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# Decentralised control of nonlinear dynamical systems

Firdaus E. Udwadia[a,*], Prasanth B. Koganti[b], Thanapat Wanichanon[c] and Dušan M. Stipanović[d]

[a]*Department of Aerospace and Mechanical Engineering, Civil Engineering, Mathematics, and Information and Operations Management, 430K Olin Hall, University of Southern California, Los Angeles, CA 90089-1453, USA;* [b]*Department of Civil Engineering, University of Southern California, Los Angeles, CA 90089, USA;* [c]*Department of Mechanical Engineering, Mahidol University, 25/25 Phutthamonthon, Nakhon Pathom 73170, Thailand;* [d]*Department of Industrial and Enterprise Systems Engineering and Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801, USA*

In this paper, we provide a simple novel approach to decentralised control design. Each subsystem of an interconnected interacting system is controlled in a decentralised manner using locally available information related only to the state of that particular subsystem. The method is developed in two steps. In the first step, we define what we call a 'nominal system', which consists of 'nominal subsystems'. The nominal subsystems are assumed to be acted upon by forces that can be computed using only locally available information. We obtain an asymptotically stable control for each nominal subsystem which minimises a suitable, desired norm of the control effort at each instant of time. In the second step, we determine the control force that needs to be applied to the actual (interconnected) subsystem in addition to the control force calculated for the nominal subsystem, so each actual subsystem tracks the state of the controlled nominal subsystem as closely as desired. This additional compensating controller is obtained using the concept of a generalised sliding surface control. The design of this additional controller needs as its input an estimate of the bound on the mismatch between the nominal and the actual subsystems. Examples of non-autonomous, nonlinear, distributed systems are provided that demonstrate the efficacy and ease of implementation of the control method.

**Keywords:** decentralised control; Lyapunov function; nominal system; generalised sliding surface; nonlinear system



## 1. Introduction

In engineering applications, many a times, we come across complex systems whose dynamics are coupled together. Often, information about the state of the entire system may not be available, or if available it would be so enormous as to prevent real-time control because of data gathering and information processing overheads. Thus, for large complex systems, one is often constrained to using only the 'locally available' information about each subsystem that comprises the entire interacting conglomerate in order to control the conglomerate in a desired manner. Such problems of decentralised control arise in numerous fields where large complex systems are involved such as in process control, formation flight of multiple unmanned aerial vehicles (UAVs) project management, and in the analysis of economic and social systems, to name a few. Thus, decentralised control design is an important problem when dealing with large complex systems. For this reason, the development of methods to ensure effective and efficient control of an interconnected, conglomerate dynamical system through the control of each of its composite subsystems by using information that is only locally available has become a topic of intensive research in recent years. As mentioned before, the obvious difficulty in designing localised, or decentralised, control is that we have limited information regarding the global state of the system. This, in particular, raises considerable issues about the stability of the entire controlled system when it is controlled by decentralised controllers, each of which does not have information about the entire state of the system and the behaviour of any of the other controllers. In this paper, we show that such a control design is possible and can simply be effected for general nonlinear systems.

While the literature in the area of decentralised control of linear systems is enormous, that dealing with nonlinear systems is extremely scant. We mention only a few relevant and representative results. Wang et al. (2007) applied decentralised control techniques to reduce the response of a building subjected to earthquakes. Fallah and Taghikhany (2011) applied decentralised control to reduce the response of a cable-stayed bridge under seismic loads. Lu, Loh, Yang, and Lin (2008) investigated the application of sliding mode control in the control of a building using magneto-rheological (MR) dampers. They have also provided the damper configuration required for decentralised control. The building and bridge structures being controlled are modelled as linear systems. We also mention valuable results on decentralised

*Corresponding author. Email: fudwadia@usc.edu

control provided in a recent book by Zečević and Šiljak (2010) and a long list of references reported therein. Most of the above-mentioned work uses linear system models and tries to minimise a cost function that is quadratic in the state variable and in the control cost integrated over time. They achieve decentralised control by applying a constraint over the structure of the control gain matrix. In other words, not much has changed since Sandell, Varaiya, Athans, and Safonov (1978) noted that linear problems are the ones mostly studied, since nonlinear feedback control theory is not developed nearly as far even for the case of centralised control. Witsenhausen (Sandell et al., 1978; Witsenhausen, 1968) formulated a very simple counter-example problem that shows that the performance of linear feedback control is inferior compared to nonlinear feedback control strategies when full information is not shared between the subsystems.

It is important to highlight that the design in this paper is developed for nonlinear systems and the resulting controllers are nonlinear. This is very different from linear matrix inequalities (LMIs) control design (Boyd, Ghaoui, Feron, & Balakrishnan, 1994; Ghaoui & Niculescu, 2000) which is based on convex programming. LMIs-based designs produce linear controllers and at best may be used for nonlinear systems with linear nominal parts (Šiljak & Stipanović, 2000; Šiljak, Stipanović, & Zečević, 2002; Stanković, Stipanović, & Šiljak, 2007) or for nonlinear systems which are bounded by comparison systems that are linear (Boyd et al., 1994). In either case both the controllers and the analysis are linear in nature. Another drawback of using LMIs for designing decentralised controllers in particular is that when imposing decentralised information structure on the controllers even in the case of pure gain feedback controllers, the LMIs require a necessary parameterisation of the gain matrices to be imposed (Šiljak & Stipanović, 2000). This parameterisation introduces a significant restriction and thus the LMIs decentralised control design is shown to fail to produce decentralised stabilising controllers even in the case when they are known to exist (Šiljak & Stipanović, 2000). In our design no such parameterisation is needed, and the decentralised information structure constraint is directly incorporated into the design of decentralised controllers.

An approach that extends the small-gains theorem to certain decentralised systems in which characterisation of the subsystems is mixed – the stability of some of the subsystems is characterised using summation of gains and stability of other subsystems is characterised using maximisation of gains – has been recently proffered in Dashkovskiy, Kosmykov, and Wirth (2011). The usefulness of this result lies in the fact that the obtained condition is less conservative and hence applicable to more general systems. The standard control design approach is used in which the control is first designed and its stability then verified. For example, in Polushin, Dashkovskiy, Takhmar, and Patel (2013) that uses this approach, controls are designed for a networked cooperative force-reflecting teleoperator system based on a version of the nonlinear small-gain theorem. The controls are assumed to be of the proportional-derivative (PD) type and the parameters are fine-tuned, so the interconnected system is stable.

In the approach presented herein no a-priori structure is imposed on the controller, and since use of a composite Lyapunov function is made to obtain the control, the need to check stability is obviated. Thus the current method is simpler and easier to implement than the small-gains approach and needs less effort on the part of the control designer. More importantly, the small-gains approach uses input to state stability (ISS), which merely requires that trajectories be bounded under bounded inputs, while in the current approach trajectories are asymptotically attracted to a region which can be made as close to the origin as desired. Other differences are that in the current approach the subsystems can be unstable and non-autonomous and such examples are provided in Section 3.

The authors have not found in the current literature any general methods for the decentralised control of non-autonomous nonlinear systems each of whose subsystems may be unstable, nonlinear, and have nonlinear couplings between them. Specifically, the control approach developed in this paper differs from the current state of the art in the sense that: (1) it is a general approach applicable to non-autonomous nonlinear systems, (2) the subsystems may be unstable, (3) a linear structure is not imposed on the controller, and (4) rather than minimising the integral of a cost over the time duration over which the control is effected, the control cost is minimised at *each* instant of time.

In what follows, we shall refer to the mechanical system that we want to control as the 'actual system'. We develop the control design in two steps. In the first step, we define a 'nominal system' which is an imaginary system that does not exist in reality, but is an approximation of the real-life 'actual system' in some sense. The nominal system consists of 'nominal subsystems' whose equations of motion can be independently integrated. We obtain the control forces to be applied to this nominal system, so the controlled nominal system has an asymptotically stable equilibrium at the origin. Closed-form controllers are obtained which use user-prescribed positive definite functions defined over local domains. These control forces are computed in such a way that user-prescribed cost functions are also simultaneously minimised at each instant of time. The control of each nominal subsystem is done so that stability of the nominal system is assured from the manner in which the subsystems are controlled. One advantage of doing this is that we do not need to search for a Lyapunov function to ensure the stability of the entire coupled nonlinear system under the decentralised control scheme developed herein. This is done by using a composite function which is related to vector Lyapunov functions (Lakshmikantham, Matrosov, & Sivasundaram, 1991; Šiljak, 1978, 1991). In most complex systems the nominal subsystems are usually known fairly well; however, the couplings (interactions between the subsystems) are often difficult to assess both in their qualitative nature and in the parameter values used to describe them. We choose the vector Lyapunov function approach for designing decentralised controllers since we subsume that though the subsystems' dynamics are known, the interconnections are known unreliably so. In that case the subsystem-based design which leads to decentralised control based on (vector) Lyapunov functions is known to be more reliable than the centralised one (Šiljak, 1978, 1991).

In the second step, we design additional compensating controllers that ensure that each controlled actual subsystem tracks the trajectory of the corresponding nominal subsystem to within pre-specified error bounds (Wanichanon, 2012). Since the nominal subsystem satisfies the control objective, this ensures that each controlled actual subsystem satisfies the same. The additional controllers required for each subsystem are designed using the concept of generalised sliding surfaces. For more on regular sliding surfaces we refer to the pioneering work by Utkin (1978). This gives us an additional advantage that the controlled actual system is robust to uncertainties. A limitation of this approach is that we need a bound on the mismatch between the nominal and the actual subsystems. This can be overcome in large measure by having a very crude estimate of the bound, and then multiplying it by a suitable factor of safety, since overestimating this bound does not have a significant impact on the magnitude of the additional compensating control.

## 2. Decentralised control

### 2.1 *Actual system*

Consider a general mechanical system consisting of $p$ nonlinear, non-autonomous mechanical subsystems, which are mutually coupled, and whose dynamics are described by the equations,

$$\begin{aligned} M^{(1)}(x^{(1)}, t)\ddot{x}^{(1)} &= F^{(1)}(x, \dot{x}, t), \\ M^{(2)}(x^{(2)}, t)\ddot{x}^{(2)} &= F^{(2)}(x, \dot{x}, t), \\ &\vdots \\ M^{(p)}(x^{(p)}, t)\ddot{x}^{(p)} &= F^{(p)}(x, \dot{x}, t), \end{aligned} \tag{1}$$

where $M^{(i)}$ is the $p_i$ by $p_i$ symmetric, positive definite mass matrix that describes the $i$th subsystem, $i = 1, 2, \ldots, p$, the vector $x^{(i)} \in R^{p_i}$ is a vector describing the configuration of the $i$th subsystem, and $F^{(i)} \in R^{p_i}$ is the external force vector acting on it. In what follows, the superscript '$(i)$' over a quantity refers to that quantity pertinent to the $i$th subsystem. The vector $x = \left[x^{(1)^T}, x^{(2)^T}, \ldots, x^{(p)^T}\right]^T$ in Equation (1) is the configuration vector of the interacting conglomerate system and it has a dimension $P = \sum_{i=1}^{p} p_i$. As noted from the right-hand side of Equation (1), in addition to the externally applied forces on the system that may depend on its global state, each subsystem can also exert, in general, forces on every other. The dots on top of the variables denote derivatives with respect to time. Equation (1) can be written more compactly as

$$M\ddot{x} := \begin{bmatrix} M^{(1)}(x^{(1)}, t) & & & \\ & M^{(2)}(x^{(2)}, t) & & \\ & & \ddots & \\ & & & M^{(p)}(x^{(p)}, t) \end{bmatrix} \ddot{x} = \begin{bmatrix} F^{(1)} \\ F^{(2)} \\ \vdots \\ F^{(p)} \end{bmatrix} := F(x, \dot{x}, t), \tag{2}$$

where $M$ is the $P$ by $P$ positive definite mass matrix and $F$ is a $P$-vector each of whose components depends on $t$, $x$, and $\dot{x}$. Equation (2) is defined over the domain $D \times R^+$, where $D \subseteq R^P \times R^P$. We shall assume that the $M^{(i)}$'s and $F^{(i)}$'s are at least $C^1$ functions of their arguments.

In what follows we shall refer to the real-life mechanical system that we are trying to control described by Equation (1) as the 'actual system'. Our aim is to control it in such a way that the controlled actual system has an equilibrium point at $x = 0$ and $\dot{x} = 0$. We do this in two steps. First, we define what we are going to call a 'nominal system'. We do this in the next subsection and we derive a decentralised control for this nominal system that ensures its asymptotic stability while minimising user-prescribed control costs. In the subsequent subsection, we derive an additional controller that forces the actual system to track the trajectories of the nominal system as closely as desired, thus ensuring the stability of controlled actual system.

