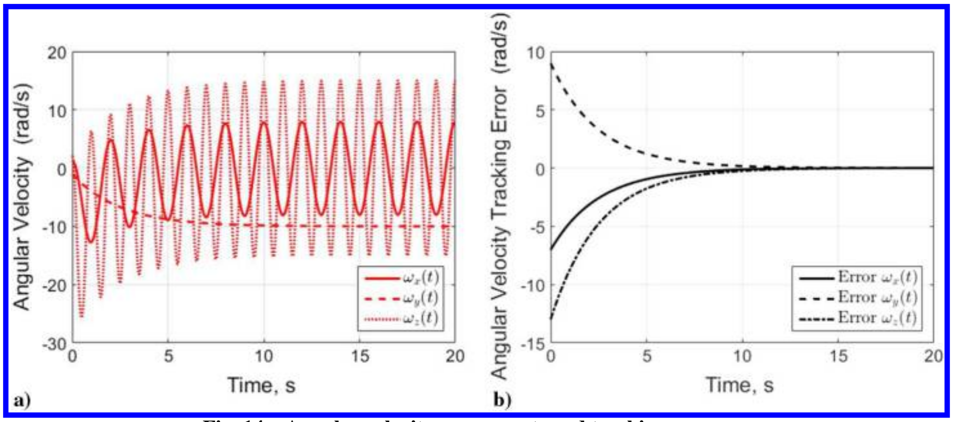

Fig. 14 **Angular velocity components and tracking error e_ω.**

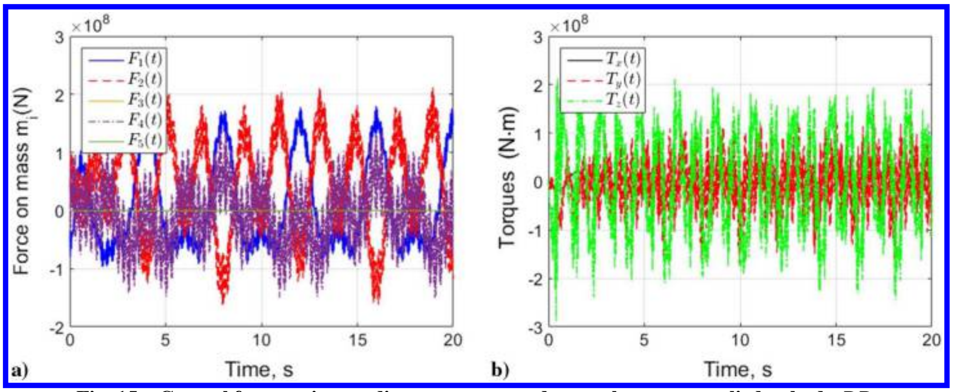

Fig. 15 **Control force acting on discrete masses and control torques applied to body BR.**

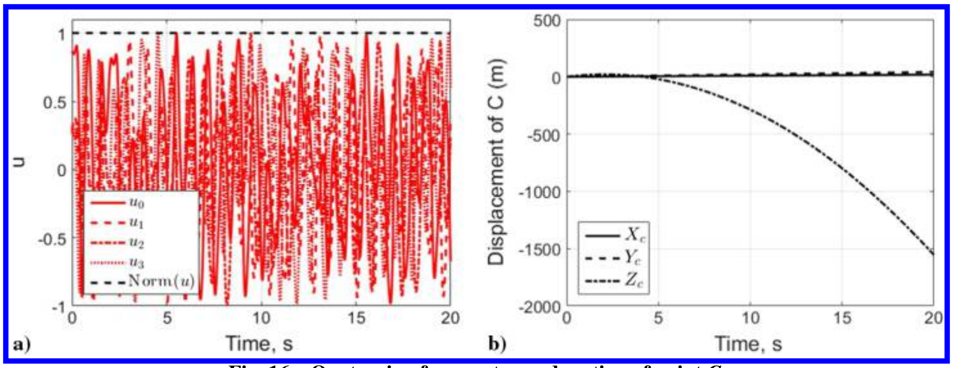

Fig. 16 Quaternion four vector and motion of point *C*.

VIII. Conclusions

In this paper, a generic model of a tumbling rigid body is developed that has an arbitrary number of internal degrees of freedom. The internal motions are modeled by discrete masses that move along an axis that has an arbitrary orientation to the body and that are connected by nonlinear spring elements. The explicit quaternion equations of motion for the system are obtained. They are shown to be highly nonlinear, coupling the translational and rotational motions of the rigid body to those of the vibrating discrete masses.

