

# Convergence of Iterative Methods for Nonclassically Damped Dynamic Systems

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## ABSTRACT

This paper deals with two computationally efficient iterative methods for determining the response of nonclassically damped dynamic systems. Rigorous analytical convergence results related to these iterative methods are provided. Sufficient conditions under which these two iterative schemes are convergent are derived. Three different kinds of damping matrices, namely, (i) strongly diagonally dominant, (ii) irreducible and weakly diagonally dominant, and (iii) symmetric and positive definite damping matrices, are considered. Asymptotic rates of convergence are discussed. Theoretical results are illustrated with numerical examples that show vastly improved rates of convergence when compared to earlier iterative schemes.

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## 1. INTRODUCTION

Nonclassically damped dynamic structural systems are modeled by the following linear second-order differential equations of motion.

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = a(t); x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0, t \in (t_0, T) \quad (1)$$

where the constant  $N \times N$  matrices  $M$ ,  $K$ , and  $C$  are the mass, the stiffness, and the damping matrices, respectively. The vectors  $x(t)$  and  $a(t)$  are  $N \times 1$  vectors of displacement and force, respectively. For most of the physical systems arising in the area of structural dynamics, the mass matrix  $M$  is real, symmetric, and positive definite, and the stiffness matrix  $K$  is real, symmetric, and positive semidefinite. Under these circumstances, we can find a transformation matrix  $\Phi$  that simultaneously diagonalizes  $M$  and  $K$ ; for this transformation to diagonalize  $C$  also, the matrix  $C$  has to be of a special form [1, 2]. In the literature, this kind of damping is referred to as classical damping or proportional damping. The response of classically damped systems is obtained by the modal superposition method.

Yet in practice, proportional damping is usually a rare occurrence rather than a common one. This is because most large-scale, real-life, dynamic systems are comprised of different subcomponents. Even if we were to ascribe a viscous damping character to each of these subcomponents, the final damping matrix  $C$ , constructed through, say, a finite element model for the whole system, would generally be of the nonproportional type. This would of course be more so true when these subcomponents themselves are comprised of widely differing materials, as is found, for example, in the area of soil-structure interaction, and in the area of aerospace structures (which are usually optimized for their weight).

We assume that  $C$  is a real general matrix. When the matrices  $M$ ,  $K$ , and  $C$  cannot be simultaneously diagonalized by a suitable matrix transformation, one is left with the following coupled set of second-order linear differential equations.

$$\ddot{z}(t) + F\dot{z}(t) + \Lambda z(t) = h(t); z(t_0) = z_0, \dot{z}(t_0) = \dot{z}_0, t \in (t_0, T) \quad (2)$$

where  $h(t) = \Phi^T a(t)$ . The matrix  $\Phi$  has columns which are the eigenvectors of the undamped system; the damping matrix  $F$ , in general, is now a full matrix, and the diagonal matrix  $\Lambda = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ , where  $\lambda_1, \lambda_2, \dots, \lambda_N$  are the eigenvalues of undamped system. Over the years, a considerable amount of research effort has been expended in the determination of the response of such MDOF systems whose damping is of non-classical type. The reader may refer to the extensive literature survey on this topic provided in Udwardia and Esfandiari [3], Shahruz and Langari [4], Shahruz and Packard [5], Felszeghy [6], and Claret and Venancio-Filho [7].

In this paper, we introduce two different sets of iterative schemes for determining the response of non-classically damped dynamic systems. They are superior to the previously proposed scheme [3] in that they are applicable to a much wider class of matrices  $F$ , and/or are computationally more

efficient. The range of applicability of both schemes has been significantly extended to include; (a) irreducible and weakly diagonally dominant  $F$  matrices, and (b) *all* symmetric and positive definite  $F$  matrices. Analytical results guaranteeing convergence of these iterative schemes are provided. The first set of schemes results in an uncoupled set of equations; it thus yields additional insights into the physics of the structural response. The second set of schemes, while not uncoupling the system, is, in general, computationally far superior to the first.

Section 2 of this paper introduces the basic underlying iterative approach for both sets of schemes. Section 3 provides the analytical results related to the convergence of these two sets of schemes. Section 4 provides convergence rates and error bounds. Section 5 contains some numerical examples to show the validity of the proposed methods. For the different cases covered in the paper, it is shown that the second set of schemes converges faster than the first. The examples considered have been chosen with some care, in the sense that these examples when handled by the usual uncoupling techniques used to date, have presented some measure of difficulty to previous investigators. Finally, Section 6 contains a discussion and comparison of these two sets of schemes. We also compare them with some of the previously proposed iterative methods.

## 2. ITERATIVE SCHEMES

We start from equation (2) by partitioning matrix  $F$  as:

$$F = \alpha D + A + B \quad (3)$$

where  $D = \text{Diag}(d_1, d_2, \dots, d_N)$  is the diagonal matrix obtained by taking the diagonal elements of matrix  $F$ , and the real parameter  $\alpha$  ( $\alpha \neq 0$ ) is as yet unspecified. Substituting this decomposition of the matrix  $F$  in equation (2), we get

$$\left. \begin{aligned} \ddot{z}(t) + (\alpha D + A)\dot{z}(t) + \Lambda z(t) &= h(t) - B\dot{z}(t) \\ z(t_0) = z_0, \dot{z}(t_0) &= \dot{z}_0, t \in (t_0, T) \end{aligned} \right\} \quad (4)$$

Our purpose is to generate a *cluster* of iterative schemes depending on: (1) the specific split-down of the matrix  $F$  (i.e., the matrices chosen to be  $A$  and  $B$ ) and (2) the value of the parameter  $\alpha$  chosen.

We first replace equation (4) by the following system:

$$\begin{aligned} \ddot{u}(t) + (\alpha D + A)\dot{u}(t) + \Lambda u(t) &= f(t); \\ u(t_0) = z_0, \dot{u}(t_0) &= \dot{z}_0, t \in (t_0, T) \end{aligned} \quad (5)$$

where the function  $f(t)$  is as yet unknown function. Let  $\delta(t) = z(t) - u(t)$  denote the error vector in the responses determined from equations (4) and (5). Subtracting equation (5) from equation (4) we get

$$\begin{aligned} \ddot{\delta}(t) + (\alpha D + A)\dot{\delta}(t) + \Lambda \delta(t) &= h(t) - B\dot{z}(t) - f(t) \\ \delta(t_0) = \dot{\delta}(t_0) &= 0, t \in (t_0, T) \end{aligned} \quad (6)$$

Since equation (6) is a set of second order linear differential equations with zero initial conditions, we can conclude that  $\delta(t) = 0, t \in (t_0, T)$  for all functions,  $h(t)$ , if and only if the right-hand side of equation (6) is zero, i.e.,

$$f(t) = h(t) - B\dot{z}(t) \quad (7)$$

This implies that the solution of equations (2), (4), and (5) will be identical; i.e.,  $z(t) = u(t)$ , for all  $t$ , if and only if  $f(t)$  is as defined in equation (7). The only difficulty involved is that the time derivative of the response  $z(t)$  is not known, and in fact, is obtained through the solution of equation (2), which is what we want to solve for, in the first place.

To overcome this problem, we consider the following iterative procedure which uses successive approximations for  $\dot{z}(t)$ . The scheme can be best described in the following equation form where the superscript  $n$  denotes quantities related to the  $n^{\text{th}}$  iteration.

$$\ddot{u}^{(n)}(t) + (\alpha D + A)\dot{u}^{(n)}(t) + \Lambda u^{(n)}(t) = \hat{f}^{(n-1)}(t), \quad t \in (t_0, T) \quad (8)$$

where

$$\hat{f}^{(n)}(t) = h(t) - B\dot{u}^{(n)}(t), \quad t \in (t_0, T) \quad (9)$$

and  $\dot{u}^{(0)}(t) = 0$ .

Different iterative schemes can now be generated from the general procedure outlined above by making different choices of the matrices,  $A$  and  $B$ , and the parameter  $\alpha$ . In the sequel we shall, in particular, concentrate on two specific sets of iterative schemes.

SCHEME I. The parameters that define this scheme are as follows:

- (1)  $A = 0$ ; and,
- (2)  $B = (1 - \alpha)D + P$ , where the matrix  $P$  contains the off-diagonal terms of the matrix  $F$  and has zeros along the diagonal.

We note that different values of the parameter  $\alpha$  will generate different iterative schemes, all belonging generically to Scheme I. Thus, the matrix  $F$  is split as

$$F = \alpha D + B := \alpha D + \{(1 - \alpha)D + P\} \quad (10)$$

SCHEME II. The parameters that define this scheme are as follows:

- (1)  $A = L$ , where  $L$  is the lower triangular part of the matrix  $F$ , and
- (2)  $B = (1 - \alpha)D + U$ , where  $U$  is the upper triangular part of the matrix  $F$ .

Again, different values of the parameter  $\alpha$  will generate different iterative schemes all generically belonging to Scheme II. In this set of schemes, the matrix  $F$  is split as

$$F = \alpha D + A + B := \alpha D + L + (1 - \alpha)D + U \quad (11)$$

The first set of iterative schemes gives rise to a set of uncoupled differential equations. Therefore, it may be possible to think of the response of the system represented by equation (2) as being separable into different “modes” provided that it is subjected to the pseudo-force  $f(t)$  rather than the actual forcing function  $h(t)$ . As pointed out in the work of Udvardia and Esfandiari [3], unlike in this scheme, past efforts for uncoupling equation (2) have concentrated mainly on diagonalizing the damping matrix  $F$  without making appropriate modifications to the forcing function,  $h(t)$ , on the right-hand side of the equation. Without such an adjustment, it is obvious that, in general, inaccurate responses will result for nonclassically damped systems.

We next investigate the conditions under which convergence to the exact response is guaranteed. In other words, conditions under which the error vector  $\delta^{(n)}(t) = z(t) - u^{(n)}(t) \rightarrow 0$  as  $n \rightarrow \infty$  will be investigated.