### 2.2 *Nominal system*

Let us take a typical nominal subsystem whose mass matrix is $M^{(i)}$, displacement vector is $x_n^{(i)}$, and velocity vector is $\dot{x}_n^{(i)}$. The subscript $n$ indicates quantities that correspond to the nominal system. The global displacement vector for the nominal system is $x_n = \left[x_n^{(1)^T}, x_n^{(2)^T}, \ldots, x_n^{(p)^T}\right]^T$. Let us define $\tilde{x}_n^i$, an approximation of the global displacement vector, by substituting in $x_n$ zeros for displacement of all subsystems except the $i$th subsystem ($x_n^{(j)} = 0$, $j \in [1, p]$, $j \neq i$),

$$\tilde{x}_n^{(i)} = \left[\ldots, 0^T, x_n^{(i)^T}, 0^T, \ldots\right]^T. \tag{3}$$

Similarly we can define an approximation of the global velocity vector as

$$\dot{\tilde{x}}_n^{(i)} = \left[\ldots, 0^T, \dot{x}_n^{(i)^T}, 0^T, \ldots\right]^T \tag{4}$$

and in like manner the external force on the $i$th nominal subsystem by substituting these approximate global vectors into the expression for the external force as

$$\tilde{F}_n^{(i)}\big(x_n^{(i)}, \dot{x}_n^{(i)}, t\big) := F^{(i)}\big(\tilde{x}_n^{(i)}, \dot{\tilde{x}}_n^{(i)}, t\big). \qquad (5)$$

It should be noted that the force on a nominal subsystem depends only on the state of that particular subsystem. Then, the equation of motion for the entire nominal system can be written in a simplified form as

$$M\ddot{x}_n = \begin{bmatrix} \tilde{F}_n^{(1)}\big(x_n^{(1)}, \dot{x}_n^{(1)}, t\big) \\ \tilde{F}_n^{(2)}\big(x_n^{(2)}, \dot{x}_n^{(2)}, t\big) \\ \vdots \\ \tilde{F}_n^{(p)}\big(x_n^{(p)}, \dot{x}_n^{(p)}, t\big) \end{bmatrix} := \tilde{F}_n. \qquad (6)$$

We apply control forces $Q_c^{(i)}\big(x_n^{(i)}, \dot{x}_n^{(i)}, t\big) \in R^{p_i}$ on each nominal subsystem, so that the controlled nominal system (i) has an equilibrium point at $x_n = \dot{x}_n = 0$ and (ii) the control minimises a user-prescribed control cost as described later on in Equation (13). Let us define the global control force vector for the nominal system as

$$Q_c := \Big[Q_c^{(1)^T}, Q_c^{(2)^T}, \ldots, Q_c^{(p)^T}\Big]^T. \qquad (7)$$

In the presence of this control force, the equation of motion for each of the controlled nominal subsystems is

$$M^{(i)}\ddot{x}_n^{(i)} = \tilde{F}_n^{(i)} + Q_c^{(i)}, i = 1, 2, \cdots, p. \qquad (8)$$

We shall refer to $Q_c$, for short, as the nominal control force.

Let us consider a Lyapunov function $V(x_n, \dot{x}_n, t)$ for this controlled nominal system, which is described by Equation (8), such that

$$\text{(i)} \qquad V_L(x_n, \dot{x}_n) \le V(x_n, \dot{x}_n, t) \le V_U(x_n, \dot{x}_n), \qquad (9)$$

where $V_L(x_n, \dot{x}_n)$ and $V_U(x_n, \dot{x}_n)$ are positive definite functions on the domain $D$ and

$$\text{(ii)} \qquad \dot{V} := \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_n}\dot{x}_n + \frac{\partial V}{\partial \dot{x}_n}\ddot{x}_n = -w(x_n, \dot{x}_n) \qquad (10)$$

in $D$, where $w(x_n, \dot{x}_n)$ is a positive definite function in $D$. Any controller that causes the dynamics of the entire controlled nominal system to satisfy Equation (10), for a given candidate Lyapunov function that satisfies Equation (9), ensures that the controlled nominal system has an asymptotically stable equilibrium point at $x_n = 0$, $\dot{x}_n = 0$ (Khalil, 2002; Lefschetz, 1977; Perko, 1996; Sontag, 1998; Vidyasagar, 1993; Zubov, 1997).

Our first aim is to design the distributed controllers $Q_c^{(i)}$, by considering a user-prescribed candidate Lyapunov function $V$ – a function that satisfies only relation (9) above – and a user-prescribed positive definite function $w$. Since we are interested in localised control, we further assume that the candidate Lyapunov function $V$ is obtained as the sum of $p$ prescribed local candidate Lyapunov functions, $V^{(i)}$, $i = 1, 2, \ldots, p$, for each of the $p$ subsystems that depend only on the locally available states so that

$$V(x_n, \dot{x}_n, t) = \sum_{i=1}^{p} V^{(i)}\big(x_n^{(i)}, \dot{x}_n^{(i)}, t\big). \qquad (11)$$

Similarly, the function $w$ is obtained as the sum of $n$ local positive definite functions $w^{(i)}$, $i = 1, 2, \ldots, p$, one for each of the $n$ nominal subsystems, so that

$$w(x_n, \dot{x}_n) = \sum_{i=1}^{p} w^{(i)}\big(x_n^{(i)}, \dot{x}_n^{(i)}\big). \qquad (12)$$

Our second aim is to require each of the distributed controllers, $Q_c^{(i)}$, to minimise user-prescribed cost functions $J^{(i)}(x_n^{(i)}, \dot{x}_n^{(i)}, t)$, $i = 1, 2, \ldots, p$, at each instant of time $t$. The cost function for the $i$th controller is assumed to be of the form

$$J^{(i)}\big(x_n^{(i)}, \dot{x}_n^{(i)}, t\big) = Q_c^{(i)^T} N^{(i)}\big(x_n^{(i)}, \dot{x}_n^{(i)}, t\big) Q_c^{(i)}, \qquad (13)$$

where $N^{(i)}(x_n^{(i)}, \dot{x}_n^{(i)}, t)$, $i = 1, 2, \ldots, p$, are again user-prescribed positive definite matrices. From here on, we will not explicitly show the arguments of the various quantities unless required for clarity.

#### 2.2.1 *Derivation of the 'nominal' control force on the ith nominal subsystem*

We begin by considering the $i$th nominal subsystem. To ensure stability of the proposed control, we shall require the time derivative of our candidate Lyapunov functions $V^{(i)}$ along the trajectories of the solution of the controlled nominal subsystem to be negative. To do this, let us enforce the following $p$ constraints on the nominal system described by Equation (6):

$$\begin{aligned} \dot{V}^{(i)} &:= \frac{\partial V^{(i)}}{\partial t} + \frac{\partial V^{(i)}}{\partial x_n^{(i)}}\dot{x}_n^{(i)} + \frac{\partial V^{(i)}}{\partial \dot{x}_n^{(i)}}\ddot{x}_n^{(i)} \\ &= -w^{(i)}\big(x_n^{(i)}, \dot{x}_n^{(i)}\big), i = 1, 2, \ldots, p. \end{aligned} \qquad (14)$$

Denoting

$$A^{(i)} := \frac{\partial V^{(i)}}{\partial \dot{x}_n^{(i)}}, \qquad (15)$$

Equation (14) can be rearranged as

$$A^{(i)}\ddot{x}_n^{(i)} = -w^{(i)} - \frac{\partial V^{(i)}}{\partial x_n^{(i)}}\dot{x}_n^{(i)} - \frac{\partial V^{(i)}}{\partial t}, i = 1, 2, \ldots, p, \tag{16}$$

which upon the use of Equation (8) can be rewritten as

$$A^{(i)}M^{(i)^{-1}}Q_c^{(i)} = -w^{(i)} - \frac{\partial V^{(i)}}{\partial x_n^{(i)}}\dot{x}_n^{(i)} - \frac{\partial V^{(i)}}{\partial t} - A^{(i)}M^{(i)^{-1}}\tilde{F}_n^{(i)} := b^{(i)}, i = 1, 2, \ldots, p. \tag{17}$$

Equation (17) can be expressed more compactly as

$$B^{(i)}Q_c^{(i)} = b^{(i)}, i = 1, 2, \ldots, p, \tag{18}$$

where

$$B^{(i)} = A^{(i)}M^{(i)^{-1}}. \tag{19}$$

By defining the vector

$$z^{(i)} = N^{(i)^{1/2}}Q_c^{(i)} \tag{20}$$

the cost function given in Equation (13) becomes

$$J^{(i)} = z^{(i)^T}z^{(i)}. \tag{21}$$

Observing the form of the cost function in Equation (21), let us rewrite Equation (18) as

$$G^{(i)}z^{(i)} := B^{(i)}N^{(i)^{-1/2}}N^{(i)^{1/2}}Q_c^{(i)} = b^{(i)}, \tag{22}$$

where

$$G^{(i)} := B^{(i)}N^{(i)^{-1/2}}. \tag{23}$$

Thus, we desire $z^{(i)}$ that satisfies Equation (22), *and* at the same time minimises the cost function given in Equation (21). This is obtained as (Udwadia, 2003; Udwadia, 2008; Udwadia & Kalaba, 1992)

$$z^{(i)} = G^{(i)^+}b^{(i)}, \tag{24}$$

where the '+' in the superscript indicates the Moore–Penrose inverse. Then, by Equation (20), the required control force that satisfies the constraint in Equation (14) *and* minimises the cost function given in Equation (13) is obtained as

$$Q_c^{(i)} = N^{(i)^{-1/2}}G^{(i)^+}b^{(i)}. \tag{25}$$

Observing that $G^{(i)}$ is a row vector, we can further simplify Equation (25) to yield

$$\begin{aligned} Q_c^{(i)}(t) &= N^{(i)^{-1/2}}\frac{G^{(i)^T}}{G^{(i)}G^{(i)^T}}b^{(i)} \\ &= N^{(i)^{-1/2}}\frac{\left(B^{(i)}N^{(i)^{-1/2}}\right)^T}{\left(B^{(i)}N^{(i)^{-1/2}}\right)\left(B^{(i)}N^{(i)^{-1/2}}\right)^T}b^{(i)} \\ &= N^{(i)^{-1}}\frac{B^{(i)^T}}{B^{(i)}N^{(i)^{-1}}B^{(i)^T}}b^{(i)}. \end{aligned} \tag{26}$$

**Result:** The nominal control forces $Q_c^{(i)}$, $i = 1, 2, \ldots, p$, obtained in Equation (25) ensure that the entire controlled nominal system represented by Equation (8) has an asymptotically stable equilibrium point at $x_n = 0$ and $\dot{x}_n = 0$.

**Proof:** Using the positive definite candidate Lyapunov function given in Equation (11), its time derivative along the trajectories of the solution of Equation (8) is given by

$$\dot{V} = \sum_{i=1}^{p}\dot{V}^{(i)} = -\sum_{i=1}^{p}w^{(i)} = -w. \tag{27}$$

Thus, the controlled nominal system of Equation (8) has an asymptotically stable equilibrium point at $x_n = 0$, $\dot{x}_n = 0$. □

For this control scheme to work, we need Equation (18) to be consistent. In Appendix 1, we have provided a class of positive definite functions $V^{(i)}$'s and corresponding $w^{(i)}$'s for which this is true.

### 2.3 *Controlled actual system*

Since the controller of an actual subsystem does not have the knowledge of the entire global state vectors, it cannot accurately obtain the external force $F^{(i)}$ in Equation (1). Our nominal system adduces the force $\tilde{F}_n^{(i)}$ solely based on locally available information, namely $x_n^{(i)}(t)$. This disregard of non-local information regarding the force $F^{(i)}$ could make the actual system unstable, and this in fact is the crux of the problem of decentralised control.

To ensure the stability of controlled actual system, we add an additional compensating controller. This controller utilises the concept of generalised sliding surfaces to ensure that the controlled actual system tracks the solution trajectories of the nominal system within pre-specified error bounds, thus ensuring its stability. As we will see shortly, this controller needs a bound on the difference between the force acting on the actual subsystem $F^{(i)}$ and the approximate force $\tilde{F}^{(i)}$ acting on the nominal subsystem, to ensure that the controlled actual system can adequately track the trajectories of controlled nominal system.