A simple approach, which can also be used for nonautonomous dynamical systems, is based on recent developments from analytical dynamics, has been used for precision control of the tumbling, vibrating system. Control requirements imposed on the system are as follows:

1) The angular velocity of the entire system should track user-prescribed time-dependent angular velocity trajectories.

2) The motion of the discrete masses internal to the tumbling body should track user-prescribed time-dependent trajectories.

The control approach has the following characteristics:

1) It uses the entire nonlinear quaternion dynamical equations with no approximations or linearizations.

2) The (generalized) control forces are obtained in closed form.

3) This leads to negligible computational costs, making the approach attractive for real-time control of tumbling objects, like space debris.

4) At *each* instant of time, a user-specified quadratic control cost is minimized.

5) It is shown that the control is both simple to obtain and efficacious.

6) Precision control is achieved, and the errors in tracking the control requirements are shown to be asymptotically of the same order of magnitude as the numerical error tolerances with which the equations of motion of the controlled system are integrated. This is due to the exact nature of the (generalized) control forces obtained here for enforcing the control requirements (constraints).

Appendix A: Derivation of Lagrange's Equations of Motion

To prove the relations in Eqs. (50–53), we breakdown Eq. (49) into its three components: equations arising from partial derivatives with respect to the three partitions of the configuration vector q shown in Eq. (44). Thus, the first three equations of the equation set [Eq. (49)] are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{R}}\right) - \frac{\partial T}{\partial R} = -\frac{\partial U}{\partial R} \tag{A1}$$

The first term in Eq. (A1) is computed using Eq. (39) in the following manner:

$$\begin{aligned}\frac{\partial T}{\partial \dot{R}} &= m_{BR}\left(\frac{\partial \dot{R}}{\partial \dot{R}}\right)^T \dot{R} + \left(\frac{\partial \omega}{\partial \dot{R}}\right)^T J\omega + \left(\frac{\partial \dot{r}}{\partial \dot{R}}\right)^T M_D \dot{r} \\ &= m_{BR} I_3 \dot{R} + 0 + (\mathbf{1}_n^T \otimes I_3) M_D \dot{r} = m_{BR}\dot{R} + (\mathbf{1}_n^T \otimes I_3) M_D \dot{r}\end{aligned} \tag{A2}$$

Noting that $\mathbf{1}_n^T \otimes I_3 M_D$ can be simplified using the identity given in Eq. (8) as

$$(\mathbf{1}_n^T \otimes I_3) M_D = (\mathbf{1}_n^T \otimes I_3)(\mathrm{diag}(m) \otimes I_3) = (\mathbf{1}_n \circ m)^T \otimes I_3 = m^T \otimes I_3 \tag{A3}$$

we can simplify Eq. (A2) as

$$\frac{\partial T}{\partial \dot{R}} = m_{BR}\dot{R} + (m^T \otimes I_3)\dot{r}. \tag{A4}$$

Thus, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{R}}\right) &= m_{BR}\ddot{R} + (m^T \otimes I_3)\ddot{r} \\ &= m_{BR}\ddot{R} + (m^T \otimes I_3)\Big(\mathbf{1}_n \otimes \ddot{R} + \ddot{p} \otimes Sa + 2\dot{p} \otimes L_a \dot{u} \\ &\quad + p \otimes [\dot{L}_a \dot{u} + L_a \ddot{u}] + \mathbf{1}_n \otimes [\dot{L}_d \dot{u} + L_d \ddot{u}]\Big) \\ &= m_{BR}\ddot{R} + (m^T \mathbf{1}_n)\ddot{R} + (m^T \ddot{p}) Sa + 2(m^T \dot{p}) L_a \dot{u} \\ &\quad + (m^T p)(\dot{L}_a \dot{u} + L_a \ddot{u}) + (m^T \mathbf{1}_n)(\dot{L}_d \dot{u} + L_d \ddot{u})\end{aligned} \tag{A5}$$

We have substituted for $\ddot{r}$ from Eq. (40) in the second equality and repeatedly used the identity given in Eq. (7) to obtain the third equality.