### 3. CONVERGENCE OF THE ITERATIVE SCHEMES

The algorithm that was explained in the previous section can be written in an equation form as

$$\begin{aligned} \ddot{u}^{(n)}(t) + (\alpha D + A)\dot{u}^{(n)}(t) + \Lambda u^{(n)}(t) &= h(t) - B\dot{u}^{(n-1)}(t) \\ u^{(n)}(t_0) &= z_0, \quad n = 1, 2, 3, \dots, \\ \dot{u}^{(n)}(t_0) &= \dot{z}_0, \quad n = 1, 2, 3, \dots \end{aligned} \quad (12)$$

Subtracting equation (12) from equation (4) and noting that  $\delta^{(n)} = z(t) - u^{(n)}(t)$ , we obtain the following error differential equation.

$$\begin{aligned} \ddot{\delta}^{(n)}(t) + (\alpha D + A)\dot{\delta}^{(n)}(t) + \Lambda \delta^{(n)}(t) &= -B\dot{\delta}^{(n-1)}(t) \\ \delta^{(n)}(t_0) = \dot{\delta}^{(n)}(t_0) &= 0, \quad n = 1, 2, \dots, \text{ and,} \\ \delta^{(0)}(t) &= z(t) - u^{(0)}(t), \quad t \in (t_0, T), \\ \dot{\delta}^{(0)}(t) &= \dot{z}(t) - \dot{u}^{(0)}(t), \quad t \in (t_0, T) \end{aligned} \quad (13)$$

The vector  $\delta^{(0)}(t)$  corresponds to the error in the initial iteration. Defining the Fourier transform as

$$\tilde{\delta}(\omega) = \int_{t_0}^{\infty} \delta(t) e^{-i\omega t} dt \quad (14)$$

we obtain from the first equality in the set of equation (13),

$$\left[ \omega(\alpha D + A) + i(\omega^2 I - \Lambda) \right] \tilde{\delta}^{(n)}(\omega) = -\omega B \tilde{\delta}^{(n-1)}(\omega) \quad (15)$$

where  $I$  is the identity matrix. When  $D$  is nonsingular, equation (15) yields the following recursion.

$$\tilde{\delta}^{(n)}(\omega) = [S(\omega)]^n \tilde{\delta}^{(0)}(\omega) \quad (16)$$

where

$$S(\omega) = \left[ \omega(\alpha D + A) + i\omega G \right]^{-1} [-\omega B] \quad (17)$$

and

$$\omega G = \text{Diag}\{\omega^2 - \lambda_1, \omega^2 - \lambda_2, \dots, \omega^2 - \lambda_N\} \quad (18)$$

We now specify the iteration matrix,  $S(\omega)$ , for the two sets of schemes described in Section 2. For the set of iterative methods described by Scheme I, noting equations (17) and (10), the iteration matrix  $S(\omega)$ , for different values of the parameter  $\alpha$  takes the form

$$S_{1;\alpha}(\omega) = -\omega[\omega\alpha D + i\omega G]^{-1}\{(1 - \alpha)D + P\} \quad (19)$$

For the set of iterative methods described by Scheme II, after substituting  $A = L$  and  $B = (1 - \alpha)D + U$  in equation (17), we get the following iteration matrix for various values of the parameter  $\mu$  ( $\mu = 1/\alpha$ ):

$$S_{2;\mu}(\omega) = [\omega(D + \mu L) + i\omega\mu G]^{-1}[-\omega\mu U + (1 - \mu)\omega D] \quad (20)$$

Having specified the iteration matrices (19) and (20) for these two sets of iterative schemes, we now present the following convergence results.

**THEOREM 3.1.**  $u^{(n)}(t) \rightarrow z(t)$  almost everywhere if and only if either:

- (i)  $[S(\omega)]^n \rightarrow 0$ , for all  $\omega$ , as  $n \rightarrow \infty$ , or equivalently,
- (ii) the spectral radius of  $S(\omega)$ , denoted by  $\rho_s(\omega)$ , is less than unity for all  $\omega$ .

**PROOF.**  $u^{(n)}(t) \rightarrow z(t)$  almost everywhere, as  $n \rightarrow \infty$  implies that  $\delta^{(n)}(\omega) \rightarrow 0$  for all  $\omega$ , and for arbitrary  $\delta^{(0)}(\omega)$ . Noting equation (16) the result follows. Also if  $[S(\omega)]^n \rightarrow 0$ , then  $\delta^{(n)}(\omega) \rightarrow 0$ . ■

**LEMMA 3.1.** If the matrix  $\Lambda = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$  is nonsingular and  $\mu \neq 0$  then

- (i)  $\rho_{S_{1;\alpha}}(0) = \rho_{S_{2;\mu}}(0) = 0$ , and,
- (ii)  $\rho_{S_{1;\alpha}}(\pm\infty) = \rho_{S_{2;\mu}}(\pm\infty) = 0$ .

**PROOF.** Using relation (19) and taking appropriate limits we see that  $\rho_{S_{1;\alpha}}(0) = 0$  and  $\rho_{S_{1;\alpha}}(\pm\infty) = 0$ . Similarly, from equation (20), after taking the appropriate limits the results follow.

We note from equation (16) that  $\tilde{\delta}^{(n)}(\omega = 0, \omega = \pm\infty) \rightarrow 0$  as  $n \rightarrow \infty$ . So far, we have shown that if  $\omega$  is equal to 0 or tends to  $\pm\infty$ , both sets of iterative schemes converge to the exact response results. We now obtain results for  $0 < |\omega| < \infty$ . ■

LEMMA 3.2. *The spectral radius of the matrix  $S_{1;\alpha}(\omega)$  has the following property.*

$$\begin{aligned} \rho_{S_{1;\alpha}}(\omega) &\leq \max_{\mathbf{v}_i} \left[ \frac{|\omega|}{|\phi_i|} \left\{ |(1-\alpha)d_i| + \sum_{j=1}^N |p_{ij}| \right\} \right] \\ &\leq \frac{|(1-\alpha)| + \pi}{|\alpha|} := \pi_1(\alpha) \end{aligned} \quad (21)$$

where

$$|\phi_i| = \sqrt{\alpha^2 \omega^2 d_i^2 + (\omega^2 - \lambda_i)^2}, \quad (22)$$

$$\pi = \max_{\mathbf{v}_i} \left[ \frac{\sum_{j=1}^N |p_{ij}|}{|d_i|} \right], \quad (23)$$

and  $p_{ij}$  is the  $(i, j)^{th}$  element of the matrix  $P$ .

PROOF. The matrix  $(\omega\alpha D + i\omega G)$  is diagonal, and the matrix  $D$  contains the diagonal entries of the matrix  $F$ . The matrix  $P$  contains the off-diagonal elements of  $F$  and has zeros along the diagonal. Hence, the  $(i, j)^{th}$  element of  $[S_{1;\alpha}(\omega)]$ ,

$$[S_{1;\alpha}(\omega)]_{ij} = \begin{cases} \frac{-\omega(1-\alpha)f_{ii}}{\alpha\omega f_{ii} + i(\omega^2 - \lambda_i)}, & i = j \\ \frac{-\omega f_{ij}}{\alpha\omega f_{ii} + i(\omega^2 - \lambda_i)}, & i \neq j \end{cases}, \quad 1 \leq i, j \leq N. \quad (24)$$



Using

$$\rho_{S_{1,\alpha}}(\omega) \leq \|(\alpha\omega D + i\omega G)^{-1}(-\omega B)\|_\infty = \|S_{1,\alpha}(\omega)\|_\infty, \quad (25)$$

the first inequality in equation (21) is obtained. Since for any row  $i$ , and all  $\omega$ ,  $|\phi_i| \geq |\alpha\omega d_i|$ , the result follows. ■

**THEOREM 3.2.**

(1) If  $F = (f_{ij})$ ,  $1 \leq i, j \leq N$ , is a strictly diagonally dominant matrix then Scheme I is convergent for all values of  $\alpha > (1 + \pi)/2$ , where  $\pi$  is defined in equation (23). Specifically, for  $\alpha = 1$ ,  $\pi_1(\alpha)$  is a minimum and then we have  $\rho_{S_{1,\alpha}}(\omega) \leq \pi_1 = \pi < 1$ .

(2) If  $F = (f_{ij})$ ,  $1 \leq i, j \leq N$ , is an irreducible matrix with weak diagonal dominance, then Scheme I is convergent for all values of  $\alpha > 1$ .

**PROOF.**

(1) Since  $F$  is strictly diagonally dominant,  $\pi$  as defined in equation (23) is less than unity. Noting relation (21), a sufficient condition for  $\rho_{S_{1,\alpha}}(\omega) < 1$ , for all  $\omega$ , is that

$$\pi_1(\alpha) := \frac{|(1 - \alpha)| + \pi}{|\alpha|} < 1. \quad (26)$$

When  $\alpha$  is nonpositive,  $\pi_1(\alpha)$  is greater than unity. Thus  $\alpha$  must be positive. Now, from the above equation, we have

$$\frac{1 - |\alpha| + \pi}{|\alpha|} \leq \frac{|(1 - \alpha)| + \pi}{|\alpha|} < 1, \quad (27)$$

which readily yields

$$\alpha > \frac{1 + \pi}{2}. \quad (28)$$

Recalling that  $\alpha$  is positive, the first result follows.

Furthermore, if  $\alpha$  is less than unity then

$$\pi_1(\alpha) := \frac{1 - \alpha + \pi}{\alpha}, \quad (29)$$

or

$$\pi_1(\alpha) := \frac{1 + \pi}{\alpha} - 1. \quad (30)$$

Differentiating equation (30) with respect to  $\alpha$ , we get

$$\frac{d\pi_1(\alpha)}{d\alpha} = -\frac{1 + \pi}{\alpha^2}. \quad (31)$$

This shows that the slope of the function  $\pi_1(\alpha)$  for values of  $\alpha$  less than unity is negative. Similarly, it can be proved that for  $\alpha = 1$  and  $\alpha > 1$ , the slope of the function  $\pi_1(\alpha)$  is 0 and positive, respectively. Therefore, the *minimum* of  $\pi_1(\alpha)$  is achieved when  $\alpha = 1$  and the value of this minimum (from equation (26)) is  $\pi$ . We note that this approximate optimum value of  $\alpha$  (i.e.,  $\alpha = 1$ ), corresponding to which the *bound* on the spectral radius is a minimum, is independent of frequency,  $\omega$ .