The equation of motion of the controlled actual subsystem in the presence of a compensating controller $Q_u^{(i)}$ is then

$$M^{(i)}\ddot{x}^{(i)} = F^{(i)}(x, \dot{x}, t) + Q_c^{(i)}(t) + Q_u^{(i)}(x^{(i)}, \dot{x}^{(i)}, t), \quad i = 1, 2, \ldots, p. \tag{28}$$

To obtain the additional controller $Q_u^{(i)}$, we first define the tracking error for a typical subsystem (difference between the state of the controlled actual subsystem and the controlled nominal subsystem),

$$e^{(i)} = x^{(i)} - x_n^{(i)}, i = 1, 2, \ldots, p. \tag{29}$$

The nominal subsystem and the controlled actual subsystem are given the same initial conditions, i.e. $e^{(i)}(0) = \dot{e}^{(i)}(0) = 0, i = 1, \ldots, p$.

Let us define a sliding surface for the subsystem as

$$s^{(i)} = L^{(i)}e^{(i)} + \dot{e}^{(i)}, i = 1, 2, \ldots, p, \tag{30}$$

where the $L^{(i)}$s are positive scalars. We denote $e := [e^{(1)}, e^{(2)}, ..., e^{(p)}]^T$ and $s := [s^{(1)}, s^{(2)}, ..., s^{(p)}]^T$, where $e^{(i)}, s^{(i)} \in R^{p_i}$. If we could restrict the dynamics of the controlled actual subsystem to be on the sliding surface $s = 0$, it would slide along this surface to the asymptotic equilibrium point $e = 0$ and $\dot{e} = 0$. Thus, the controlled actual system would track the trajectories of controlled nominal system. But, to restrict the system to the sliding surface, we need discontinuous control forces. Instead, we provide a continuous control force that restricts the system to a region enclosing the sliding surface which can be made as close to the sliding surface as we desire. Let us denote this region by $\Omega_\varepsilon^{(i)}$.

To ensure that the controlled actual subsystem is restricted to a region ($\Omega_\varepsilon^{(i)}$) enclosing the sliding surface, we apply an additional compensating control force, which is explicitly given as (see Appendix 2)

$$Q_u^{(i)} = M^{(i)}\ddot{u}^{(i)} = -M^{(i)}\left(L^{(i)}\dot{e}^{(i)} + \gamma^{(i)} f^{(i)}\left(s^{(i)}\right)\right), \quad i = 1, 2, \ldots, p. \tag{31}$$

In the above equation, $\gamma^{(i)}$ is a positive constant chosen such that $\gamma^{(i)} > p_i \left\| M^{(i)^{-1}} \right\| \left\| F^{(i)} - \tilde{F}_n^{(i)} \right\|, \forall t > 0$, where $\|\cdot\|$ represents the infinity norm. The parameter $\gamma^{(i)}$ determines the maximum control acceleration provided by the additional compensating controller so that the controlled actual subsystem's state always stays inside a user-prescribed region, $\Omega_\varepsilon^{(i)}$, around the state of the nominal subsystem (see Equation (B15) in Appendix 2). We point out that since the actual subsystem and the nominal subsystem have the same initial conditions, they start out inside the region $\Omega_\varepsilon^{(i)}$, and the compensating controller then ensures that they always stay inside it.

The function $f^{(i)}(s^{(i)})$ is a vector-valued function, whose $j$th component is defined as

$$f_j^{(i)}\left(s^{(i)}\right) = g_\varepsilon^{(i)}\left(s_j^{(i)}/\varepsilon\right), \tag{32}$$

where $s_j^{(i)}$ is the $j$th component of $s^{(i)}$, $g_\varepsilon^{(i)}$ is an odd, continuous, monotonically increasing function such that $g_\varepsilon^{(i)}\left(s_j^{(i)}/\varepsilon\right) > 1$ if $s^{(i)} \notin \Omega_\varepsilon^{(i)}$. We explicitly point out the dependence of $g$ on $\varepsilon$ by displaying its subscript $\varepsilon$.

The controller thus requires an estimate of the quantity $\gamma^{(i)}$ over the time horizon over which the control is applied. Providing an overestimate of this quantity, however, has a small influence on the magnitude of the control force $Q_u^{(i)}$ (see Example 2). Later in this section we will show how to obtain an estimate of $\gamma^{(i)}$.

This control force ensures that the controlled actual subsystem is restricted to a region (which could be made as close to the surface $s^{(i)} = 0$ as we desire) around the sliding surface. The proof is given in Appendix 2; we also show that the asymptotic bound on the error in tracking the displacements of the nominal subsystem is given by

$$\lim_{t\to\infty} \left\| e^{(i)}(t) \right\| \le \frac{\rho_\varepsilon^{(i)}}{L^{(i)}}, \tag{33}$$

where

$$\rho_\varepsilon^{(i)} := \varepsilon g_\varepsilon^{(i)^{-1}}(1). \tag{34}$$

Similarly, the bound on the error in tracking the velocities is

$$\lim_{t\to\infty} \left\| \dot{e}^{(i)}(t) \right\| \le 2\rho_\varepsilon^{(i)}. \tag{35}$$

It should be noted that we can choose functions $g_\varepsilon^{(i)}$, to satisfy user-prescribed error bounds on tracking errors. This is demonstrated in Example 1 in the following section.

Now, we go back to the problem of estimating $\gamma^{(i)}$. Since $\dot{V}^{(i)}$ is negative throughout the duration of control for the nominal system, we must have $V_L^{(i)}(x_n^{(i)}, \dot{x}_n^{(i)}) \le V^{(i)}(x_n^{(i)}, \dot{x}_n^{(i)}, t) < V^{(i)}\left(x_n^{(i)}(0), \dot{x}_n^{(i)}(0), 0\right) \le V_U^{(i)}\left(x_n^{(i)}(0), \dot{x}_n^{(i)}(0)\right) := V_0^{(i)}$. This implies that $x_n^{(i)}, \dot{x}_n^{(i)}$ at any time $t$, will lie in a local domain $D_0^{(i)}$ which is defined as

$$D_0^{(i)} := \left\{\left(x_n^{(i)}, \dot{x}_n^{(i)}\right) \middle| V_L^{(i)}\left(x_n^{(i)}, \dot{x}_n^{(i)}\right) < V_0^{(i)}\right\}. \tag{36}$$

Thus, the global state of the nominal system is restricted to the domain

$$D_0 := D_0^{(1)} \times D_0^{(2)} \times \cdots \times D_0^{(p)}. \tag{37}$$

Since we assume that the norms of the tracking errors in displacement and velocity $\|x^{(i)} - x_n^{(i)}\|$,

$\|\dot{x}^{(i)} - \dot{x}_n^{(i)}\|$ are very small, we can obtain an approximate estimate for $\gamma^{(i)}(t)$ initially using the supremum of $\|M^{(i)^{-1}}\|\|F^{(i)} - \tilde{F}_n^{(i)}\|$ over this domain as

$$\gamma^{(i)} \geq p_i \sup_{\left(x_n^{(i)}, \dot{x}_n^{(i)}\right) \in D_0, t \geq 0} \left\{ \|M^{(i)^{-1}}\| \left\| F^{(i)}(x_n, \dot{x}_n, t) - \tilde{F}_n^{(i)}\left(x_n^{(i)}, \dot{x}_n^{(i)}, t\right) \right\| \right\}. \quad (38)$$

## 3. Numerical examples

In this section we illustrate the efficacy of the control method developed by applying it to two very different multi-degrees-of-freedom, non-autonomous systems. The motivation for the first system comes from the area of astronautics while the motivation for the second comes from the areas of civil and mechanical engineering.

The first example illustrates a spacecraft system in which the mass matrix of the system changes during a flight manoeuvre, so that elements of the mass matrix gradually reduce in time as fuel is spent during the manoeuvre. Thus, the mass matrix is a function of time. The system is subjected to external forces from sources such as reaction wheels, solar pressure, etc., which we have taken to be sinusoidal for purposes of illustration. The model consists of a three-degree-of-freedom nonlinear system that is coupled to a two-degree-of-freedom system, so that the entire system has dimension 10 in $R^n$. This example is used to illustrate the more general situation that could arise when the mass matrices of the subsystems become functions of time, as can happen when they are modelled using Lagrange's equations with generalised coordinates.

In the second example, we consider two 'chain' structural subsystems, which arise in various applications such as when modelling two building structures standing side by side. One of the subsystems has five degrees of freedom (a five-storey building structure), the other has four. Three of the corresponding masses (floor levels) of the two structural subsystems are connected with one another ('bridges' between them), which are modelled by nonlinear springs and damping elements. The dimension of the differential equation system in $R^n$ in this example is 18. More generally, the example could apply to various mechanical systems where decentralised vibration control is the objective.

**Example 1:** We consider a non-autonomous, nonlinear mechanical system consisting of the two mutually coupled subsystems described by the equations

$$\begin{aligned} M^{(1)}\ddot{x}^{(1)} &= -k^{(1)}x^{(1)} - u^{(1)}(x^{(1)}) + v^{(1)}(x, \dot{x}, t) \\ &\quad + h^{(1)}(t) := F^{(1)}(x, \dot{x}, t) \\ M^{(2)}\ddot{x}^{(2)} &= -k^{(2)}x^{(2)} - u^{(2)}(x^{(2)}) + v^{(2)}(x, \dot{x}, t) \\ &\quad + h^{(2)}(t) := F^{(2)}(x, \dot{x}, t). \end{aligned} \quad (39)$$

In the above equation

$$x^{(1)} = \left[x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right]^T \in R^3, \; x^{(2)} = \left[x_1^{(2)}, x_2^{(2)}\right]^T \in R^2, \quad (40)$$

$$M^{(1)} = \text{diag}\left(m_1^{(1)}, m_2^{(1)}, m_3^{(1)}\right), \; M^{(2)} = \text{diag}\left(m_1^{(2)}, m_2^{(2)}\right), \quad (41)$$

where

$$\begin{aligned} m_1^{(1)} &= \frac{t+3}{2(t+1)}, \; m_2^{(1)} = \frac{t+3}{t+2}, \; m_3^{(1)} = \frac{3(t+1)}{4(2t+1)}, \\ m_1^{(2)} &= \frac{t+4}{3(t+2)}, \; m_2^{(2)} = \frac{t+1}{5t+3}, \\ v^{(1)}(x, \dot{x}, t) &= x_1^{(2)}x_2^{(2)} \begin{bmatrix} \dot{x}_1^{(1)} \\ \dot{x}_2^{(1)} \\ \dot{x}_3^{(1)} \end{bmatrix}, \\ v^{(2)}(x, \dot{x}, t) &= \begin{bmatrix} x_2^{(1)}x_3^{(1)}\dot{x}_1^{(2)} \\ x_3^{(1)}x_1^{(1)}\dot{x}_2^{(2)} \end{bmatrix}, \end{aligned} \quad (42)$$

$$\begin{aligned} u^{(1)}(x^{(1)}) &= \begin{bmatrix} \left(x_1^{(1)} - x_2^{(1)}\right)^3 \\ \left(x_2^{(1)} - x_3^{(1)}\right)^3 \\ \left(x_3^{(1)} - x_1^{(1)}\right)^3 \end{bmatrix}, \\ u^{(2)}\left(x^{(2)}\right) &= \begin{bmatrix} \left(x_1^{(2)} - x_2^{(2)}\right)^3 \\ \left(x_2^{(2)} - x_1^{(2)}\right)^3 \end{bmatrix}, \end{aligned} \quad (43)$$

$$\begin{aligned} h^{(1)}(t) &= \left[10\sin(t-1), 7\sin(0.5t+3), 10\sin(0.75t+1)\right]^T \text{ and} \\ h^{(2)}(t) &= \left[9\cos(t), 12\cos(t+2)\right]^T, \end{aligned} \quad (44)$$

and $k^{(1)}$ and $k^{(2)}$ are symmetric, semi-positive definite stiffness matrices of the same dimension as $x^{(1)}$, $x^{(2)}$, respectively, given by $k^{(1)} = \begin{bmatrix} 100 & -100 & 0 \\ -100 & 150 & -50 \\ 0 & -50 & 100 \end{bmatrix}$ and $k^{(2)} = \begin{bmatrix} 150 & -50 \\ -50 & 100 \end{bmatrix}$.

For this choice of parameters *both* the subsystems described in Equation (39) are *unstable*.