The second term in Eq. (A1) is simply

$$\frac{\partial T}{\partial R} = 0 \tag{A6}$$

and the last one is obtained using Eq. (34):

$$\begin{aligned}\frac{\partial U}{\partial R} &= m_{BR} g e_3 + g\left(\frac{\partial r}{\partial R}\right)^T (m \otimes e_3) \\ &= m_{BR} g e_3 + g(\mathbf{1}_n^T \otimes I_3)(m \otimes e_3) \\ &= m_{BR} g e_3 + g(\mathbf{1}_n^T m) e_3\end{aligned} \tag{A7}$$

Using Eqs. (A5–A7), Eq. (A1) is rewritten as

$$\begin{aligned}&\{m_{BR} + m^T \mathbf{1}_n\}\ddot{R} + \Big[\{m^T p\} L_a + \{m^T \mathbf{1}_n\} L_d\Big]\ddot{u} + Sam^T \ddot{p} \\ &= -2\{m^T \dot{p}\} L \dot{u} - \Big[\{m^T p\}\dot{L}_a + \{m^T \mathbf{1}_n\}\dot{L}_d\Big]\dot{u} \\ &\quad - \Big(m_{BR} + \{m^T \mathbf{1}_n\}\Big) g e_3\end{aligned} \tag{A8}$$

where, for convenience, scalars have been shown by curly brackets. It can be immediately verified that the first block row of mass matrix M and generalized force vector Q in Eqs. (50–52) is correct using Eq. (A8).

We next look at the Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{\partial U}{\partial u} \tag{A9}$$

Noting Eqs. (15) and (42), the first term in Eq. (A9) is computed as

$$\begin{aligned}\frac{\partial T}{\partial \dot{u}} &= \left(\frac{\partial \omega}{\partial \dot{u}}\right)^T J\omega + \left(\frac{\partial \dot{r}}{\partial \dot{u}}\right)^T M_D \dot{r} \\ &= H^T J H \dot{u} + (p^T \otimes L_a^T + 1_n^T \otimes L_d^T) M_D \dot{r}\end{aligned} \tag{A10}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{u}}\right) &= \dot{H}^T J H \dot{u} + H^T J \dot{H}\,\dot{u} + H^T J H \ddot{u} + (\dot{p}^T \otimes L_a^T) M_D \dot{r} \\ &+ (p^T \otimes \dot{L}_a^T) M_D \dot{r} + (1_n^T \otimes \dot{L}_d^T) M_D \dot{r} \\ &+ (p^T \otimes L_a^T) M_D \ddot{r} + (1_n^T \otimes L_d^T) M_D \ddot{r} \\ &= \dot{H}^T J H \dot{u} + H^T J H \ddot{u} + \left(\dot{p}^T \otimes L_a^T + p^T \otimes \dot{L}_a^T + 1_n^T \otimes \dot{L}_d^T\right) M_D \dot{r} \\ &+ \left[(p^T \otimes L_a^T) + (1_n^T \otimes L_d^T)\right] M_D \ddot{r}\end{aligned} \tag{A11}$$

because $\dot{H}\,\dot{u} = 0$.

Using Eqs. (15) and (42) again, the second term is calculated as,

$$\begin{aligned}\frac{\partial T}{\partial u} &= \left(\frac{\partial \omega}{\partial u}\right)^T J\omega + \left(\frac{\partial \dot{r}}{\partial u}\right)^T M_D \dot{r} \\ &= -\dot{H}^T J H \dot{u} + \left(\dot{p}^T \otimes L_a^T + p^T \otimes \dot{L}_a^T + 1_n^T \otimes \dot{L}_d^T\right) M_D \dot{r}.\end{aligned} \tag{A12}$$

Using Eq. (34), the final term is computed as

$$\begin{aligned}\frac{\partial U}{\partial u} &= g\left(\frac{\partial r}{\partial u}\right)^T (m \otimes e_3) = g(p^T \otimes L_a^T + 1_n^T \otimes L_d^T)(m \otimes e_3) \\ &= g\left[\{p^T m\} L_a^T + \{m^T 1_n\} L_d^T\right] e_3\end{aligned} \tag{A13}$$