(2) For irreducible and weakly diagonally dominant matrices  $F$ ,  $\pi$  in equation (23) is unity [8]. Now, to satisfy relation (26)  $\alpha$  should be greater than unity. From relation (21), for all  $\alpha > 1$ ,  $\pi_1 = 1$  and  $\rho_{S_{1,\alpha}}(\omega) \leq 1$ . Again, from equation (24) we have for  $1 \leq i \leq N$ ,

$$\sum_{j=1}^N |[S_{1,\alpha}(\omega)]_{ij}| = \frac{\sum_{j \neq i}^N |\omega f_{ij}| + |(1 - \alpha)| |\omega f_{ii}|}{|\alpha \omega f_{ii} + i(\omega^2 - \lambda_i)|}, \quad (32)$$

or

$$\sum_{j=1}^N |[S_{1,\alpha}(\omega)]_{ij}| \leq \frac{\sum_{j \neq i}^N |\omega f_{ij}| + |(1 - \alpha)| |\omega f_{ii}|}{|\alpha| |\omega f_{ii}|}. \quad (33)$$

Noting that  $F$  is irreducible and weakly diagonally dominant, we get

$$\sum_{j=1}^N |[S_{1,\alpha}(\omega)]_{ij}| \leq \frac{|1 - \alpha| + 1}{|\alpha|}. \quad (34)$$

But  $\alpha > 1$ , therefore,

$$\sum_{j=1}^N \left| [S_{1;\alpha}(\omega)]_{ij} \right| \leq 1 \text{ for all } \omega \text{ and for all } 1 \leq i \leq N. \quad (35)$$

Denoting by  $|S_{1;\alpha}(\omega)|$  the nonnegative matrix whose entries are the moduli of the corresponding entries of the matrix  $S_{1;\alpha}(\omega)$ , we then have from equation (24) that for  $\alpha > 1$ ,  $|S_{1;\alpha}(\omega)|$  is irreducible because  $F$  is irreducible and hence  $\rho(|S_{1;\alpha}(\omega)|) < 1$  [9]. Since  $0 \leq |S_{1;\alpha}(\omega)| \leq |S_{1;\alpha}(\omega)|$ , we therefore get

$$\rho(S_{1;\alpha}(\omega)) \leq \rho(|S_{1;\alpha}(\omega)|) < 1. \quad (36)$$

Hence the result. ■

The application of these two results is shown in numerical Examples 1 and 2, respectively.

LEMMA 3.3. *The spectral radius of the matrix  $S_{2;\mu}(\omega)$  has the following property.*

$$\rho_{S_{2;\mu}}(\omega) \geq \frac{|1 - \mu|}{\left[ \prod_{j=1}^N (1 + \mu^2 g_{1j}^2) \right]^{1/2N}} \quad (37)$$

where  $g_{1j} = \left( \frac{\omega^2 - \lambda_j}{\omega d_j} \right)$  and  $d_j$  is the  $j^{\text{th}}$  diagonal element of the matrix  $F$ .

PROOF. We know that the product of the eigenvalues of  $S_{2;\mu}(\omega)$  is equal to  $\det[S_{2;\mu}(\omega)]$ . By equation (20) and assuming that  $D^{-1}$  exists, we have

$$S_{2;\mu}(\omega) = (I + \mu L_1 + i\mu G_1)^{-1} (-\mu U_1 + (1 - \mu)I) \quad (38)$$

where  $L_1 = D^{-1}L$ ,  $U_1 = D^{-1}U$ , and  $G_1 = D^{-1}G$ . And recalling that determinant of the inverse of a matrix is the inverse of the determinant of the matrix, we get

$$\det[S_{2;\mu}(\omega)] = \frac{(1 - \mu)^N}{\prod_{j=1}^N (1 + i\mu g_{1j})}. \quad (39)$$

Let  $\gamma_i$ ,  $i = 1, 2, \dots, N$  be the eigenvalues of the matrix  $S_{2,\mu}(\omega)$  then

$$|\gamma_1 \gamma_2 \cdots \gamma_N| = \left| \frac{(1 - \mu)^N}{\prod_{j=1}^N (1 + i\mu g_{1j})} \right|, \quad (40)$$

which yields

$$|\gamma_1| |\gamma_2| \cdots |\gamma_N| = \frac{|(1 - \mu)|^N}{\prod_{j=1}^N |(1 + i\mu g_{1j})|}. \quad (41)$$

Now it follows that

$$\max_{\forall i} |\gamma_i| \geq \frac{|1 - \mu|}{\prod_{j=1}^N |(1 + i\mu g_{1j})|^{1/N}}. \quad (42)$$

Therefore, from the definition of spectral radius, we get

$$\rho_{S_{2,\mu}}(\omega) \geq \frac{|1 - \mu|}{\prod_{j=1}^N |(1 + i\mu g_{1j})|^{1/N}} \quad (43)$$

and the lemma follows. Moreover, if Scheme II converges, then for all  $\omega$  we should have

$$\frac{|1 - \mu|}{\left[ \prod_{j=1}^N (1 + \mu^2 g_{1j}^2) \right]^{1/2N}} < 1. \quad (44)$$

Noting that  $\mu$  and  $g_{1j}$ ,  $j = 1, 2, \dots, N$  are real quantities, the above inequality will be satisfied for  $0 < \mu < 2$ . ■

**THEOREM 3.3.** *If  $F$  is an irreducible and weakly diagonally dominant matrix, then Scheme II is convergent for  $0 < \mu \leq 1$ ,  $0 < |\omega| < \infty$ , i.e.,  $\rho_{S_{2,\mu}}(\omega) < 1$ .*

PROOF. Let us assume the contrary, namely that the spectral radius of  $S_{2,\mu}(\omega)$  is greater than or equal to unity and  $0 < \mu \leq 1$ . Then for some eigenvalue  $\lambda$  of  $S_{2,\mu}(\omega)$ , we have  $|\lambda| \geq 1$  and

$$\det[S_{2,\mu}(\omega) - \lambda I] = \det Q = 0, \tag{45}$$

where,

$$Q = \frac{\lambda + \mu - 1}{\mu\lambda}D + iG + L + \frac{1}{\lambda}U. \tag{46}$$

Let  $\lambda = re^{i\theta}$ , where  $r$  and  $\theta$  are real. We have

$$\text{Real}\left(\frac{\lambda + \mu - 1}{\mu\lambda}\right) = \frac{1}{\mu} - \frac{\cos \theta}{r}\left(\frac{1}{\mu} - 1\right) \geq 1 \tag{47}$$

since  $r \geq 1$  and  $0 < \mu \leq 1$ . Now  $|\lambda| \geq 1$  and the diagonal matrix  $iG$  has all its elements as imaginary quantities; thus, the matrix  $Q$  is weakly diagonally dominant since  $F$  is weakly diagonally dominant. Also  $F$  is irreducible and from equation (46) and relation (47), so is  $Q$ . Hence, the determinant of  $Q$  cannot be zero. We therefore have a contradiction pointing out that our assumption is incorrect. Hence, the result. ■

Example 2 of Section 5 illustrates the result of this theorem.

**THEOREM 3.4.** *If  $F$  is strictly diagonally dominant matrix, then Scheme II is convergent for  $0 < \mu \leq 1$ ,  $0 < |\omega| < \infty$ , i.e.,  $\rho_{S_{2,\mu}}(\omega) < 1$ .*

PROOF. The proof follows exactly as in Theorem 3.3, and finally we say the following. If  $F = D + L + U$  is strongly diagonally dominant matrix, then from relations (46) and (47) it is evident that  $Q$  is also strongly diagonally dominant, and hence its determinant cannot be zero. We therefore have a contradiction pointing out that our assumption of  $|\lambda| \geq 1$ , is incorrect. Hence, the result. ■

The result of Theorem 3.4 has been applied to a 60 degree-of-freedom nonclassically damped system in Example 1 of Section 5.

So far, we have shown that when the damping matrix  $F$  is strongly diagonally dominant, and irreducible and weakly diagonally dominant, the iterative schemes are convergent. We next consider the convergence of these

two sets of schemes when the matrix  $F$  is symmetric and positive definite. We start with the following lemmas.

LEMMA 3.4. *When the matrix  $F$  is symmetric and positive definite, the spectral radius  $\rho_{S_{1,\alpha}}(\omega) \leq \rho_{(\alpha D)^{-1}N_1}$ , where  $N_1 = (1 - \alpha)D + P$ .*

PROOF. Let  $\lambda$  be the eigenvalue corresponding to the spectral radius of  $S_{1,\alpha}(\omega)$  and  $x$  be the corresponding eigenvector. Then writing the eigenvalue problem as

$$S_{1,\alpha}(\omega)x = -(\alpha D + iG)^{-1}N_1x = \lambda x, \quad (48)$$

or

$$-\lambda(\alpha D + iG)x = N_1x. \quad (49)$$

Premultiplying equation (49) by  $x^H$  and noting that  $|\lambda| = \rho_{S_{1,\alpha}}(\omega)$ , we get

$$\begin{aligned} \rho_{S_{1,\alpha}}(\omega) = |\lambda| &= \frac{|(N_1x, x)|}{|(\alpha Dx, x) + i(Gx, x)|} \leq \frac{|(N_1x, x)|}{|(\alpha Dx, x)|} \\ &\leq \max_{\forall y \neq 0} \left[ \frac{|(N_1y, y)|}{|(\alpha Dy, y)|} \right] = \rho_{(\alpha D)^{-1}N_1}. \end{aligned} \quad (50)$$

■

LEMMA 3.5.

$$\rho_{(\alpha D)^{-1}N_1} < 1 \Leftrightarrow \rho_{(\alpha D)^{-1}F} < 2.$$

PROOF. Let  $\lambda$  be any eigenvalue of  $(\alpha D)^{-1}N_1$  and  $y$  be the corresponding eigenvector. Then since  $F = N_1 + \alpha D$ , we get

$$(\alpha D)^{-1}Fy = (\lambda + 1)y, \quad (51)$$

so that  $(\lambda + 1)$  is an eigenvalue of  $(\alpha D)^{-1}F$ . The result now follows. ■

LEMMA 3.6. *When the matrix  $F$  is symmetric and positive definite, for*

$$\alpha > \frac{\lambda_{\max}(D^{-1}F)}{2}, \text{ the spectral radius } \rho_{(\alpha D)^{-1}N_1} < 1. \quad (52)$$

■

PROOF. Noting Lemma 3.5 and the fact that  $\rho_{(\alpha D)^{-1}F} = \rho_{D^{-1}F}/\alpha$ , the result follows. ■