The force on the first nominal subsystem is obtained by substituting $x_n^{(2)} = \dot{x}_n^{(2)} = 0$ in the expression for $F^{(1)}$ in Equation (39) as

$$\tilde{F}^{(1)} := -k^{(1)}x_n^{(1)} - u^{(1)}\left(x_n^{(1)}\right) + h^{(1)}(t). \quad (45)$$

Similarly, the force on the second nominal subsystem is obtained by substituting $x_n^{(1)} = \dot{x}_n^{(1)} = 0$ in the expression for $F^{(2)}$ as

$$\tilde{F}^{(2)} := -k^{(2)}x_n^{(2)} - u^{(2)}\left(x_n^{(2)}\right) + h^{(2)}(t). \quad (46)$$

The equations of motion of the nominal system are

$$\begin{aligned} M^{(1)}\ddot{x}_n^{(1)} &= -k^{(1)}x_n^{(1)} - u^{(1)}\big(x_n^{(1)}\big) + h^{(1)}(t) \\ &:= \tilde{F}_n^{(1)}\big(x_n^{(1)}, \dot{x}_n^{(1)}, t\big) \\ M^{(2)}\ddot{x}_n^{(2)} &= -k^{(2)}x_n^{(2)} - u^{(2)}\big(x_n^{(2)}\big) + h^{(2)}(t) \\ &:= \tilde{F}_n^{(2)}\big(x_n^{(2)}, \dot{x}_n^{(2)}, t\big) \end{aligned} \tag{47}$$

and they depend only on the local states. We next compute the control forces $Q_c^{(1)}$ and $Q_c^{(2)}$ using Equation (25). For that, let us choose the required parameters as $N^{(1)} = M^{(1)^{-1}}$ and $N^{(2)} = M^{(2)^{-1}}$. Let us choose the positive definite functions $V^{(1)}$ and $V^{(2)}$ as

$$\begin{aligned} V^{(1)} &= \frac{1}{2}a_1^{(1)}x_n^{(1)^T}x_n^{(1)} + \frac{1}{2}a_2^{(1)}\dot{x}_n^{(1)^T}\dot{x}_n^{(1)} + a_{12}^{(1)}x_n^{(1)^T}\dot{x}_n^{(1)} \\ V^{(2)} &= \frac{1}{2}a_1^{(2)}x_n^{(2)^T}x_n^{(2)} + \frac{1}{2}a_2^{(2)}\dot{x}_n^{(2)^T}\dot{x}_n^{(2)} + a_{12}^{(2)}x_n^{(2)^T}\dot{x}_n^{(2)} \end{aligned} \tag{48}$$

and $w^{(1)}$and $w^{(2)}$ as

$$w^{(1)} = \alpha^{(1)}V^{(1)}, w^{(2)} = \alpha^{(2)}V^{(2)}, \tag{49}$$

where $a_1^{(1)} = 1,\ a_2^{(1)} = 8,\ a_{12}^{(1)} = 1,\ \alpha^{(1)} = \frac{1}{4},\ a_1^{(2)} = 1,$ $a_2^{(2)} = 4$, $a_{12}^{(2)} = 1$, $\alpha^{(2)} = \frac{1}{2}$. These values cause Equation (18) to be consistent (see Appendix 1).

With this choice of positive definite functions, we obtain

$$A^{(1)} = a_2^{(1)}\dot{x}_n^{(1)^T} + a_{12}^{(1)}x_n^{(1)^T}, A^{(2)} = a_2^{(2)}\dot{x}_n^{(2)^T} + a_{12}^{(2)}x_n^{(2)^T} \tag{50}$$

and the explicit control forces from Equation (26) are given by

$$\begin{aligned} Q_c^{(1)} &= \frac{A^{(1)^T}}{A^{(1)}M^{(1)^{-1}}A^{(1)^T}}\big(-\alpha^{(1)}V^{(1)} - a_1^{(1)}x^{(1)^T}\dot{x}^{(1)} \\ &\quad - a_{12}^{(1)}\dot{x}^{(1)^T}\dot{x}^{(1)} - A^{(1)}M^{(1)^{-1}}\tilde{F}_n^{(1)}\big) \\ Q_c^{(2)} &= \frac{A^{(2)^T}}{A^{(2)}M^{(2)^{-1}}A^{(2)^T}}\big(-\alpha^{(2)}V^{(2)} - a_1^{(2)}x^{(2)^T}\dot{x}^{(2)} \\ &\quad - a_{12}^{(2)}\dot{x}^{(2)^T}\dot{x}^{(2)} - A^{(2)}M^{(2)^{-1}}\tilde{F}_n^{(2)}\big). \end{aligned} \tag{51}$$

Thus, the equation of motion of the controlled nominal system is

$$\begin{aligned} M^{(1)}\ddot{x}^{(1)} &= \tilde{F}_n^{(1)}\big(x_n^{(1)}, \dot{x}_n^{(1)}, t\big) + Q_c^{(1)}, \\ M^{(2)}\ddot{x}^{(2)} &= \tilde{F}_n^{(2)}\big(x_n^{(2)}, \dot{x}_n^{(2)}, t\big) + Q_c^{(2)}. \end{aligned} \tag{52}$$

We can define the tracking errors between the controlled nominal system and the controlled actual system as in Equation (29) by

$$e^{(1)} = x^{(1)} - x_n^{(1)} \quad \text{and} \quad e^{(2)} = x^{(2)} - x_n^{(2)}. \tag{53}$$

Similarly, we define the tracking errors in velocities as

$$\dot{e}^{(1)} = \dot{x}^{(1)} - \dot{x}_n^{(1)} \quad \text{and} \quad \dot{e}^{(2)} = \dot{x}^{(2)} - \dot{x}_n^{(2)}. \tag{54}$$

We choose the following parameters for the additional compensating controllers in Equation (31): $L^{(1)} = L^{(2)} = 10$, $\gamma^{(1)} = \gamma^{(2)} = 5000$.

We note that the computations require estimates of $\gamma^{(1)}$ and $\gamma^{(2)}$; however, the additional control forces $Q_u^{(i)}$ are relatively insensitive to the values of these estimates, as long as these values exceed $p_i \left\| M^{(i)^{-1}} \right\| \left\| F^{(i)} - \tilde{F}_n^{(i)} \right\|$. That is, using overestimates of $\gamma^{(1)}$ and $\gamma^{(2)}$ does not significantly affect the magnitudes of the additional control forces to be applied.

The sliding surfaces are defined as

$$s^{(1)} = L^{(1)}e^{(1)} + \dot{e}^{(1)} \quad \text{and} \quad s^{(2)} = L^{(2)}e^{(2)} + \dot{e}^{(2)}. \tag{55}$$

Here, $s^{(1)} \in R^3$ and $s^{(2)} \in R^2$. We define the functions $f^{(1)}\left(s^{(1)}\right)$, $f^{(2)}\left(s^{(2)}\right)$ in Equation (31) as

$$\begin{aligned} f_j^{(1)}\left(s_j^{(1)}\right) &= g_\varepsilon^{(1)}\left(s_j^{(1)}/\varepsilon\right) := \left(s_j^{(1)}/\varepsilon\right)^3, \ j = 1, 2, 3 \\ f_j^{(2)}\left(s_j^{(2)}\right) &= g_\varepsilon^{(2)}\left(s_j^{(2)}/\varepsilon\right) := \left(s_j^{(2)}/\varepsilon\right)^3, \ j = 1, 2, \end{aligned} \tag{56}$$

where subscript $j$ represents the $j$th component of a vector. The parameter $\varepsilon$ is a small number that can be chosen depending on how closely we want to track the nominal system. Observing the bound in Equations (33)–(35), the bound on the tracking error for this particular choice of $g_\varepsilon^{(i)}$, $i = 1, 2$ is

$$\lim_{t\to\infty} \left\| e^{(i)} \right\| \le \frac{\varepsilon}{L^{(i)}}, i = 1, 2, \tag{57}$$

$$\lim_{t\to\infty} \left\| \dot{e}^{(i)} \right\| \le 2\varepsilon, i = 1, 2. \tag{58}$$

Thus if the user provides a desired error bound, the additional compensating controller can be designed by choosing appropriate values for $\varepsilon$. For this example, we choose $\varepsilon$ to be $1 \times 10^{-4}$. Substituting this value of $\varepsilon$ and $L^{(1)} = L^{(2)} = 10$ in Equation (33), the bound on the tracking error in displacements is $1 \times 10^{-5}$. Similarly, using Equation (35), the bound on the tracking error in velocities can be obtained as $2 \times 10^{-4}$.

With the above-defined quantities, the explicit expressions for the additional compensating control forces on each

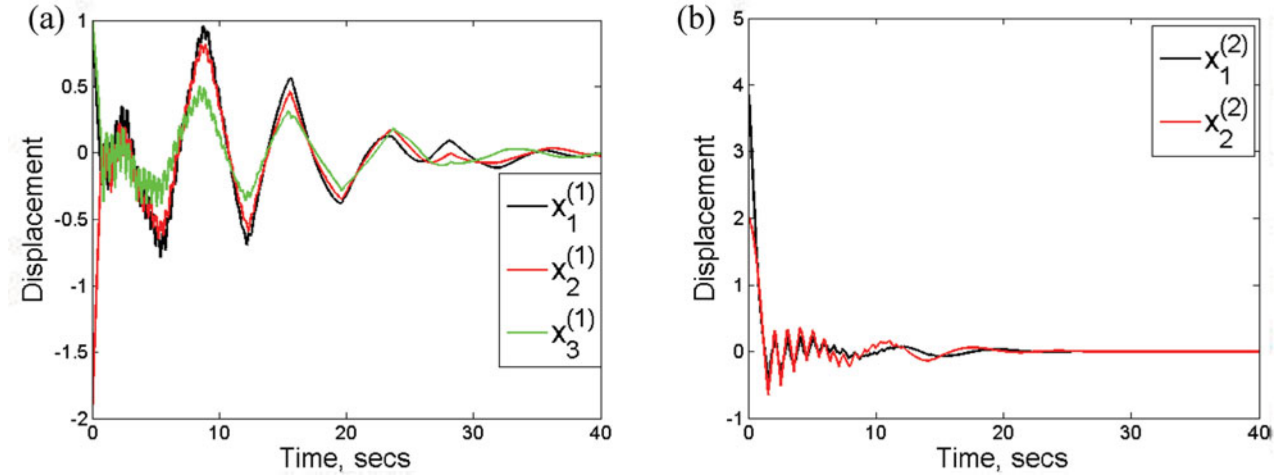


Figure 1. (a) Displacement history of controlled actual subsystem 1. (b) Displacement history of controlled actual subsystem 2.

subsystem are given by

$$Q_u^{(1)} = M^{(1)}\ddot{u}^{(1)} = -M^{(1)}\left(L^{(1)}\dot{e}^{(1)} + \gamma^{(1)} f^{(1)}\left(s^{(1)}\right)\right)$$
$$Q_u^{(2)} = M^{(2)}\ddot{u}^{(2)} = -M^{(2)}\left(L^{(2)}\dot{e}^{(2)} + \gamma^{(2)} f^{(2)}\left(s^{(2)}\right)\right). \tag{59}$$

The equations of motion for the controlled actual system are then

$$M^{(1)}\ddot{x}^{(1)} = F^{(1)}(x, \dot{x}, t) + Q_c^{(1)}(t) + Q_u^{(1)}(x^{(1)}, \dot{x}^{(1)}, t)$$
$$M^{(2)}\ddot{x}^{(2)} = F^{(2)}(x, \dot{x}, t) + Q_c^{(2)}(t) + Q_u^{(2)}(x^{(2)}, \dot{x}^{(2)}, t). \tag{60}$$

We use the ODE15s numerical solver in the MATLAB environment to perform numerical integration of Equations (60) and (52) using a relative error tolerance of $10^{-8}$ and an absolute error tolerance of $10^{-12}$. Figure 1(a) shows the displacement response of the controlled actual subsystem 1 as a function of time.

The displacement history of the second subsystem as a function of time is plotted and shown in Figure 1(b).

Figure 2(a) shows the decentralised control force to be applied (as a function of time) calculated for the nominal system and compares it against the additional compensating control force shown in Figure 2(b). For brevity, we show only the forces on the first mass of the two subsystems ($m_1^{(1)}$ and $m_1^{(2)}$). The additional compensating control force is seen to be quite small when compared with the nominal control forces (less than 10% of nominal control forces).