Identity (7) has been used again in obtaining the third equality. Following the same steps shown in Eq. (A3), we can simplify $(p^T \otimes L_a^T) M_D$ and $(1_n^T \otimes L_d^T) M_D$ as

$$(p^T \otimes L_a^T) M_D = (p \circ m)^T \otimes L_a^T$$

and

$$(1_n^T \otimes L_d^T) M_D = (1_n \circ m)^T \otimes L_d^T \tag{A14}$$

The left-hand side of Eq. (A9) then simplifies, using identity (7), to

$$\begin{aligned}& H^T J H \ddot{u} + [(p^T \otimes L_a^T) + (1_n^T \otimes L_d^T)] M_D \ddot{r} + 2\dot{H}^T J H \dot{u} \\ &= H^T J H \ddot{u} + (p \circ m)^T \otimes L_a^T [1_n \otimes \ddot{R} + \ddot{p} \otimes Sa + 2\dot{p} \otimes L_a \dot{u} \\ &+ p \otimes (\dot{L}_a \dot{u} + L_a \ddot{u}) + 1_n \otimes (\dot{L}_d \dot{u} + L_d \ddot{u})] \\ &+ (1_n \circ m)^T \otimes L_d^T [1_n \otimes \ddot{R} + \ddot{p} \otimes Sa + 2\dot{p} \otimes L_a \dot{u} \\ &+ p \otimes (\dot{L}_a \dot{u} + L_a \ddot{u}) + 1_n \otimes (\dot{L}_d \dot{u} + L_d \ddot{u})] + 2\dot{H}^T J H \dot{u} \\ &= H^T J H \ddot{u} + 2\dot{H}^T J H \dot{u} + \{(p \circ m)^T 1_n\} L_a^T \ddot{R} + \{m^T 1_n\} L_d^T \ddot{R} \\ &+ \{(p \circ m)^T \ddot{p}\} L_a^T Sa + \{m^T \ddot{p}\} L_d^T Sa + 2\{(p \circ m)^T \dot{p}\} L_a^T L_a \dot{u} \\ &+ \{(p \circ m)^T p\} L_a^T (\dot{L}_a \dot{u} + L_a \ddot{u}) + \{(p \circ m)^T 1_n\} L_a^T (\dot{L}_d \dot{u} + L_d \ddot{u}) \\ &+ 2\{m^T \dot{p}\} L_d^T L_a \dot{u} + \{m^T p\} L_d^T (\dot{L}_a \dot{u} + L_a \ddot{u}) \\ &+ \{m^T 1_n\} L_d^T (\dot{L}_d \dot{u} + L_d \ddot{u})\end{aligned} \tag{A15}$$

To facilitate reading, curly brackets in the last equality show inner products (scalars). Thus, Eq. (A9) can be rewritten using Eqs. (A13) and (A15) as

$$\begin{aligned}& \left[\{(p \circ m)^T 1_n\} L_a^T + \{m^T 1_n\} L_d^T\right] \ddot{R} + \Big(H^T J H + \{(p \circ m)^T p\} L_a^T L_a \\ &+ \{(p \circ m)^T 1_n\} L_a^T L_d + \{m^T p\} L_d^T L_a + \{m^T 1_n\} L_d^T L_d\Big) \ddot{u} \\ &+ \left(L_a^T Sa (p \circ m)^T + L_d^T Sam^T\right) \ddot{p} \\ &= -2\dot{H}^T J H \dot{u} - 2\Big(\{(p \circ m)^T \dot{p}\} L_a^T L_a + \{m^T \dot{p}\} L_d^T L_a\Big) \dot{u} \\ &- g\left(\{p^T m\} L_a^T + \{m^T 1_n\} L_d^T\right) e_3 - \Big(\{(p \circ m)^T p\} L_a^T \dot{L}_a \\ &+ (p \circ m)^T 1_n\} L_a^T \dot{L}_d + \{m^T p\} L_d^T \dot{L}_a + \{m^T 1_n\} L_d^T \dot{L}_d\Big) \dot{u}\end{aligned} \tag{A16}$$

Equation (A16) can be used to verify that the second block row of mass matrix M and generalized force vector Q in Eqs. (51) and (52) is correct. In the third and final step, we look at the Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = -\frac{\partial U}{\partial p} \tag{A17}$$

Noting Eq. (43), the first term is computed as

$$\frac{\partial T}{\partial \dot{p}} = \left(\frac{\partial \dot{r}}{\partial \dot{p}}\right)^T M_D \dot{r} = (I_n \otimes (Sa)^T) M_D \dot{r} = \mathrm{diag}(m) \otimes (Sa)^T \dot{r} \tag{A18}$$

and where we have used identity (9) to simplify the expression $(I_n \otimes (Sa)^T) M_D$ as

$$\begin{aligned}\left(I_n \otimes (Sa)^T\right) M_D &= \left(\mathrm{diag}(1_n) \otimes (Sa)^T\right)(\mathrm{diag}(m) \otimes I_3) \\ &= \mathrm{diag}(1_n \circ m) \otimes (Sa)^T = \mathrm{diag}(m) \otimes (Sa)^T\end{aligned} \tag{A19}$$