LEMMA 3.7. *When the matrix  $F$  is symmetric and positive definite, the value of  $\alpha$  for which  $\rho_{(\alpha D)^{-1}N_1}$  is a minimum is given by*

$$\alpha_{opt} = \frac{\lambda_{\max}(D^{-1}F) + \lambda_{\min}(D^{-1}F)}{2}, \quad (53)$$

*and the value of this spectral radius, for this approximate optimum value of  $\alpha = \alpha_{opt}$ , is given by*

$$\kappa = \frac{\lambda_{\max}(D^{-1}F) - \lambda_{\min}(D^{-1}F)}{\lambda_{\max}(D^{-1}F) + \lambda_{\min}(D^{-1}F)} \quad (54)$$

PROOF. See [10]. Note that  $\kappa$  is always less than unity. ■

THEOREM 3.5. *When the matrix  $F$  is symmetric and positive definite, the set of methods described by Scheme I will always converge for  $\alpha > (\lambda_{\max}(D^{-1}F))/2$ . Specifically for  $\alpha = \alpha_{opt}$ , where  $\alpha_{opt}$  is defined in equation (53), we have*

$$\rho_{S_{1; \alpha_{opt}}}(\omega) \leq \kappa < 1. \quad (55)$$

PROOF. We note that

$$\begin{aligned} \rho_{S_{1,\alpha}}(\omega) &\leq \rho_{(\alpha D)^{-1}N_1}, \\ \rho_{(\alpha D)^{-1}N_1} | \forall \alpha > \frac{\lambda_{max}(D^{-1}F)}{2} &< 1, \text{ and,} \\ \kappa = \rho_{(\alpha_{opt}D)^{-1}N_1} &\leq \rho_{(\alpha D)^{-1}N_1} | \forall \alpha > \frac{\lambda_{max}(D^{-1}F)}{2} < 1, \end{aligned} \quad (56)$$

where the first inequality follows from Lemma 3.4, the second from Lemma 3.6 and the last from Lemma 3.7. The theorem now follows. We note that the approximate optimum value of  $\alpha$  (i.e.,  $\alpha = \alpha_{opt}$ ), which makes the bound on spectral radius a minimum, is independent of frequency,  $\omega$ . ■

The results of this theorem are illustrated in Example 3. We also show the effect on the convergence rate of choosing a value of  $\alpha$  that is widely different from the approximate optimum value obtained here.

LEMMA 3.8. *Let the matrix  $F$  be symmetric and positive definite. Let  $\lambda$  be an eigenvalue of the matrix  $S_{2,\mu}(\omega)$  and  $x$  be the corresponding eigenvector. Then*

$$|\lambda| = \frac{|\sigma(1 - \mu) - a\mu + i\{b\mu\}|}{|\sigma + a\mu + i\{(g + b)\mu\}|} \quad (57)$$

where  $\mu > 0$ , and

$$\begin{aligned} x^H Ux &= a(\mu, \omega) - ib(\mu, \omega), \\ x^H Lx &= a(\mu, \omega) + ib(\mu, \omega), \\ x^H Dx &= \sigma(\mu, \omega) > 0, \text{ and,} \\ x^H Gx &= g(\mu, \omega). \end{aligned} \quad (58)$$

PROOF. Using equation (20), we get the relation

$$\lambda x^H [\omega(D + \mu L) + i\mu\omega G] x = x^H [-\omega\mu U + (1 - \mu)\omega D] x \quad (59)$$



Taking the modulus on both sides and noting equation (58), the result follows. ■

We note that the quantities  $a$ ,  $b$ ,  $g$ , and  $\sigma$  in equation (58) are real functions of  $\mu$  and  $\omega$ .

LEMMA 3.9. *The quantity (see equation (58))  $(\sigma + 2a) > 0$ .*

PROOF. The result follows because  $F$  is symmetric and positive definite and therefore  $x^H F x > 0$ . ■

LEMMA 3.10. *Let  $\lambda$  be the eigenvalue corresponding to the spectral radius of  $S_{2,\mu}(\omega)$ , and  $x$  be the corresponding eigenvector. A necessary and sufficient condition for  $|\lambda| < 1$ , is:*

$$\omega(g) := \mu^2 g^2 + 2\mu^2 b g - \sigma \mu (\mu - 2)(\sigma + 2a) > 0. \quad (60)$$

PROOF. The result follows by taking the modulus of the numerator and denominator on the right-hand side of equation (57) and then requiring that  $|\lambda| < 1$ . ■

LEMMA 3.11. *When  $\omega \rightarrow 0, \pm \infty$ , relation (60) is satisfied for  $\mu \neq 0$ .*

PROOF. The eigenvectors,  $x$ , can always be normalized so that  $x^H x = 1$ . As  $\omega \rightarrow 0$ , and  $\pm \infty$ ,  $g^2 \rightarrow \infty$  because (see equation (18)) the elements of the matrix  $G \rightarrow \infty$ . ■

LEMMA 3.12. *Relation (60) will always be satisfied as long as*

$$0 < \mu < \frac{2}{\xi + 1}, \quad \text{where } \xi(\mu, \omega) := \frac{b^2}{\sigma(\sigma + 2a)} > 0. \quad (61)$$

PROOF. By Lemma 3.11, we know that relation (60) is satisfied for  $\omega \rightarrow 0, \pm \infty$ . For this relation *not* to be satisfied for  $0 < |\omega| < \infty$ , we require that for some  $g$ ,

$$w(g) = 0. \quad (62)$$

But this is impossible if the discriminant of the quadratic equation (62) is negative. This requires

$$b^2\mu^4 < \sigma(\sigma + 2a)\mu^3(2 - \mu). \quad (63)$$

Noting that quantities  $\sigma$  and  $(\sigma + 2a)$  are positive, we see that equation (63) cannot be satisfied if  $\mu \leq 0$  and  $\mu \geq 2$ . Using equation (63), the result now follows. ■

**THEOREM 3.6.** *When  $F$  is a symmetric and positive definite matrix, the set of iterative methods represented by Scheme II will converge for all  $\omega$ , as long as*

$$0 < \mu < \frac{2}{\xi(\mu, \omega) + 1}, \quad \text{where } \xi(\mu, \omega) := \frac{b^2}{\sigma(\sigma + 2a)} > 0. \quad (64)$$

where the quantities  $a$ ,  $b$ , and  $\sigma$  are defined in equation (58) and  $x$  is taken to be the eigenvector corresponding to the eigenvalue that yields the spectral radius of  $S_{2;\mu}(\omega)$ .

PROOF. The proof follows from the previous lemmas. We note that a sufficient condition for convergence, is that  $0 < \mu < 2/(\xi + 1) < 2$ .

Since the eigenvectors and eigenvalues of  $S_{2;\mu}(\omega)$  are dependent on the parameter  $\mu$ , the quantities  $a$ ,  $b$ , and  $\sigma$  are functions of  $\mu$ . This explicit dependence of  $\xi$  on  $\mu$  is shown in equation (64). Thus we note that equation (64), which yields the upper bound on  $\mu$  for convergence, requires to be solved in an iterative fashion. These results are verified in numerical Example 3. ■

#### 4. CONVERGENCE RATES AND ERROR BOUNDS

Let  $e_i^{(n)}$  be the  $L_2$  norm of error  $\delta_i^{(n)}(t)$  in the  $i^{th}$  component at the  $n^{th}$  iteration. Then

$$e_i^{(n)} = \left( \int_0^\infty \{\delta_i^{(n)}(t)\}^2 dt \right)^{1/2}. \quad (65)$$

Assuming that  $e_i^{(n)}$  as  $n \rightarrow \infty$  and  $e_i^{(0)}$  for all  $i$  are bounded and defining  $\|e^{(n)}\|$  as the  $L_2$  norm of the error vector whose components are as in equation (65), we have

$$\|e^{(n)}\|^2 = \sum_{i=1}^N (e_i^{(n)})^2. \quad (66)$$

Noting that for  $t < 0$ ,  $\delta_i^{(n)}(t) = 0$  for  $i = 1, 2, \dots, N$ , from equations (66) and (65)

$$\|e^{(n)}\|^2 = \sum_{i=1}^N \int_{-\infty}^{\infty} \{\delta_i^{(n)}(t)\}^2 dt. \quad (67)$$

Now from Parseval theorem

$$\int_{-\infty}^{\infty} \{\delta_i^{(n)}(t)\}^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{\delta}_i^{(n)}(\omega)|^2 d\omega. \quad (68)$$

Substituting equation (68) into (67), we get

$$\|e^{(n)}\|^2 = \frac{1}{2\pi} \sum_{i=1}^N \int_{-\infty}^{\infty} |\tilde{\delta}_i^{(n)}(\omega)|^2 d\omega. \quad (69)$$

Interchanging the order of the summation and integration, we obtain

$$\|e^{(n)}\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ |\tilde{\delta}_1^{(n)}(\omega)|^2 + |\tilde{\delta}_2^{(n)}(\omega)|^2 + \dots + |\tilde{\delta}_N^{(n)}(\omega)|^2 \right] d\omega, \quad (70)$$

or

$$\|e^{(n)}\|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\tilde{\delta}^{(n)}(\omega)\|^2 d\omega. \quad (71)$$

From equation (16), the error vector at the  $n^{th}$  iteration is given as

$$\tilde{\delta}^{(n)}(\omega) = [S(\omega)]^n \tilde{\delta}^{(0)}(\omega). \quad (72)$$

Taking  $L_2$  norm of the above equation, we get

$$\|\tilde{\delta}^{(n)}(\omega)\| \leq \| [S^n(\omega)] \| \|\tilde{\delta}^{(0)}(\omega)\|. \quad (73)$$

From equations (73) and (71), we have

$$\|e^{(n)}\|^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|S^n(\omega)\|^2 \|\tilde{\delta}^{(0)}(\omega)\|^2 d\omega \quad (74)$$

Now for an arbitrary  $N \times N$  complex matrix  $S(\omega)$  for which  $\rho(S(\omega)) > 0$  [9], we have

$$\|S^n(\omega)\| \sim v \binom{n}{p-1} [\rho(S(\omega))]^{n-(p-1)}, \quad n \rightarrow \infty \quad (75)$$

where  $p$  is the largest order of all diagonal submatrices  $J_r$  of the Jordan normal form of  $S(\omega)$  with  $\rho(J_r) = \rho(S(\omega))$ ,  $\binom{n}{p-1}$  is the binomial coefficient, and  $v$  is a positive constant. Now from equations (75) and (74) for  $n \rightarrow \infty$ , we have

$$\|e^{(n)}\|^2 \leq \frac{1}{2\pi} \left[ v \binom{n}{p-1} \right]^2 \int_{-\infty}^{\infty} [\rho^2(S(\omega))]^{n-(p-1)} \|\tilde{\delta}^{(0)}(\omega)\|^2 d\omega. \quad (76)$$

If  $\rho(S(\omega)) \leq k < 1$ , where  $k$  is the upper bound on the spectral radius  $\rho(S(\omega))$ , then we have as  $n \rightarrow \infty$

$$\|e^{(n)}\|^2 \leq \left[ v \binom{n}{p-1} \right]^2 [k^2]^{n-(p-1)} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\tilde{\delta}^{(0)}(\omega)\|^2 d\omega \right\}. \quad (77)$$

Noting that the expression in the curly brackets in the above relation is nothing but  $\|e^{(0)}\|^2$ , then we will have as  $n \rightarrow \infty$ ,

$$\frac{\|e^{(n)}\|}{\|e^{(0)}\|} \leq v \binom{n}{p-1} k^{n-(p-1)}. \quad (78)$$

When  $0 < k < 1$ , for  $n \rightarrow \infty$ , the right hand side of relation (78) goes to zero, and hence the absolute error in the calculated response goes to zero.