The tracking errors in displacement and velocity between the nominal and the actual subsystems given in

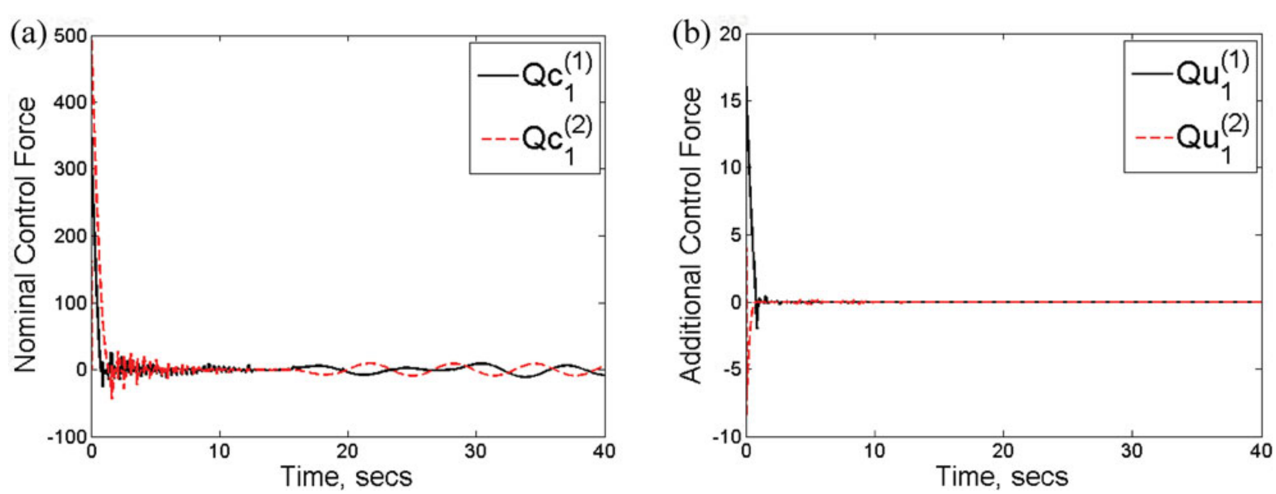


Figure 2. (a) Control force on the first mass of the subsystems 1and 2 computed from response of nominal system (first components of $Q_c^{(1)}$, $Q_c^{(2)}$). (b) Additional compensating control force on the first mass of the subsystems 1 and 2 (first components of vectors $Q_u^{(1)}$ and $Q_u^{(2)}$) with $\varepsilon = 10^{-4}$.

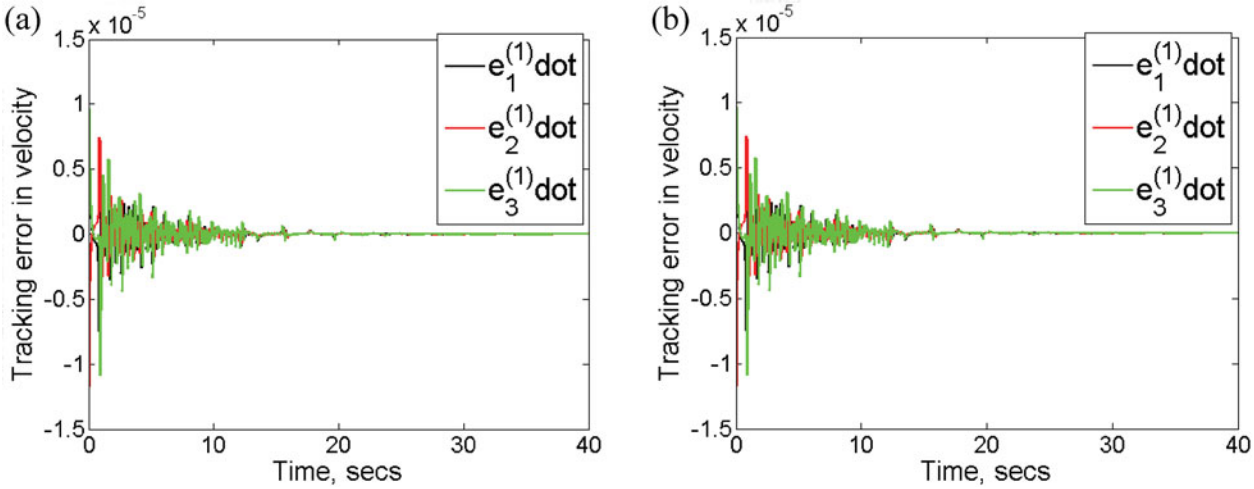


Figure 3. (a) Tracking errors in displacement, $e^{(1)}(t)$, for subsystem 1 with $\varepsilon = 10^{-4}$. (b) Tracking errors in velocity, $\dot{e}^{(1)}(t)$, for subsystem 1 with $\varepsilon = 10^{-4}$.

Equations (53) and (54) are shown in Figures 3 and 4. These are smaller than the error bounds given in Equations (33) and (35), which are $1 \times 10^{-5}$ and $2 \times 10^{-4}$, respectively, in our particular case.

Figure 5 (a) shows the variation of Lyapunov functions given in Equation (48) with time. Figure 5 (b) shows the error in satisfying the constraints imposed on the nominal system given in Equation (14).

To demonstrate how to choose appropriate functions $g_\varepsilon^{(i)}$ to satisfy a pre-specified tracking error tolerance, let us assume that we want our displacement tracking error for each subsystem, $e^{(i)}$, to be less than $1 \times 10^{-7}$. Observing Equation (57), we can choose an appropriate value for $\varepsilon$ as

$$\varepsilon = L^{(i)} \times 1 \times 10^{-7} = 10 \times 10^{-7} = 1 \times 10^{-6}. \qquad (61)$$

The corresponding velocity tracking error, $\dot{e}^{(i)}$, is then $2 \times 10^{-6}$ (see Equation (58)). Thus, we use this value for $\varepsilon$, keeping all other parameters the same, and show the results below. The displacements of the two subsystems look quite similar to the ones shown for the case when $\varepsilon = 1 \times 10^{-4}$, and are not shown for brevity. Figure 6 shows, as before, the additional compensating control forces on the first mass of subsystems 1 and 2 as a function of time, for the case when $\varepsilon = 1 \times 10^{-6}$. It can be seen from Figure 6 that the magnitude of additional control force does not change much with the parameter $\varepsilon$ (see Figure 2(b) for comparison).

Figure 7 shows the tracking errors for subsystem 1. As seen, the tracking errors in displacement are less than $1 \times 10^{-7}$, and tracking errors in velocity are less than $2 \times 10^{-6}$, as expected. For subsystem 2, the tracking errors in displacement and velocity were of the order of $10^{-9}$ and

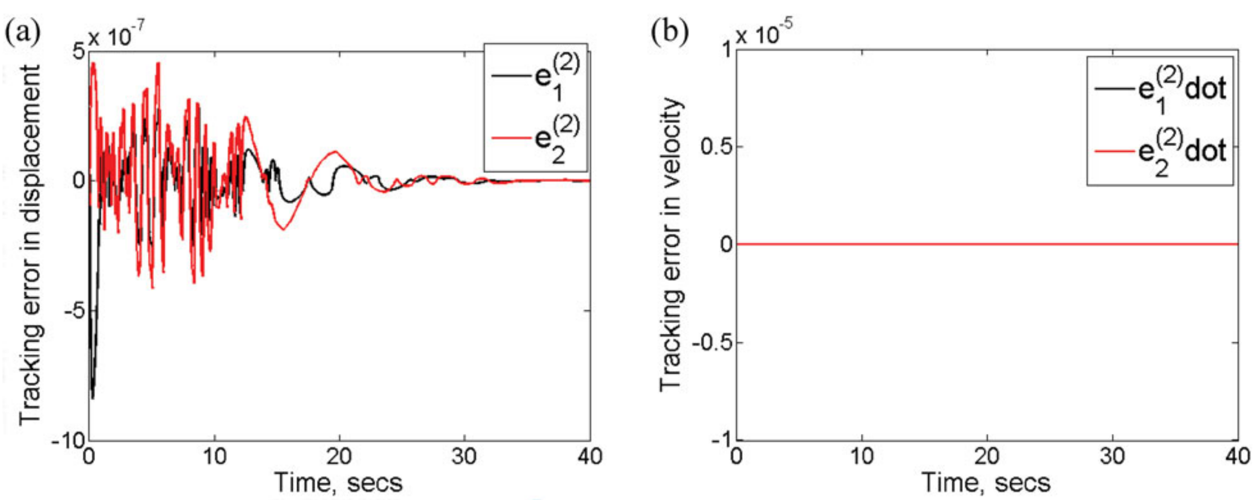


Figure 4. (a) Tracking errors in displacement, $e^{(2)}(t)$, for subsystem 2 with $\varepsilon = 10^{-4}$. (b) Tracking errors in velocity, $\dot{e}^{(2)}(t)$, for subsystem 2 with $\varepsilon = 10^{-4}$.

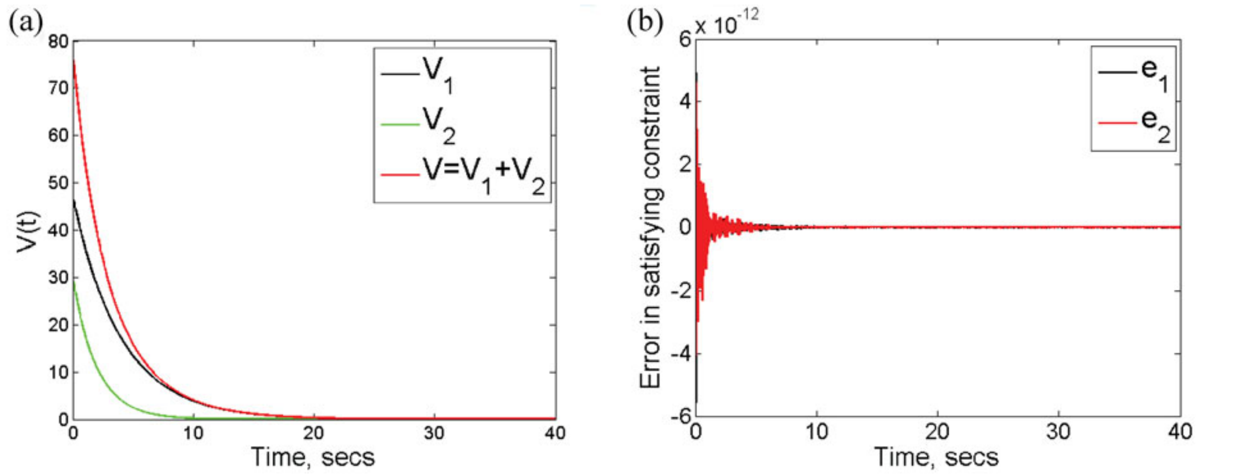


Figure 5. (a) Variation of Lyapunov functions with time. (b) Error in satisfying constraint on the nominal system as a function of time.

$10^{-8}$, respectively, which are much below the pre-specified values (see Equation (61)).