Differentiating Eq. (A18) with respect to time yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{p}}\right) = \mathrm{diag}(m) \otimes (\dot{S}a)^T \dot{r} + \mathrm{diag}(m) \otimes (Sa)^T \ddot{r} \tag{A20}$$

Also using Eq. (43), we obtain

$$\frac{\partial T}{\partial p} = \left(\frac{\partial \dot{r}}{\partial p}\right)^T M_D \dot{r} = (I_n \otimes (\dot{S}a)^T) M_D \dot{r} = \mathrm{diag}(m) \otimes (\dot{S}a)^T \dot{r} \tag{A21}$$

Furthermore, using Eqs. (34) and (10), we obtain the third term as

$$\begin{aligned}\frac{\partial U}{\partial p} &= g\left(\frac{\partial r}{\partial p}\right)^T m \otimes e_3 + \frac{\partial U_s}{\partial p} = g(I_n \otimes Sa)^T m \otimes e_3 + \frac{\partial U_s}{\partial p} \\ &= g(1_n \circ m)(Sa)^T e_3 + \frac{\partial U_s}{\partial p} = mg(Sa)^T e_3 - F^S\end{aligned} \tag{A22}$$

where, the ith element of the column vector F^S is the generalized force applied by the spring elements:

$$\begin{aligned}F_i^S &= -\frac{\partial U_s}{\partial p} = -k_i^l (p^i - p^{i-1} - p_e^i + p_e^{i-1}) \\ &- k_i^n (p^i - p^{i-1} - p_e^i + p_e^{i-1})^3 + k_{i+1}^l (p^{i+1} - p^i - p_e^{i+1} + p_e^{i-1}) \\ &+ k_{i+1}^n (p^{i+1} - p^i - p_e^{i+1} + p_e^{i-1})^3\end{aligned} \tag{A23}$$

Combining Eqs. (A20) and (A21), and using Eq. (40), the left-hand side of Eq. (A17) is computed by recalling Eq. (10) as

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} &= \mathrm{diag}(m) \otimes (Sa)^T \ddot{r} \\
&= \mathrm{diag}(m) \otimes (Sa)^T \Big[1_n \otimes \ddot{R} + \ddot{p} \otimes Sa + 2\dot{p} \otimes L_a \dot{u} \\
&\quad + p \otimes (\dot{L}_a \dot{u} + L_a \ddot{u}) + 1_n \otimes (\dot{L}_d \dot{u} + L_d \ddot{u}) \Big] \\
&= (m \circ 1_n)(Sa)^T \ddot{R} + (m \circ \ddot{p})(Sa)^T Sa + 2(m \circ \dot{p})(Sa)^T L_a \dot{u} \\
&\quad + (m \circ p)(Sa)^T (\dot{L}_a \dot{u} + L_a \ddot{u}) + (m \circ 1_n)(Sa)^T (\dot{L}_d \dot{u} + L_d \ddot{u}) \\
&= m(Sa)^T \ddot{R} + \Big((p \circ m)(Sa)^T L_a + m(Sa)^T L_d \Big) \ddot{u} \\
&\quad + \{(Sa)^T Sa\} \mathrm{diag}(m) \ddot{p} + 2\{(Sa)^T L_a \dot{u}\}(m \circ \dot{p}) \\
&\quad + \Big((p \circ m)(Sa)^T \dot{L}_a + m(Sa)^T \dot{L}_d \Big) \dot{u}
\end{aligned} \tag{A24}
$$

In the third equality, we have used identity (10) repeatedly. Thus, using Eqs. (A22) and (A24), Eq. (A17) can be rewritten as

$$
\begin{aligned}
& m(Sa)^T \ddot{R} + \Big((p \circ m)(Sa)^T L_a + m(Sa)^T L_d \Big) \ddot{u} + \{(Sa)^T Sa\} \mathrm{diag}(m) \ddot{p} \\
&= -2\{(Sa)^T L_a \dot{u}\}(m \circ \dot{p}) - \{(Sa)^T \dot{L}_a \dot{u}\}(p \circ m) \\
&\quad - \{(Sa)^T \dot{L}_d \dot{u}\} m + F_s - g\{(Sa)^T e_3\} m
\end{aligned} \tag{A25}
$$

From Eq. (A25), the last block row of the mass matrix M and the generalized force vector Q in Eqs. (51) and (52) are verified to be correct. □