Defining

$$R_{av} = -\frac{1}{n} \log \left[ \frac{\|e^{(n)}\|}{\|e^{(0)}\|} \right] \quad (79)$$

as the average rate of convergence over  $n$  iterations [9], then

$$\lim_{n \rightarrow \infty} R_{av} = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \left[ v \binom{n}{p-1} k^{n-(p-1)} \right] \quad (80)$$

or

$$\lim_{n \rightarrow \infty} R_{av} = -\log k \equiv R_{\infty}, \quad (81)$$

where  $R_{\infty}$  is the lower bound on the asymptotic rate of convergence. Thus, we are able to establish a relation between the lower bound on the asymptotic rate of convergence and the upper bound on the spectral radius of the iteration matrix,  $S(\omega)$ . Now we can particularize this relation for the following cases.

CASE 1. When  $F$  is a strictly diagonally dominant matrix, then for the approximate optimum value of parameter  $\alpha$  (Scheme I), we have  $k = \pi$ , where  $\pi$  is defined in equation (23). The lower bound on the asymptotic rate of convergence is then

$$R_{\infty} \equiv -\log \pi \quad (82)$$

CASE 2. When  $F$  is a symmetric and positive definite matrix, then for the approximate optimum value of parameter  $\alpha$  (Scheme I), we have  $k = \kappa$ , where  $\kappa$  is defined in equation (54). The lower bound on the asymptotic rate of convergence is then

$$R_{\infty} = -\log \kappa \quad (83)$$

In addition to the lower bound on the asymptotic rate of convergence for Scheme I, when  $F$  is a strictly diagonally dominant matrix, we can also give the upper bound on the error for the individual components. We follow as below. Taking the  $\infty$  norm of relation (16), we get

$$\|\tilde{\delta}^{(n)}(\omega)\|_{\infty} \leq \|S^n(\omega)\|_{\infty} \|\tilde{\delta}^{(0)}(\omega)\|_{\infty}, \quad (84)$$

or (from the definition of  $\infty$  norm of a vector)

$$\max_{\forall i} |\tilde{\delta}_i^{(n)}(\omega)| \leq \|S(\omega)\|_{\infty}^n \max_{\forall i} |\tilde{\delta}_i^{(0)}(\omega)|. \quad (85)$$

For the approximate optimum value of  $\alpha$ , from equations (21) and (25), we then have

$$\max_{\forall i} \left| \bar{\delta}_i^{(n)}(\omega) \right| \leq \pi^n \max_{\forall i} \left| \bar{\delta}_i^{(0)}(\omega) \right|. \quad (86)$$

From the above inequality, we note that the upper error bound is frequency dependent.

## 5. NUMERICAL RESULTS

This section covers some numerical results for nonclassically damped systems to show the effectiveness of the two iterative schemes proposed in this paper. These results belong to systems having large degrees of freedom and serve as supplements to the numerical results provided in Udwardia and Kumar [17]. For the examples considered here, the system responses are strongly coupled through the damping terms. Customary uncoupling methods in  $N$ -space fail miserably in these situations [11, 12]. We find that in all examples studied, the iterated results, after only a few iterations are almost the same as those obtained from using the fourth-order Runge Kutta integration scheme [13]. In what follows, we will refer to the results obtained by the fourth-order Runge Kutta integration scheme as "exact" for short. For all the matrices  $F$  that have been covered in our numerical examples, Scheme II converges faster than Scheme I. Some of the results produced here correspond to the approximate optimum values of the parameter  $\alpha$  that make the bounds on the corresponding spectral radius a *minimum*. The approximate optimum values of  $\alpha$  for different kinds of matrices  $F$  are analytically obtained in Section 3. It has also been verified through numerical experiments that these theoretical estimates of the approximate optimum values of  $\alpha$  do yield rapid convergence.

Direct use of the fourth-order Runge Kutta procedure to obtain response results requires approximately  $(12N^2 + 18N)$  multiplications for each time step, where  $N$  is the number of equations. The iterative techniques, developed in this paper, utilize the Nigam-Jennings algorithm [14] for numerical integration. This algorithm requires  $8N$  multiplications per iteration for each time step. In addition to this, Scheme I needs  $N^2$  multiplications per iteration for each time step to uncouple the set of equations, i.e., to compute  $[(1 - \alpha)D + P]i^{(n-1)}$ . Scheme II also requires an additional  $N^2$  multiplications per iteration per time step to compute  $Li^{(n)}$  and  $[(1 - \alpha)D + U]i^{(n-1)}$ . We note that these additional number of multiplications are the same for both the schemes. Thus, for each time step, a total of  $(N^2 + 8N)I$  multipli-

cations are required to obtain the response results, where  $I$  is the number of iterations. Hence, for large  $N$ , when  $I$  is less than 12 for achieving the required convergence, the two iterative schemes developed herein become computationally efficient. Throughout this section it is assumed that the various parameter values are provided in consistent physical units.

EXAMPLE 1. Consider a 60 degree-of-freedom nonclassically damped system whose parameters are defined in the Appendix. The initial time  $t_0$  and final time  $T$  are taken to be zero and 10 units, respectively. The matrix  $F$ , chosen, is nonsymmetric and strongly *diagonally dominant*.

It should be noted that the diagonal elements of matrix  $\Lambda$  (which correspond to the squares of the undamped natural frequencies of vibration) are clustered, several of them being equal to 20 units. The choice of identical values for these diagonal elements causes us to expect intense interaction [12] through the coupling created by the matrix  $F$ . This would be even more prominent because the excitation is also taken to have a frequency of  $\sqrt{20}$  units [15]. We will see later that Examples 2 and 3 also have these critical features. Standard uncoupling methods used to date have been known to provide erroneous results in such situations [12, 15].

We define the normalized root mean square (RMS) error in the response at the  $n^{th}$  iteration in component  $i$  as

$$\text{normalized RMS error at iteration } n \text{ in component } i = \frac{\text{RMS of } \{u_i^{(n)} - z_i^{\text{RK}}\}}{\text{RMS of } \{z_i^{\text{RK}}\}}, \quad (87)$$

where  $u_i^{(n)}$  is the  $n^{th}$  iterate of component  $i$ , and  $z_i^{\text{RK}}$  is the  $i^{th}$  component of the response of equation (2) calculated using the Runge-Kutta integration scheme.

In all the numerical examples, we have shown the graphs of the normalized RMS error for those components of the response which converge most slowly. Therefore, the normalized RMS error (Figures 1 and 2) in these components at various iterations gives a bound on the error for other components at corresponding iterations. The RMS values of the displacement responses obtained using the Runge-Kutta method are also provided in the figures. For this example, the stopping criterion for our iterative schemes is that the right hand side of equation (87) be less than  $10^{-4}$ . We also have provided the time history plots for the most slowly convergent velocity component for all the examples. These plots include exact results by the Runge-Kutta method and the results corresponding to the first iteration, and to some other iterations.

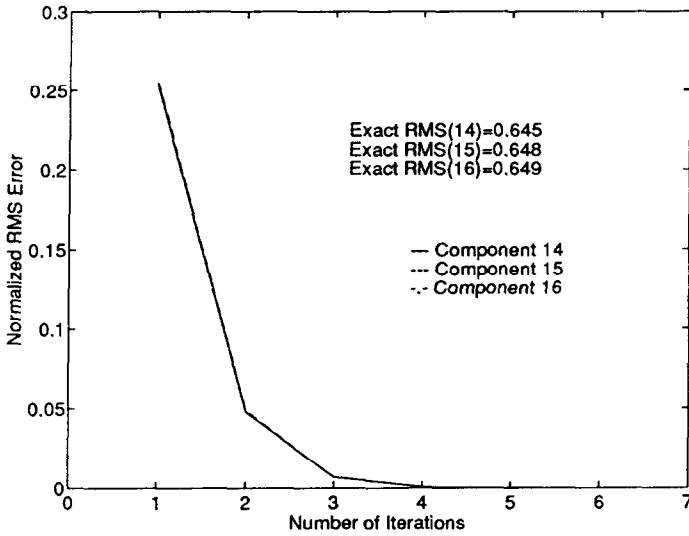


FIG. 1. Normalized RMS error of displacement components 14, 15, and 16 (Example 1, Scheme I) versus number of iterations.