**Example 2:** As our second example we consider a forced nonlinear mechanical system consisting of two 'chain-type' subsystems described by the set of equations,

$$\begin{aligned} M^{(1)}\ddot{x}^{(1)} &= -K^{(1)}x^{(1)} - C^{(1)}\dot{x}^{(1)} - K_{\text{nl}}^{(1)}(x,\dot{x},t) \\ &\quad - C_{\text{nl}}^{(1)}(x,\dot{x},t) + h^{(1)}(t) := F^{(1)}(x,\dot{x},t) \\ M^{(2)}\ddot{x}^{(2)} &= -K^{(2)}x^{(2)} - C^{(2)}\dot{x}^{(2)} - K_{\text{nl}}^{(2)}(x,\dot{x},t) \\ &\quad - C_{\text{nl}}^{(2)}(x,\dot{x},t) + h^{(2)}(t) := F^{(2)}(x,\dot{x},t). \end{aligned} \quad (62)$$

In the above equations,

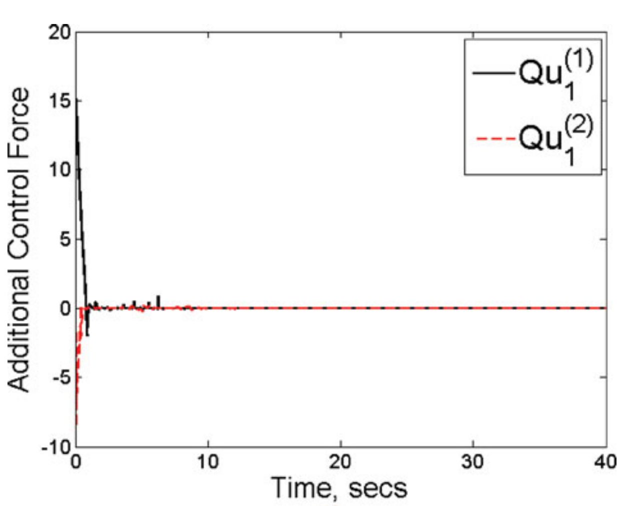


Figure 6. Additional compensating control force on the first mass of subsystems 1 and 2, with $\varepsilon = 10^{-6}$.

$$x^{(1)} = \left[x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}\right]^T \in R^4 \text{ and}$$
$$x^{(2)} = \left[x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}, x_5^{(2)}\right]^T \in R^5 \quad (63)$$

$$M^{(1)} = \text{diag}\left(m_1^{(1)}, m_2^{(1)}, m_3^{(1)}, m_4^{(1)}\right) \text{ and}$$
$$M^{(2)} = \text{diag}\left(m_1^{(2)}, m_2^{(2)}, m_3^{(2)}, m_4^{(2)}, m_5^{(2)}\right), \quad (64)$$

where $m_1^{(1)} = 1.5, m_2^{(1)} = 1.2, m_3^{(1)} = 2, m_4^{(1)} = 3, m_1^{(2)} = 2, m_2^{(2)} = 1.8, m_3^{(2)} = 3, m_4^{(2)} = 1.5, m_5^{(2)} = 2.25,$

$$K^{(1)} = \begin{bmatrix} 2k & -k & 0 & 0 \\ -k & 2k & -k & 0 \\ 0 & -k & 2k & -k \\ 0 & 0 & -k & k \end{bmatrix} \text{ and}$$

$$K^{(2)} = \begin{bmatrix} 2k & -k & & & \\ -k & 2k & -k & & \\ & -k & 2k & -k & \\ & & -k & 2k & -k \\ & & & -k & k \end{bmatrix}, \quad (65)$$

$$C^{(1)} = \sigma^{(1)}M^{(1)} + \mu^{(1)}K^{(1)} \quad \text{and}$$
$$C^{(2)} = \sigma^{(2)}M^{(2)} + \mu^{(2)}K^{(2)}, \quad (66)$$

$$K_{\text{nl}}^{(1)} = \begin{bmatrix} k_{\text{nl}}^1\left(x_1^{(1)} - x_1^{(2)}\right)^3 - k_{\text{nl}}^2\left(x_1^{(1)} - x_1^{(2)}\right)^5 \\ k_{\text{nl}}^1\left(x_2^{(1)} - x_2^{(2)}\right)^3 - k_{\text{nl}}^2\left(x_2^{(1)} - x_2^{(2)}\right)^5 \\ k_{\text{nl}}^1\left(x_3^{(1)} - x_3^{(2)}\right)^3 - k_{\text{nl}}^2\left(x_3^{(1)} - x_3^{(2)}\right)^5 \\ 0 \end{bmatrix} \text{ and}$$

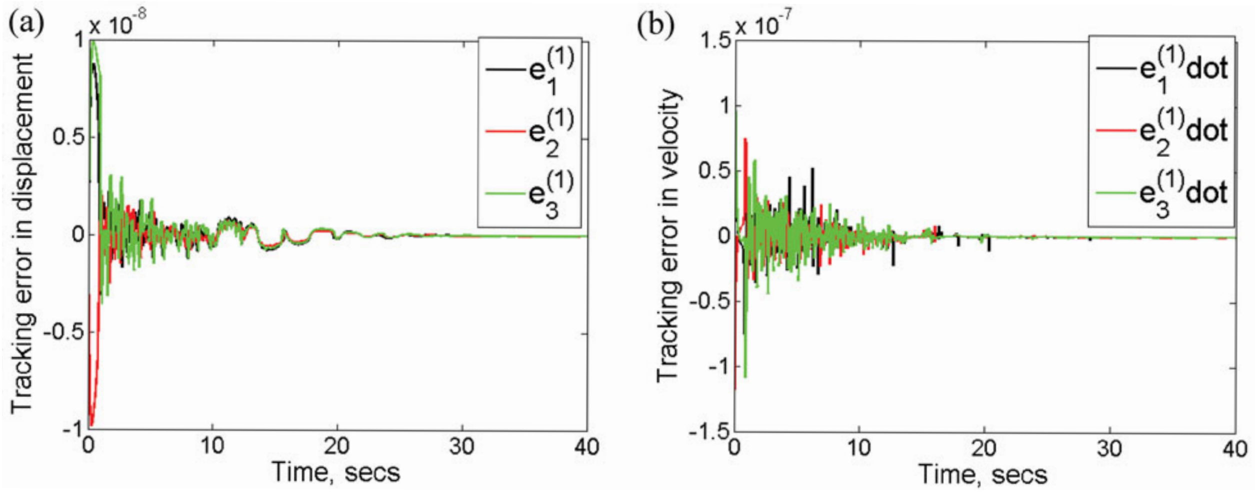


Figure 7. (a) Tracking error in displacement, $e^{(1)}(t)$, for subsystem 1 with $\varepsilon = 10^{-6}$. (b) Tracking error in velocity, $\dot{e}^{(1)}(t)$, for subsystem 1 with $\varepsilon = 10^{-6}$.

$$C_{\rm nl}^{(1)} = c_{\rm nl}\begin{bmatrix} \left(\dot{x}_1^{(1)} - \dot{x}_1^{(2)}\right)^3 \\ \left(\dot{x}_2^{(1)} - \dot{x}_2^{(2)}\right)^3 \\ \left(\dot{x}_3^{(1)} - \dot{x}_3^{(2)}\right)^3 \\ 0 \end{bmatrix}, \tag{67}$$

$$K_{\rm nl}^{(2)} = \begin{bmatrix} -K_{\rm nl}^{(1)} \\ 0 \end{bmatrix} \quad \text{and} \quad C_{\rm nl}^{(2)} = \begin{bmatrix} -C_{\rm nl}^{(1)} \\ 0 \end{bmatrix}, \tag{68}$$

$$h^{(1)}(t) = 12 \times \sin(1.5t) \times [1, 1, 1, 1]^T \quad \text{and}$$
$$h^{(2)}(t) = 10 \times \sin(2t) \times [1, 1, 1, 1, 1]^T. \tag{69}$$

We choose the various parameters as $k = 5000$, $\sigma^{(1)} = 0.2$, $\sigma^{(2)} = -0.9$, $\mu^{(1)} = 5 \times 10^{-4}$, $\mu^{(2)} = 7 \times 10^{-3}$, $k_{\rm nl}^1 = 2 \times 10^6$, $k_{\rm nl}^2 = 1 \times 10^7$ and $c_{\rm nl} = 2 \times 10^3$.

Following the first step in our approach, we will define a nominal system. The forces on the first nominal subsystem will be obtained by substituting $x_n^{(2)} = \dot{x}_n^{(2)} = 0$ in the expression for $F^{(1)}$ as

$$\tilde{F}^{(1)}\left(x_n^{(1)}, \dot{x}_n^{(1)}, t\right) := -K^{(1)}x_n^{(1)} - C^{(1)}\dot{x}_n^{(1)}$$
$$- \tilde{K}_{\rm nl}^{(1)}\left(x_n^{(1)}, \dot{x}_n^{(1)}, t\right) - \tilde{C}_{\rm nl}^{(1)}\left(x_n^{(1)}, \dot{x}_n^{(1)}, t\right) + h^{(1)}(t), \tag{70}$$

where

$$\tilde{K}_{\rm nl}^{(1)}\left(x_n^{(1)}, \dot{x}_n^{(1)}, t\right) = \begin{bmatrix} k_{\rm nl}^1 x_1^{(1)^3} - k_{\rm nl}^2 x_1^{(1)^5} \\ k_{\rm nl}^1 x_2^{(1)^3} - k_{\rm nl}^2 x_2^{(1)^5} \\ k_{\rm nl}^1 x_3^{(1)^3} - k_{\rm nl}^2 x_3^{(1)^5} \\ 0 \end{bmatrix} \text{and}$$

$$\tilde{C}_{\rm nl}^{(1)}\left(x_n^{(1)}, \dot{x}_n^{(1)}, t\right) = c_{\rm nl}\begin{bmatrix} \dot{x}_1^{(1)^3} \\ \dot{x}_2^{(1)^3} \\ \dot{x}_3^{(1)^3} \\ 0 \end{bmatrix}. \tag{71}$$

Similarly, the force on the second nominal subsystem, obtained by substituting $x_n^{(1)} = \dot{x}_n^{(1)} = 0$ in the expression for $F^{(2)}$ is

$$\tilde{F}_n^{(2)}\left(x_n^{(2)}, \dot{x}_n^{(2)}, t\right) := -K^{(2)}x_n^{(2)} - C^{(2)}\dot{x}_n^{(2)}$$
$$- \tilde{K}_{\rm nl}^{(2)}\left(x_n^{(2)}, \dot{x}_n^{(2)}, t\right) - \tilde{C}_{\rm nl}^{(2)}\left(x_n^{(2)}, \dot{x}_n^{(2)}, t\right) + h^{(2)}(t), \tag{72}$$

where

$$\tilde{K}_{\rm nl}^{(2)}\left(x_n^{(2)}, \dot{x}_n^{(2)}, t\right) = \begin{bmatrix} k_{\rm nl}^1 x_1^{(2)^3} - k_{\rm nl}^2 x_1^{(2)^5} \\ k_{\rm nl}^1 x_2^{(2)^3} - k_{\rm nl}^2 x_2^{(2)^5} \\ k_{\rm nl}^1 x_3^{(2)^3} - k_{\rm nl}^2 x_3^{(2)^5} \\ 0 \\ 0 \end{bmatrix} \quad \text{and}$$

$$\tilde{C}_{\rm nl}^{(2)}\left(x_n^{(2)}, \dot{x}_n^{(2)}, t\right) = c_{\rm nl}\begin{bmatrix} \dot{x}_1^{(2)^3} \\ \dot{x}_2^{(2)^3} \\ \dot{x}_3^{(2)^3} \\ 0 \\ 0 \end{bmatrix}. \tag{73}$$

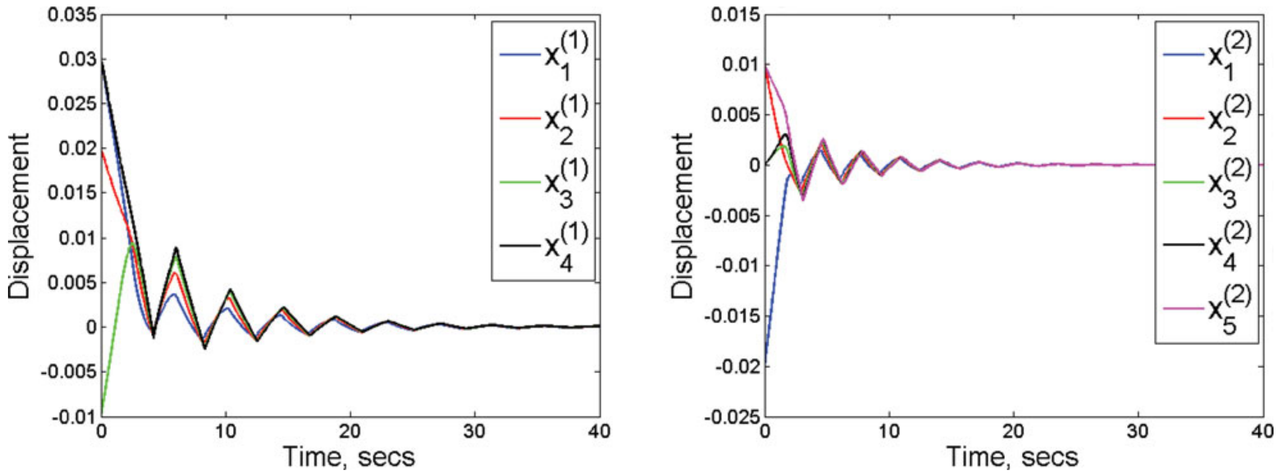


Figure 8. (a) Displacement history of controlled actual subsystem 1. (b) Displacement history of controlled actual subsystem 2.