Appendix B: Handling General Initial Conditions and Proof of Exponential Stability

This appendix addresses the use of a more general control cost $J(t)$ and exponential stability of the control approach. Consider a mechanical system for which the unconstrained equation of motion is given by

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) \tag{B1}$$

where M is an n-by-n matrix, and Q is the given generalized force n vector. The control constraints are given by

$$A(q,\dot{q},t)\dot{q} = c(q,\dot{q},t) \tag{B2}$$

where A is an m-by-n matrix, and c is an m vector. Upon differentiating Eq. (B2), we obtain

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t) \tag{B3}$$

where

$$b(q,\dot{q},t) \coloneqq -\dot{A}\,\dot{q} + \dot{c} \tag{B4}$$

On defining the error in satisfying the control constraint as

$$e \coloneqq A\dot{q} - c \tag{B5}$$

we might enforce a general modified constraint of the form

$$\dot{e} + Pe = 0 \tag{B6}$$

to allow for general initial conditions that may not satisfy the control constraint at the initial time $t = 0$. The matrix P in Eq. (B6) is a user-prescribed positive definite matrix, and hence can be diagonalized so that $P = T\Lambda T^T$, where T is an orthogonal matrix and Λ is a diagonal matrix for which the (diagonal) elements are all strictly positive. The solution of Eq. (B6) is

$$e(t) = \exp(-Pt)e_0 = T\exp(-\Lambda t)T^T e_0 \tag{B7}$$

where e_0 is the error in the satisfaction of the constraints at the initial time. Thus, $e(t) \to 0$ exponentially as $t \to \infty$, and the diagonal elements of Λ influence the rate of asymptotic convergence to $e = 0$.

The conclusion that the modified constraint in Eq. (B6) leads to stable control can also be reached in a different way by considering the Lyapunov function $V = (1/2)e^T e$ and noticing that its rate of change along the trajectory of the dynamical system is $\dot{V} = e^T\dot{e} = -e^T Pe$. Because P is a positive definite matrix, the rate of change of the Lyapunov function is always negative; hence, the controlled dynamical system converges to $e = 0$ asymptotically as $t \to \infty$.

Using Eqs. (B3–B5), Eq. (B6) can be expanded as

$$A\ddot{q} = b - Pe \tag{B8}$$

The equation of the controlled dynamical system is then given by

$$M(q,t)\ddot{q} = Q(q,\dot{q},t) + Q^C(q,\dot{q},t) \tag{B9}$$

where the control force Q^C enforces the constraint given in Eq. (B8) while simultaneously minimizing, at each instant of time, a general control cost of the form $J(t) = Q^{C^T} N(q,\dot{q},t) Q^C$, where N is a user-prescribed positive definite matrix. The explicit expression for the control force Q^C is given by [38,39]

$$Q^C = N^{-1}M^{-1}A^T(AM^{-1}N^{-1}M^{-1}A^T)^+(b - Pe - AM^{-1}Q) \tag{B10}$$

where X^+ denotes the Moore–Penrose inverse of the matrix X. It should be noted that the preceding control force can be split as

$$
\begin{aligned}
Q^C &= \underbrace{N^{-1}M^{-1}A^T(AM^{-1}N^{-1}M^{-1}A^T)^+(b - AM^{-1}Q)}_{Q_1^C} \\
&\quad - \underbrace{N^{-1}M^{-1}A^T(AM^{-1}N^{-1}M^{-1}A^T)^+ Pe}_{Q_2^C}
\end{aligned} \tag{B11}
$$

where the term Q_1^C ensures that the system does not leave the constraint manifold once it reaches it, whereas the term Q_2^C drives the system to the constraint manifold (starting from arbitrary initial conditions).

When the weighting matrix N is chosen so that $N = M^{-1}$, Eq. (B11) simplifies to

$$Q^C = \underbrace{A^T(AM^{-1}A^T)^+(b - AM^{-1}Q)}_{Q_1^C} - \underbrace{A^T(AM^{-1}A^T)^+ Pe}_{Q_2^C} \tag{B12}$$

and when $N = M^{-2}$, it simplifies to

$$Q^C = \underbrace{MA^T(AA^T)^+(b - AM^{-1}Q)}_{Q_1^C} - \underbrace{MA^T(AA^T)^+ Pe}_{Q_2^C} \tag{B13}$$

Finally, when the rank of matrix A is m, all the Moore–Penrose inverses used in Eqs. (B10–B13) can be simply replaced by (the usual) matrix inverses.

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