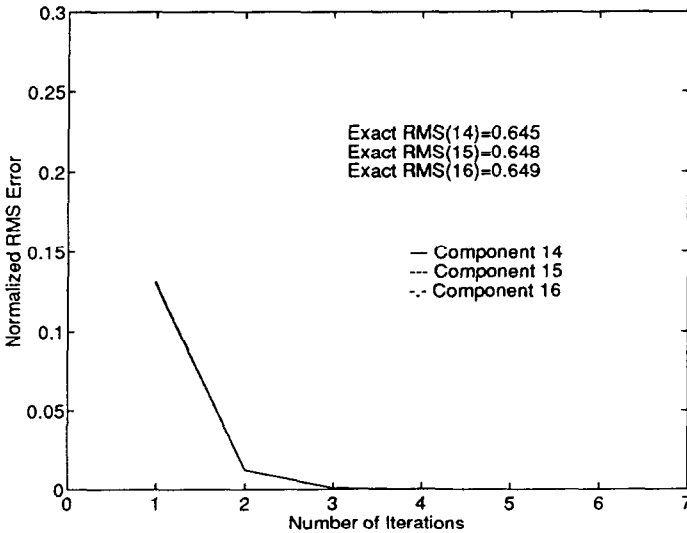


FIG. 2. Normalized RMS error of displacement components 14, 15, and 16 (Example 1, Scheme II) versus number of iterations.



In our numerical results, we also have given the estimates of average rates of convergence. In terms of actual computations, the significance of average rate of convergence [9],  $R_{av}$ , is the following. The quantity

$$\nu \equiv \left( \frac{\|e^{(n)}\|}{\|e^{(0)}\|} \right)^{1/n} \tag{88}$$

is the average reduction factor per iteration for the successive error norms, where  $\|e^{(n)}\|$  is the Euclidean norm of the error vector at  $n^{th}$  iteration and  $\|e^{(0)}\|$  is the Euclidean norm of initial error vector. The components of the error vector are as defined in equation (65). Now, from equation (79), we get

$$\nu = 10^{-R_{av}}. \tag{89}$$

Again, defining  $N_n \equiv R_{av}^{-1}$ , we see from the previous equality that

$$\nu^{N_n} = 1/10, \tag{90}$$

so that  $N_n$  is a measure of the number of iterations required to reduce the Euclidean norm of the initial error vector by a factor of 10. We have compared the two iterative schemes on the basis of their average rates of convergence,  $R_{av}$ , over a specified number of iterations.

Figures 1 and 2 show the convergence pattern of representative displacement components, as mentioned previously, for Scheme I and Scheme II, respectively. Figure 3 shows the time history of velocity component 15 (for Scheme I) at the first and fourth iterations including the exact results obtained by the Runge-Kutta method. The results from the 4<sup>th</sup> iteration cannot be distinguished in the graph from those of the Runge-Kutta method. Similarly, Figure 4 shows velocity component 15 at various iterations for Scheme II. The approximate optimum value of the parameter  $\alpha$ , which has been used in this example, is 1.0 (see Theorem 3.2). The effect of the values of  $\alpha$ , other than the approximate optimum value, on convergence of Scheme I is shown in Figure 5. It is observed that if the value of  $\alpha$  is chosen as two times the approximate optimum value, the convergence rate slows down roughly two times. It is also noted that the approximate optimum value of  $\alpha$  is very close to the numerically determined optimum value. The value of the parameter  $\mu (= 1/\alpha)$  for Scheme II has been taken as 1.0. Note that in Scheme I when  $\alpha = 1.0$ , the normalized RMS error which results at the first iteration provides a measure of the extent of the error in the system's response were all the off-diagonal terms of the matrix  $F$  ignored.

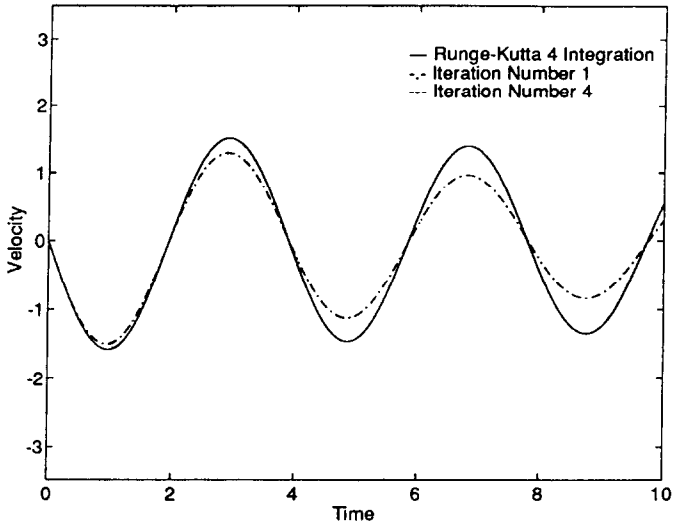


FIG. 3. Time history of the velocity component 15 (Example 1, Scheme I) corresponding to the exact response, first, and fourth iterations.

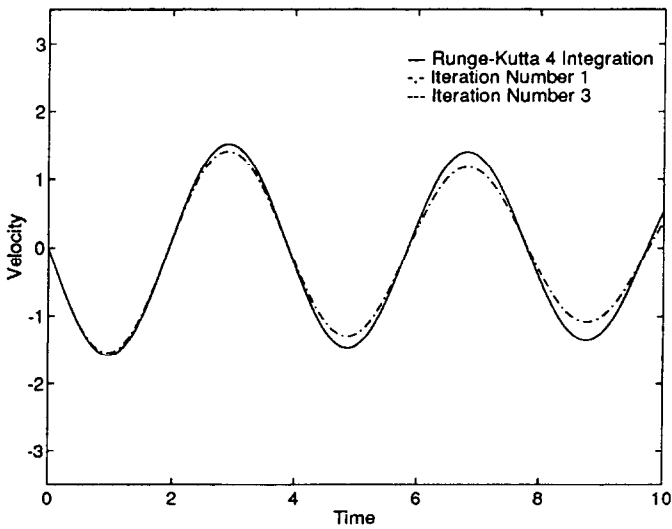


FIG. 4. Time history of the velocity component 15 (Example 1, Scheme II) corresponding to the exact response, first, and third iterations.

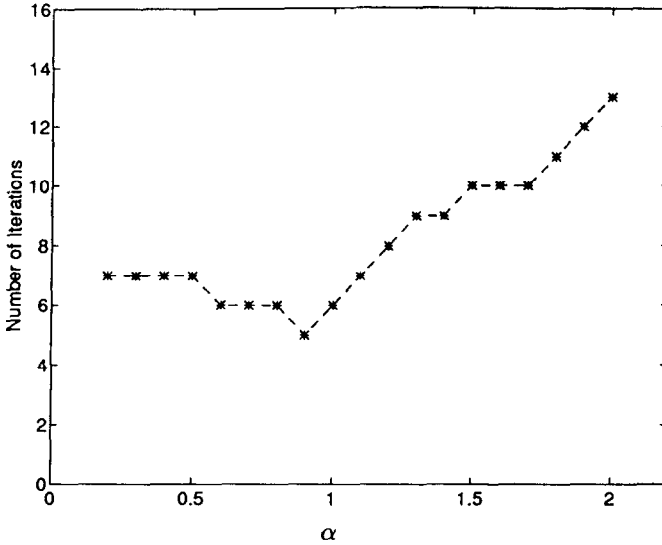


FIG. 5. Parameter  $\alpha$  versus number of iterations required to converge to the exact response results (Example 1, Scheme I).

In this example, the average rates of convergence ( $R_{av}$ ) for displacement components for Scheme I and Scheme II over 5 iterations (i.e.,  $n = 5$  in equation (88)) are 0.873 and 1.04, respectively. The reciprocals ( $N_n$ ) of these average rates of convergence show that Scheme I takes approximately 1.15 iterations to reduce the norm of the initial error vector by a factor 10 while Scheme II needs 0.97 iterations to do the same. Hence, it can be concluded that for this case Scheme II converges approximately 1.2 times faster than Scheme I. The lower bound on the asymptotic rate of convergence for Scheme I turns out to be 0.135.

EXAMPLE 2. Here we consider an irreducible and weakly diagonally dominant damping matrix  $F$  given as below:

$$F = \begin{bmatrix} 0.60 & 0.30 & 0.20 & 0.10 \\ 0.10 & 0.60 & 0.10 & 0.30 \\ 0.10 & 0.30 & 0.50 & 0.10 \\ 0.10 & 0.20 & 0.10 & 0.40 \end{bmatrix},$$

$$\Lambda = \text{Diag}\{20.0, 20.0, 20.0, 20.0\} \tag{91}$$

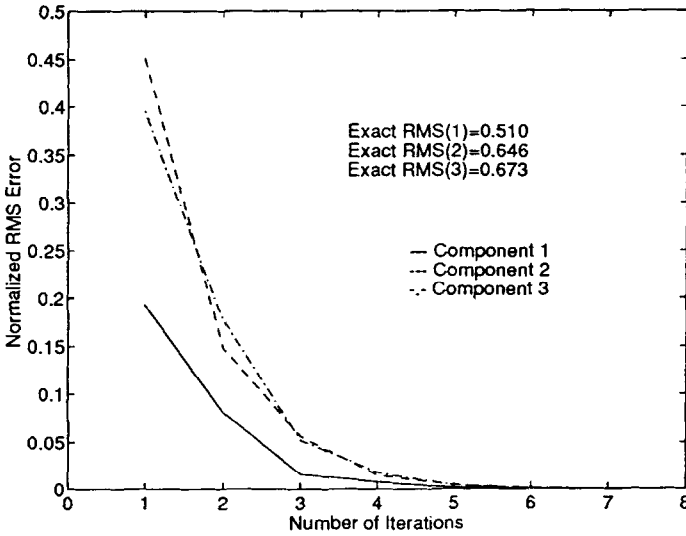


FIG. 6. Normalized RMS error of displacement components 1, 2, and 3 (Example 2, Scheme I) versus number of iterations.

The parameters  $t_0$  and  $T$  are taken to be zero and 10 units, respectively. The initial conditions are

$$z_i(0) = 0.0, 1 \leq i \leq 4; \text{ and } \dot{z}_i(0) = 1.0, 1 \leq i \leq 4. \quad (92)$$

And the system is subjected to the forcing vector  $h(t)$  given by

$$\begin{aligned} h_1(t) &= 2 \sin\sqrt{20}t, & h_2(t) &= -2 \sin\sqrt{20}t, & \text{and} \\ h_3(t) &= \sin\sqrt{20}t, & h_4(t) &= -\sin\sqrt{20}t, & t \in (0, 10) \end{aligned} \quad (93)$$

Once again the undamped natural frequencies are taken to be identical. The frequency of excitation is taken to be the same as the undamped natural frequency to ensure intense interaction of the response through the coupling caused by the nondiagonal matrix  $F$ .