With these forces thus defined, the equation of motion for the nominal system is

$$\begin{aligned} M^{(1)}\ddot{x}_n^{(1)} &= \tilde{F}_n^{(1)}\left(x_n^{(1)}, \dot{x}_n^{(1)}, t\right) \\ M^{(2)}\ddot{x}_n^{(2)} &= \tilde{F}_n^{(2)}\left(x_n^{(2)}, \dot{x}_n^{(2)}, t\right). \end{aligned} \tag{74}$$

We use the same positive definite functions used in the earlier example and given in Equations (48) and (49) with the parameters: $a_1^{(1)} = 1$, $a_2^{(1)} = 8$, $a_{12}^{(1)} = 1$, $\alpha^{(1)} = \frac{1}{4}$, $a_1^{(2)} = 1$, $a_2^{(2)} = 4$, $a_{12}^{(2)} = \frac{2}{3}$ and $\alpha^{(2)} = \frac{1}{3}$. These values are chosen, as before, in such a way as to ensure Equation (18) to be consistent. We again specify the cost functions by taking $N^{(1)} = M^{(1)^{-1}}$, $N^{(2)} = M^{(2)^{-1}}$ and calculate the explicit control forces using Equations (50) and (51). The equation for the controlled nominal system is given in Equation (52).

In the second step, we apply an additional compensating controller, obtained using Equations (53)–(59) using the parameters: $L^{(1)} = L^{(2)} = 10$, $\gamma^{(1)} = \gamma^{(2)} = 10^4$ and $\varepsilon = 10^{-4}$. The equation of motion for the controlled actual system is given in Equation (60). The equation is integrated using ODE15s in the MATLAB environment with a relative error tolerance of $10^{-8}$ and an absolute error tolerance of $10^{-12}$. Figure 8(a) shows the displacement response of the controlled actual subsystem 1 as a function of time. Figure 8(b) shows the corresponding plots for subsystem 2.

In Figure 9, the nominal control forces on the first mass of both the subsystems are contrasted with the corresponding additional compensating control forces. We observe from these plots that the magnitude of the compensating control force is comparable to the magnitude of the nominal control forces, due to the strong nonlinear coupling between the two subsystems.

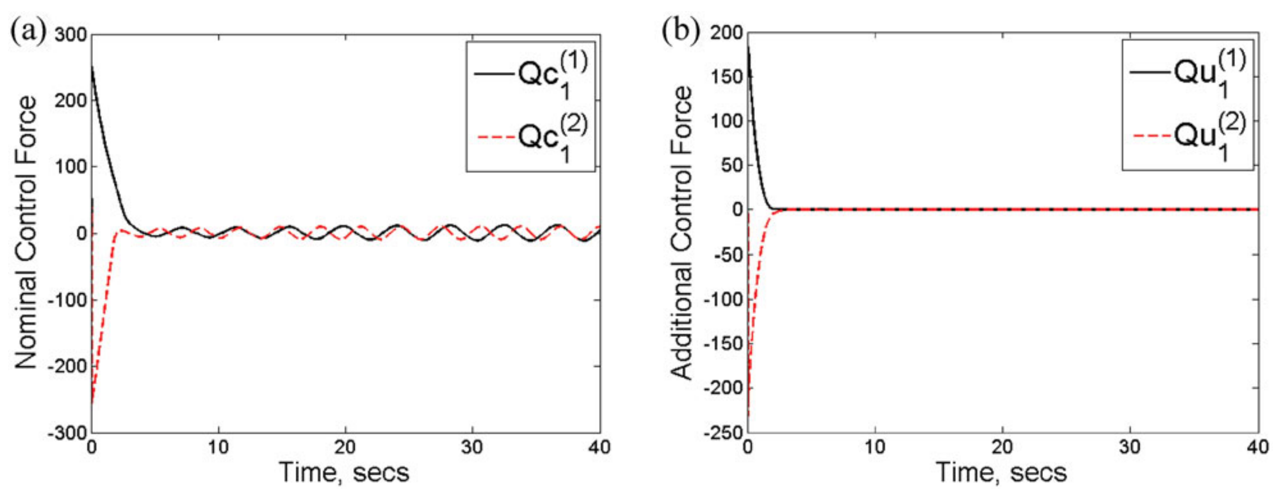


Figure 9. Comparison of nominal control forces ($Q_c$) and additional compensating control forces ($Q_u$) on the first mass of the subsystem 1 ($m_1^{(1)}$) and subsystem 2 ($m_1^{(2)}$).

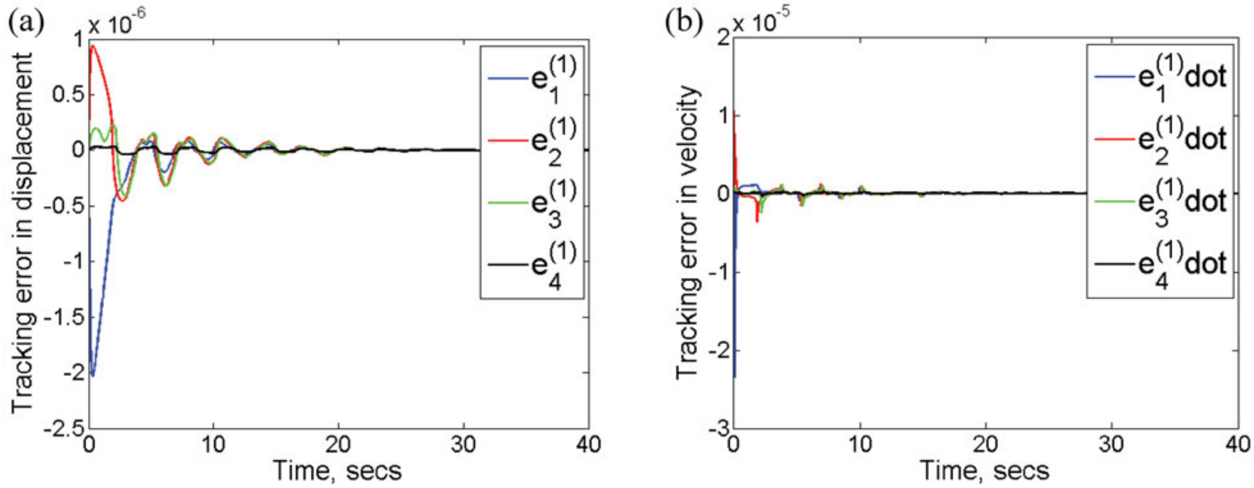


Figure 10. (a) Tracking error in displacement, $e^{(1)}(t)$, for the subsystem 1 with $\varepsilon = 10^{-4}$. (b) Tracking error in velocity, $\dot{e}^{(1)}(t)$, for the subsystem 1 with $\varepsilon = 10^{-4}$.

The errors in tracking the trajectories of nominal system are shown in Figures 10 and 11. We note that our control ensures that controlled actual system tracks the trajectories of nominal system quite well despite the strong interaction forces present between the two subsystems. The tracking errors in displacement and velocity are less than $1 \times 10^{-5}$ and $2 \times 10^{-4}$, respectively, as predicted by Equations (57) and (58) for our chosen value of $\varepsilon = 1 \times 10^{-4}$.

The values of the parameters $\gamma^{(i)}$ in our control method are chosen based on our estimates of the bounds on $p_i \| M^{(i)^{-1}} \| \| F^{(i)} - \tilde{F}_n^{(i)} \|$ (see Appendix 2); these values might therefore, at times, not be precisely known. To show that the magnitude of the additional compensating control force generated during the control effort is not very sensitive to the values of $\gamma^{(i)}$ chosen, we show the results for $\gamma^{(1)} = \gamma^{(2)} = 1 \times 10^3$ and $\gamma^{(1)} = \gamma^{(2)} = 5 \times 10^4$ keeping all other parameters the same as before. The additional control force on the first mass of the subsystem 1 ($m_1^{(1)}$) obtained by using these two choices for $\gamma^{(i)}$ is plotted in Figure 12. The plots seemingly overlap indicating that the additional compensating control forces are not significantly different. (The additional forces on the other masses show similar results.) Thus, even when our estimates of $\gamma^{(i)}$ vary by more than an order of magnitude, the additional control forces remain essentially unaltered, pointing out that only rough estimates of these quantities are needed for the control to be efficacious.

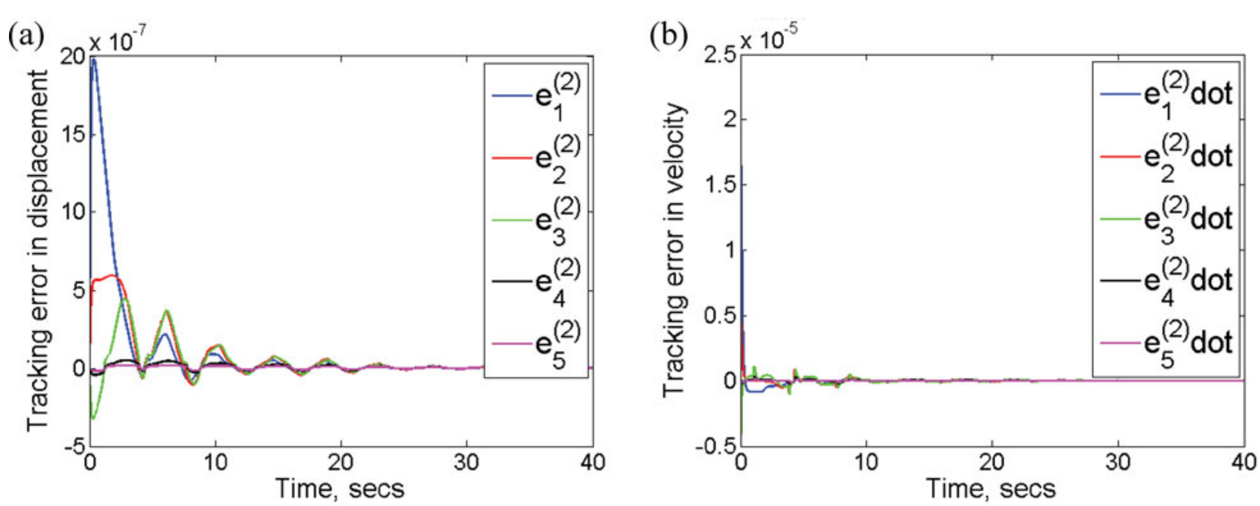


Figure 11. (a) Tracking error in displacement, $e^{(2)}(t)$, for subsystem 2 with $\varepsilon = 10^{-4}$. (b) Tracking error in velocity, $\dot{e}^{(2)}(t)$, for subsystem 2 with $\varepsilon = 10^{-4}$.

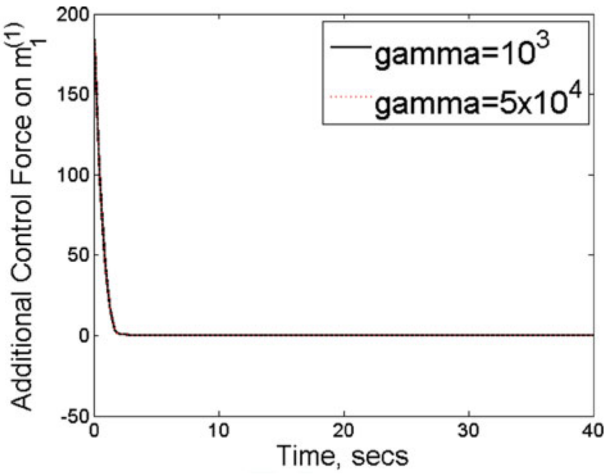


Figure 12. Additional compensating control force ($Q_u$) on $m_1^{(1)}$ for the case when $\gamma^{(1)} = \gamma^{(2)} = 1 \times 10^3$ and when $\gamma^{(1)} = \gamma^{(2)} = 5 \times 10^4$. Both the lines overlap showing low sensitivity to the values of $\gamma^{(i)}$.

## 4. Conclusions

We provide a simple approach to design decentralised controllers for distributed, non-autonomous, nonlinear mechanical systems. Our approach to the control of distributed systems is developed in two steps. First, we define a nominal system that can be constructed based on locally available information (measurements of displacements and velocities) from each subsystem. We compute the control forces to be applied to this nominal system in order to ensure that each nominal subsystem (1) has an asymptotically stable equilibrium point at the origin and (2) a user-prescribed norm of control force is minimised at each instant of time. In computing these control forces, we use user-prescribed positive definite functions $V_k$, $w_k$ defined over the domains of the local subsystems and minimise a suitable, user-prescribed norm of control force at each instant of time. No approximations/linearisations in modelling the dynamics and no a-priori assumptions regarding the structure of the control are made in this step.

In the second step of the approach, we add another additional compensating controller, which ensures that the controlled actual system tracks this nominal system as closely as desired, thereby ensuring that the controlled actual system always lies in a bounded region (that can be made as small as desired) around the origin. As demonstrated, the additional controller is designed based on a user-specified bound on the tracking error. The method requires an estimate on the bound of $p_i \|M^{(i)^{-1}}\| \|F^{(i)} - \tilde{F}_n^{(i)}\|$ over the time interval over which the control is executed. It is shown by example that the control method is not sensitive to this estimate. Two non-autonomous examples are provided of dimensions 10 and 18 in $R^n$. The first considers two unstable nonlinear subsystems, and the second considers a stable, highly coupled system. The examples illustrate the efficacy of the approach and its ease of implementation.

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## Appendix 1

For Equation (18) to be consistent, we want $b^{(i)}$ to be 0, whenever $B^{(i)}$ goes to 0, which happens only when $A^{(i)}=0$. From the definition of $b^{(i)}$ in Equation (17), we want $-w^{(i)}-\frac{\partial V^{(i)}}{\partial x_n^{(i)}}\dot{x}_n^{(i)}-\frac{\partial V^{(i)}}{\partial t}=0$ to hold whenever $A^{(i)}=0$. We give a class of functions $V^{(i)}$'s and corresponding $w^{(i)}$'s for which this is true.

Let us choose positive definite function $V^{(i)}$ to be of the form

$$V^{(i)}=\frac{1}{2}a_1^{(i)}x^{(i)^T}x^{(i)}+\frac{1}{2}a_2^{(i)}\dot{x}^{(i)^T}\dot{x}^{(i)}+a_{12}^{(i)}\dot{x}^{(i)^T}x^{(i)} \quad \text{(A1)}$$

and $w^{(i)}$ to be

$$w^{(i)}=\alpha^{(i)}V^{(i)},\alpha^{(i)}>0. \quad \text{(A2)}$$

Then, we have the following result.
**Result :** For the choice of $V^{(i)}$ in Equation (A1) and $w^{(i)}$ in Equation (A2), Equation (18) is consistent if

$$\alpha^{(i)}=\frac{2a_{12}^{(i)}}{a_2^{(i)}}>0. \quad \text{(A3)}$$

**Proof:** Since $V^{(i)}>0$ in Equation (A1) for any non-zero argument, we need

$$a_1^{(i)}>0, a_2^{(i)}>0, \quad \text{and} \quad a_{12}^{(i)}<\sqrt{a_1^{(i)}a_2^{(i)}}. \quad \text{(A4)}$$

Also, we have

$$A^{(i)}=\frac{\partial V^{(i)}}{\partial \dot{x}^{(i)}}=a_{12}^{(i)}x^{(i)^T}+a_2^{(i)}\dot{x}^{(i)^T} \quad \text{(A5)}$$

and

$$\begin{aligned}b^{(i)}&=-\alpha^{(i)}V^{(i)}-\frac{\partial V^{(i)}}{\partial x_n^{(i)}}\dot{x}_n^{(i)}\\&=-\alpha^{(i)}\left(\frac{1}{2}a_1^{(i)}x^{(i)^T}x^{(i)}+\frac{1}{2}a_2^{(i)}\dot{x}^{(i)^T}\dot{x}^{(i)}+a_{12}^{(i)}\dot{x}^{(i)^T}x^{(i)}\right)\\&\quad-\left(a_1^{(i)}x^{(i)^T}\dot{x}^{(i)}+a_{12}^{(i)}\dot{x}^{(i)^T}\dot{x}^{(i)}\right).\end{aligned} \quad \text{(A6)}$$

If $a_{12}^{(i)}=0$, then $\dot{x}^{(i)}=0\Rightarrow A^{(i)}=a_2^{(i)}\dot{x}^{(i)^T}=0$, but $b^{(i)}=-\frac{\alpha^{(i)}}{2}a_1^{(i)}x^{(i)^T}x^{(i)}\neq 0$. So, for consistency, we need $a_{12}^{(i)}\neq 0$.

Furthermore, from Equation (A5)

$$A^{(i)}=0\Rightarrow \dot{x}^{(i)}=-\frac{a_{12}^{(i)}}{a_2^{(i)}}x^{(i)}. \quad \text{(A7)}$$

Therefore, when $A^{(i)}=0$, by substituting the value of $\dot{x}^{(i)}$from Equation (A7) in Equation (A6) and simplifying, we get

$$b^{(i)}=-\left(\frac{1}{2}\alpha^{(i)}-\frac{a_{12}^{(i)}}{a_2^{(i)}}\right)\left(a_1^{(i)}-a_{12}^{(i)}\frac{a_{12}^{(i)}}{a_2^{(i)}}\right)\left(x^{(i)^T}x^{(i)}\right). \quad \text{(A8)}$$

From relation (A4), we know that the second and third terms on the right-hand side of Equation (A8) are non-negative. Therefore, to ensure that $b^{(i)}=0$, we require

$$\alpha^{(i)}=\frac{2a_{12}^{(i)}}{a_2^{(i)}}. \quad \text{(A9)}$$

□

## Appendix 2

**Result:** The compensating control force given in Equation (31) ensures that the dynamics of *i*th controlled actual subsystem asymptotically converge to a region $\Omega_\varepsilon^{(i)}$ which could be made as close to the sliding surface $s^{(i)}=0$ as we desire. The region $\Omega_\varepsilon^{(i)}$ is so defined that functions $f^{(i)}\left(s^{(i)}\right)$ defined in Equation (31) satisfy $\left\|f^{(i)}\left(s^{(i)}\right)\right\|\leq 1$ inside $\Omega_\varepsilon^{(i)}$. In what follows, we will use $\|\cdot\|$ to denote the infinity norm.

**Proof:** Let us first differentiate the tracking error in Equation (29) twice with respect to time to get

$$\ddot{e}^{(i)}=\ddot{x}^{(i)}-\ddot{x}_n^{(i)}. \quad \text{(B1)}$$

Using the equations of motion of controlled nominal system and controlled actual system given in Equations (8) and (28), respectively, we get

$$\ddot{e}^{(i)} = M^{(i)^{-1}} \left(F^{(i)}(x^{(i)}, \dot{x}^{(i)}, t) - \tilde{F}_n^{(i)}(x_n^{(i)}, \dot{x}_n^{(i)}, t)\right) + M^{(i)^{-1}} Q_u^{(i)}. \tag{B2}$$

Let us denote

$$\ddot{u}^{(i)} = M^{(i)^{-1}} Q_u^{(i)} \tag{B3}$$

and

$$\delta\ddot{x}^{(i)} = M^{(i)^{-1}} \left(F^{(i)}(x^{(i)}, \dot{x}^{(i)}, t) - \tilde{F}_n^{(i)}(x_n^{(i)}, \dot{x}_n^{(i)}, t)\right). \tag{B4}$$

We can obtain a bound on $\left\|\delta\ddot{x}^{(i)}\right\|$ as

$$\left\|\delta\ddot{x}^{(i)}\right\| \leq \left\|M^{(i)^{-1}}\right\| \left\|F^{(i)} - \tilde{F}_n^{(i)}\right\|. \tag{B5}$$

Consider the Lyapunov function with respect to sliding surface defined in Equation (30),

$$V_s^{(i)} = \frac{1}{2} s^{(i)^T} s^{(i)}. \tag{B6}$$

Differentiating Equation (30) once, we get

$$\dot{s}^{(i)} = L^{(i)}\dot{e}^{(i)} + \ddot{e}^{(i)}, \tag{B7}$$

where $L^{(i)}$ is a positive scalar. Differentiating Equation (B6) once and substituting in Equation (B7), we get

$$\dot{V}_s^{(i)} = s^{(i)^T}\dot{s}^{(i)} = s^{(i)^T}\left(L^{(i)}\dot{e}^{(i)} + \ddot{e}^{(i)}\right). \tag{B8}$$

Substituting Equation (B2) and using Equations (B3) and (B4), we obtain

$$\dot{V}_s^{(i)} = s^{(i)^T}\left(L^{(i)}\dot{e}^{(i)} + \delta\ddot{x}^{(i)} + \ddot{u}^{(i)}\right). \tag{B9}$$

Let us choose $\ddot{u}^{(i)}$ to be of the form

$$\ddot{u}^{(i)} = -L^{(i)}\dot{e}^{(i)} - \gamma^{(i)} f^{(i)}\left(s^{(i)}\right). \tag{B10}$$

As explained in Subsection 2.3, $f^{(i)}\left(s^{(i)}\right)$ is a vector-valued function, whose *j*th component is defined as

$$f_j^{(i)}\left(s^{(i)}\right) = g_\varepsilon^{(i)}\left(s_j^{(i)}/\varepsilon\right), \tag{B11}$$

where $s_j^{(i)}$ is the *j*th component of $s^{(i)}$, $g_\varepsilon^{(i)}$ is an odd, continuous, monotonically increasing function such that $g_\varepsilon^{(i)}(s_j^{(i)}/\varepsilon) > 1$ if $s^{(i)} \notin \Omega_\varepsilon^{(i)}$. Since we want to drive the system to be as close as desired to the sliding surface $s^{(i)} = 0$, we want $\dot{V}_s^{(i)}$ in Equation (B9) to be negative outside a small region $\Omega_\varepsilon^{(i)}$ around the sliding surface. This region $\Omega_\varepsilon^{(i)}$ is so defined that functions $f^{(i)}(s^{(i)})$ defined in Equation (B10) satisfy $\| f^{(i)}(s^{(i)})\| > 1$outside $\Omega_\varepsilon^{(i)}$. Substituting Equation (B10) in Equation (B9), we obtain

$$\dot{V}_s^{(i)} = s^{(i)^T}\left(\delta\ddot{x}^{(i)} - \gamma^{(i)} f^{(i)}\left(s^{(i)}\right)\right). \tag{B12}$$

Since $g_\varepsilon^{(i)}$ is an odd, monotonically increasing function

$$s^{(i)^T} f^{(i)}\left(s^{(i)}\right) \geq \left\|s^{(i)}\right\| \left\|f^{(i)}\left(s^{(i)}\right)\right\|. \tag{B13}$$

Also, noting that $\|s^{(i)^T}\| \leq p_i\|s^{(i)}\|$, we can bound $\dot{V}_s^{(i)}$ as

$$\dot{V}_s^{(i)} \leq \left\|s^{(i)}\right\| \left(p_i \left\|\delta\ddot{x}^{(i)}\right\| - \gamma^{(i)} \left\|f^{(i)}\left(s^{(i)}\right)\right\|\right). \tag{B14}$$

Since, outside the region $\Omega_\varepsilon^{(i)}$, $\left\|f^{(i)}\left(s^{(i)}\right)\right\| > 1$, we get

$$\dot{V}_s^{(i)} \leq \left\|s^{(i)}\right\| \left(p_i \left\|\delta\ddot{x}^{(i)}\right\| - \gamma^{(i)}\right). \tag{B15}$$

Thus, if we choose

$$\gamma^{(i)} > p_i \left\|M^{(i)^{-1}}\right\| \left\|F^{(i)} - \tilde{F}_n^{(i)}\right\| \geq p_i \left\|\delta\ddot{x}^{(i)}\right\|, \forall t > 0 \tag{B16}$$

we have $\dot{V}_s^{(i)} < 0$ outside $\Omega_\varepsilon^{(i)}$, and thus any trajectory $s^{(i)}(t)$, once inside, cannot escape outside $\Omega_\varepsilon^{(i)}$. Since the initial conditions for the nominal subsystem and the actual subsystem are the same, we have $s^{(i)}(0) = 0$, and therefore, every trajectory $s^{(i)}(t)$ starts inside the region $\Omega_\varepsilon^{(i)}$. Our choice of $\gamma^{(i)}$ in relation (B16) and the inequality in Equation (B15) then ensure that every trajectory stays inside $\Omega_\varepsilon^{(i)}$ for all future time.

Noting that $\| f^{(i)}(s^{(i)}\| > 1$ outside $\Omega_\varepsilon^{(i)}$, we have from Equation (B11),

$$\left\|s^{(i)}\right\| \leq \varepsilon g_\varepsilon^{(i)^{-1}}(1) := \rho_\varepsilon^{(i)}. \tag{B17}$$

Noting the definition of $s^{(i)}$ in Equation (30) and the asymptotic bound on $s^{(i)}$ in Equation (B17), the tracking error is asymptotically bounded as $\lim_{t\to\infty} \|e^{(i)}(t)\| \leq \frac{\rho_\varepsilon^{(i)}}{L^{(i)}}$ and the tracking error in velocity is bounded as $\lim_{t\to\infty} \left\|\dot{e}^{(i)}(t)\right\| \leq 2\rho_\varepsilon^{(i)}$.