Parameter  $\alpha = 1.1$  (see Theorem 3.2) and  $\mu = 1.0$  (see Theorem 3.3) have been used for the computations. The stopping criterion for this example and for Example 3 is that the right-hand side of equation (87) be less than  $6. \times 10^{-4}$ . Figures 6 and 7 show the convergence pattern of the most slowly convergent displacement components with increasing iteration numbers for the two iterative schemes. Figure 8 shows the velocity component,  $\dot{z}_3(t)$ , at different iterations along with the exact response obtained by the Runge-Kutta

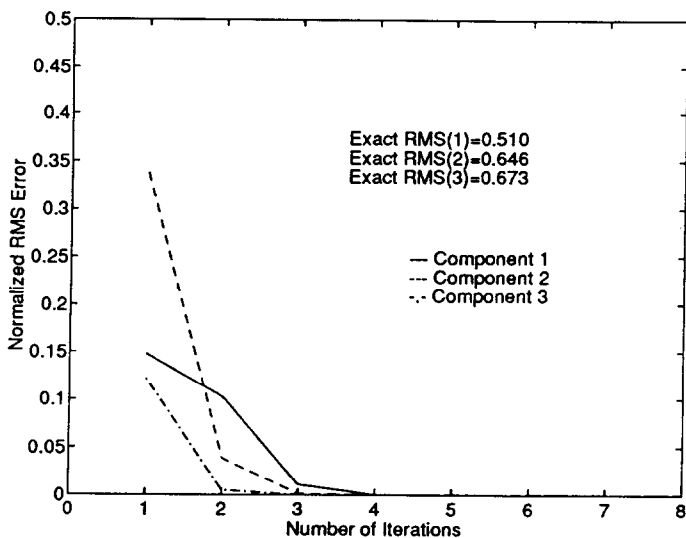


FIG. 7. Normalized RMS error of displacement components 1, 2, and 3 (Example 2, Scheme II) versus number of iterations.

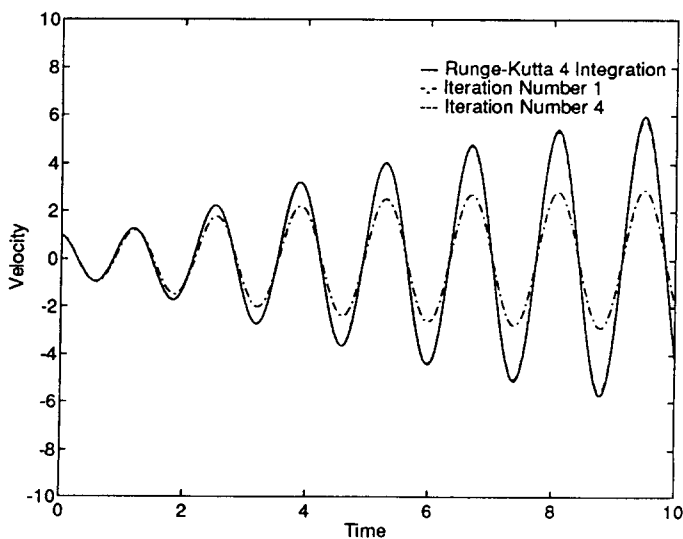


FIG. 8. Time history of the velocity component 3 (Example 2, Scheme I) corresponding to the exact response, first, and fourth iterations.

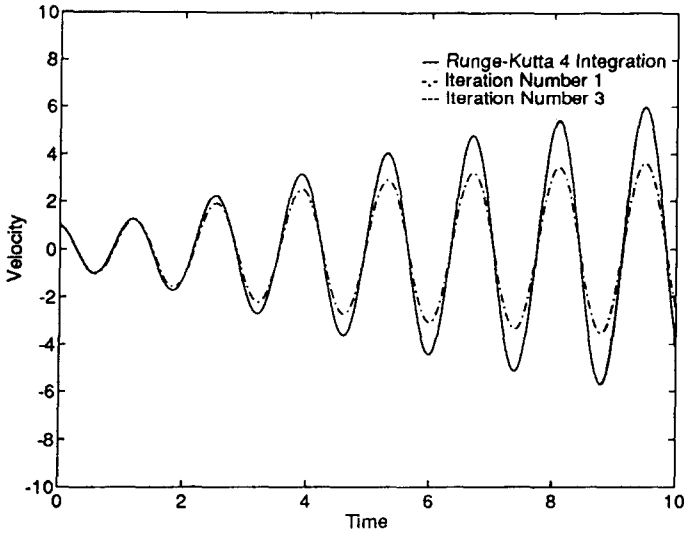


FIG. 9. Time history of the velocity component 1 (Example 2, Scheme I) corresponding to the exact response, first, and third iterations.

method (Scheme I). Similarly, Figure 9 depicts the convergence of the most slowly convergent velocity component 1 for Scheme II. It is noted here that the results at the third iteration are very close to the results obtained using the Runge-Kutta method. The average rates of convergence over five iterations for Schemes I and II turn out to be 0.49 and 0.89, respectively. Thus, for this example Scheme II converges approximately 1.8 times faster than Scheme I.

EXAMPLE 3. Let  $F$  be a  $12 \times 12$  symmetric and positive definite matrix with the elements as defined in the Appendix. The other parameters of the system are also defined in the Appendix. Here it should be noticed that  $F$  is no longer diagonally dominant. The minimum and maximum eigenvalues of matrix  $D^{-1}F$  are 0.6911 and 2.0445, respectively, where  $D$  is the diagonal part of the matrix  $F$ . Therefore, convergence for Scheme I (see Theorem 3.5) is guaranteed as long as  $\alpha > 2.0445/2$ . Here, we have given the results for  $\alpha = 1.3678$ , the approximate optimum value which has been calculated using equation (53). For Scheme II, convergence will occur if  $0 < \mu < 1.9532$  (see Lemma 3.12), but for the computations the value of  $\mu$ , which has been used, is 1.0.

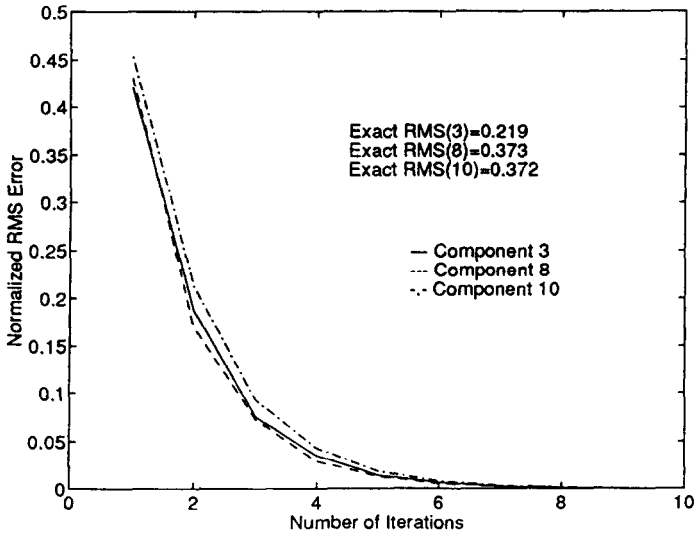


FIG. 10. Normalized RMS error of displacement components 3, 8, and 10 (Example 3, Scheme I) versus number of iterations.

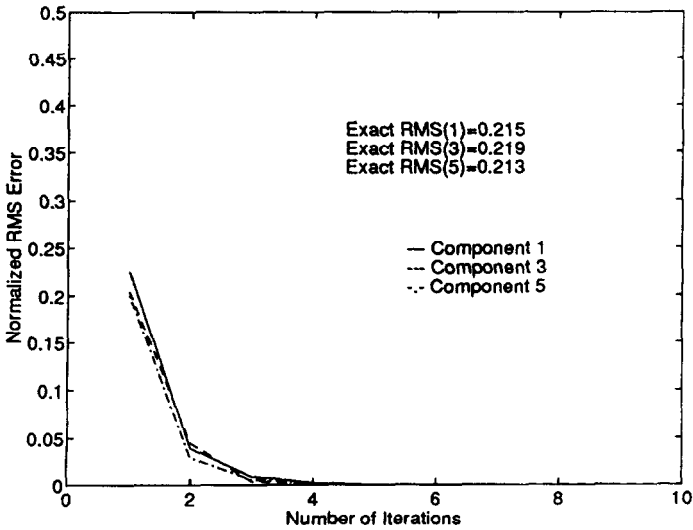


FIG. 11. Normalized RMS error of displacement components 1, 3, and 5 (Example 3, Scheme II) versus number of iterations.

Figures 10 and 11 show the normalized RMS error versus iteration number for the representative displacement components for Schemes I and II, respectively. Figure 12 contains time history plots of velocity component 10 (the most slowly converging component) at the first and fifth iterations along with the exact time history obtained by the Runge-Kutta method. Similarly, Figure 13 shows velocity component 1 (the most slowly converging component) for Scheme II at different iterations. The average rates of convergence over five iterations for Schemes I and II, for the above mentioned parameters, are calculated as 0.516 and 0.892, respectively. Thus, Scheme II converges approximately 1.7 times faster than Scheme I.

Furthermore, Figure 14 depicts the nature of convergence for Scheme I, when the value of the parameter  $\alpha$  is other than the approximate optimum value. This figure shows that choosing a value of  $\alpha$ , which is two times of the approximate optimum  $\alpha$  slows down the convergence process by almost a factor of two. Once again, we note that for this example also,  $\alpha_{opt}$  represents the numerically computed optimum value of  $\alpha$  fairly well. The lower bound on the asymptotic rate of convergence for Scheme I, using the approximate optimum value of  $\alpha$ , is computed as 0.306. Note that, for this example, the iterative scheme given by Udwadia and Esfandiari [3] does not promise convergence because  $\lambda_{max}(D^{-1}F) > 2$ , where  $\lambda_{max}$  denotes the maximum eigenvalue.

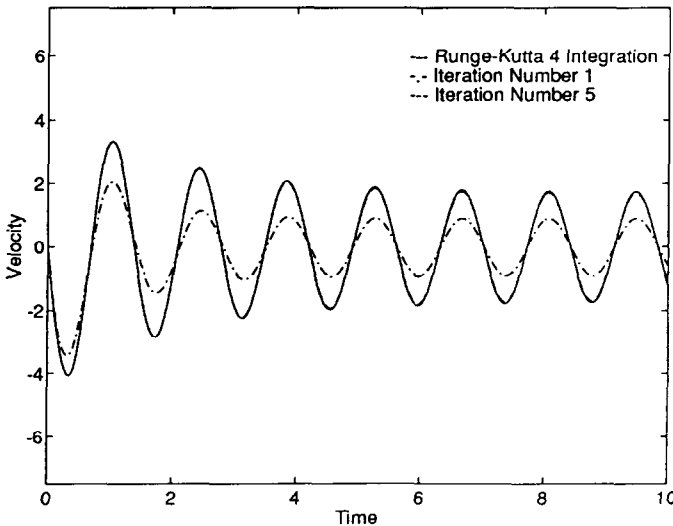


FIG. 12. Time history of the velocity component 10 (Example 3, Scheme I) corresponding to the exact response, first, and fifth iterations.



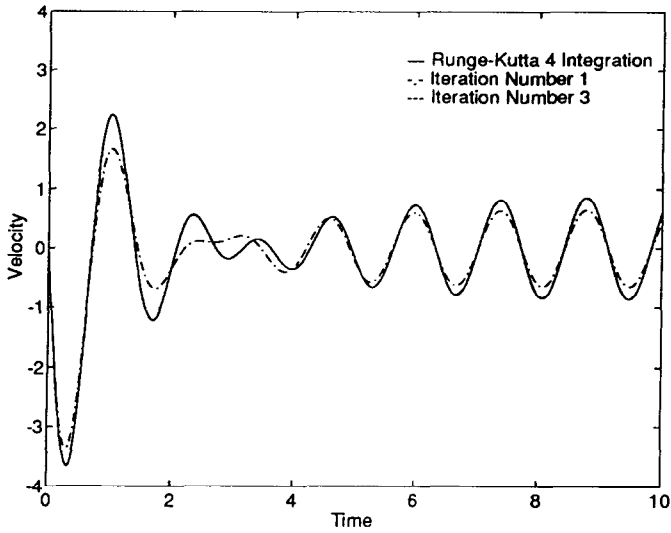


FIG. 13. Time history of the velocity component 1 (Example 3, Scheme II) corresponding to the exact response, first, and third iterations.

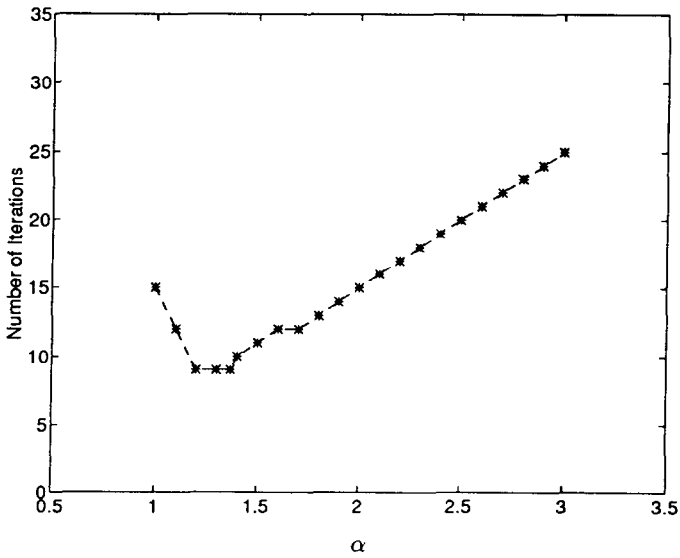


FIG. 14. Parameter  $\alpha$  versus number of iterations required to converge to the exact response results (Example 3, Scheme I).

## 6. CONCLUSIONS AND DISCUSSION

A rigorous convergence analysis for the two computationally efficient iterative schemes, proposed for the numerical solution of rather general, linear dynamic systems modeled by coupled differential equations, is presented. Sufficient conditions, under which these two schemes are convergent, are provided for three different kinds of damping matrices  $F$ . Unlike the previous work of Udwadia and Esfandiari [3] and Clart and Venancio-Filho [7], these results guarantee convergence for a wide variety of problems which are commonly met in the field of structural dynamics. We include in our considerations systems with clustered undamped natural frequencies.

For Scheme I, the approximate optimum values of the parameter  $\alpha$  are provided for two cases: (1) when the damping matrix  $F$  is strongly diagonally dominant and (2) when it is symmetric and positive definite. These approximate optimum values can be easily computed and when used help achieve a faster convergence to the exact response results for any arbitrary forcing function. Again, these approximate optimum values of the parameter  $\alpha$  are independent of the forcing function frequency  $\omega$ . Lower bounds on the asymptotic rates of convergence for the aforementioned damping matrices are also provided for Scheme I.

For symmetric and positive definite damping matrices  $F$ , the iterative scheme given by Udwadia and Esfandiari [3], which is a special case of Scheme I, guarantees convergence as long as  $\lambda_{max}(D^{-1}F) < 2$ . In this paper, we have shown that Scheme I always guarantees convergence in this situation as long as the parameter  $\alpha$  is chosen to be greater than  $\lambda_{max}(D^{-1}F)/2$ . The numerical results reported here and in Udwadia and Kumar [17] show that, in general, Scheme II is faster and computationally superior than Scheme I. However, as evident from Theorem 3.6, for symmetric and positive definite matrices  $F$ , care should be taken in selecting the value of the parameter  $\mu$  because convergence is guaranteed as long as  $0 < \mu < 2/(\xi + 1) < 2$ .

## APPENDIX

1. The damping matrix  $F$  for Example 1 is defined as

$$\begin{aligned} F(i, i) &= 0.15 & 1 \leq i \leq 60 \\ F(i, i + 1) &= -0.05 & 1 \leq i \leq 59 \\ F(i + 1, i) &= -0.06 & 1 \leq i \leq 59 \end{aligned} \tag{94}$$

and the modal stiffness matrix is given by

$$\Lambda = \text{Diag}\{1.0, 1.2, 1.2, 1.21, 1.6, 1.7, 1.9, 2.0, 2.2, 2.25, 2.3, 2.31, 2.5, 2.55, 2.6, 2.7, 2.8, 2.9, 3.0, 3.1, 3.2, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 4.0, 4.2, 4.3, 4.6, 4.8, 5.0, 5.2, 5.5, 5.6, 5.7, 6.0, 6.5, 7.0, 8.0, 10.0, 19.0, 20.0, 20.0, 20.0, 20.0, 21.0, 28.0, 28.0, 32.0, 34.0, 35.0, 36.0, 37.0, 38.0, 39.0, 40.0, 42.0, 43.0\}, \quad (95)$$

with the initial conditions

$$z_i(0) = 1.0, \quad \text{and} \quad \dot{z}_i(0) = 0.0, \quad 1 \leq i \leq 60. \quad (96)$$

This system is subjected to the forcing function  $h(t)$  given by

$$\begin{aligned} h_i(t) &= 2 \sin\sqrt{20}t, \quad i = 1, 3, 5, 7, 9, 10, \\ h_i(t) &= -2 \sin\sqrt{20}t, \quad i = 2, 4, 6, 8, \text{ and} \\ h_i(t) &= 0, \quad 11 \leq i \leq 60, \quad t \in (0, 10) \end{aligned} \quad (97)$$

2. The damping matrix  $F$  for Example 3 is [16] as follows:

$$F = \begin{bmatrix} 1.55 & 0.18 & -0.25 & -0.11 & 0.26 & 0.16 & -0.20 & -0.20 & 0.06 & 0.08 & -0.20 & -0.28 \\ 0.18 & 1.33 & 0.06 & 0.00 & 0.05 & 0.05 & -0.03 & -0.12 & -0.10 & -0.12 & -0.17 & -0.17 \\ -0.25 & 0.06 & 1.61 & 0.23 & -0.24 & -0.17 & 0.15 & 0.10 & -0.30 & -0.35 & -0.06 & 0.15 \\ -0.11 & 0.00 & 0.23 & 1.37 & 0.02 & -0.13 & -0.04 & -0.02 & -0.15 & -0.24 & -0.03 & 0.12 \\ 0.26 & 0.05 & -0.24 & 0.02 & 1.48 & 0.14 & -0.31 & -0.29 & 0.05 & 0.17 & -0.06 & -0.18 \\ 0.16 & 0.05 & -0.17 & -0.13 & 0.14 & 1.31 & -0.10 & -0.24 & 0.04 & 0.22 & 0.03 & -0.18 \\ -0.20 & -0.03 & 0.15 & -0.04 & -0.31 & -0.10 & 1.38 & 0.22 & -0.06 & -0.11 & 0.11 & 0.20 \\ -0.20 & -0.12 & 0.10 & -0.02 & -0.29 & -0.24 & 0.22 & 1.55 & 0.07 & -0.18 & 0.07 & 0.28 \\ 0.06 & -0.10 & -0.30 & -0.15 & 0.05 & 0.04 & -0.06 & 0.07 & 1.44 & 0.25 & 0.00 & -0.14 \\ 0.08 & -0.12 & -0.35 & -0.24 & 0.17 & 0.22 & -0.11 & -0.18 & 0.25 & 1.61 & 0.05 & -0.28 \\ -0.20 & -0.17 & -0.06 & -0.03 & -0.06 & 0.03 & 0.11 & 0.07 & 0.00 & 0.05 & 1.33 & 0.18 \\ -0.28 & -0.17 & 0.15 & 0.12 & -0.18 & -0.18 & 0.20 & 0.28 & -0.14 & -0.28 & 0.18 & 1.61 \end{bmatrix} \quad (98)$$

And the modal stiffness matrix is given by

$$\Lambda = \text{Diag}\{20.0, 20.0, 25.0, 20.0, 20.0, 15.0, 20.0, 20.0, 23.0, 20.0, 21.0, 22.0\}, \quad (99)$$

with the initial conditions

$$z_i(0) = 1.0, \quad \text{and} \quad \dot{z}_i(0) = 0.0, \quad 1 \leq i \leq 12. \quad (100)$$

This system is subjected to the forcing function  $h(t)$  given by

$$\left. \begin{aligned} h_i(t) &= \sin\sqrt{20}t, & i &= 1, 3, 5, 7, 9, 11, 12; \text{ and} \\ h_i(t) &= -2\sin\sqrt{20}t, & i &= 2, 4, 6, 8, 10, t \in (0, 10) \end{aligned} \right\}. \quad (101)$$

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