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Firdaus E. Udwadia

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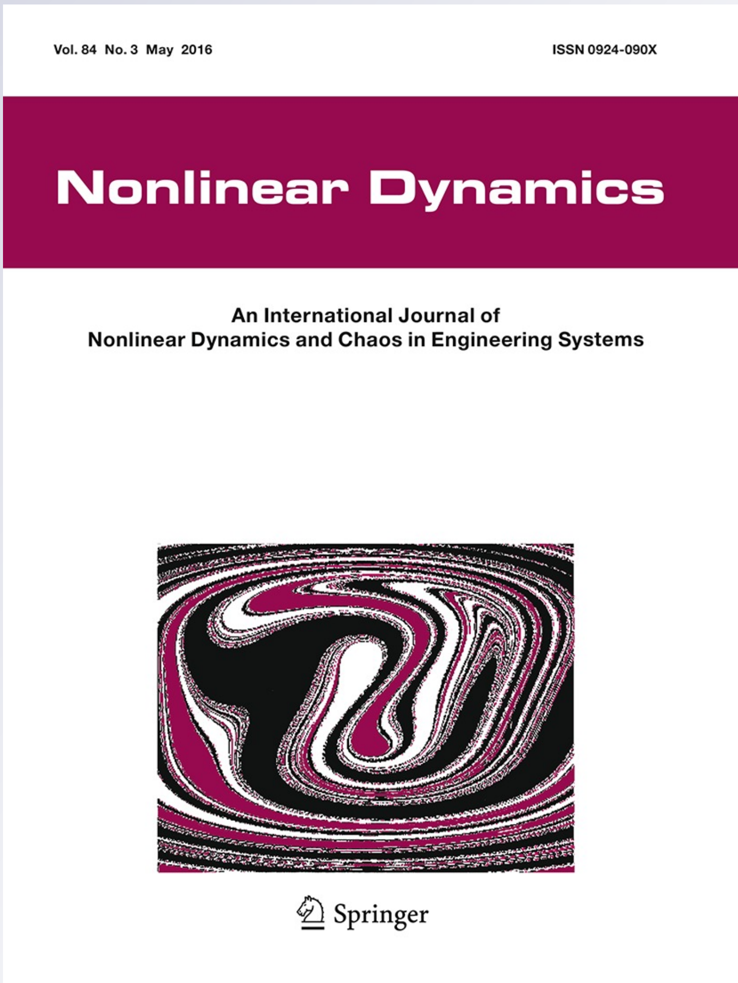

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ORIGINAL PAPER

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Abstract This paper considers dynamical systems described by Hamilton's equations. It deals with the development of the explicit equations of motion for such systems when constraints are imposed on them. Such explicit equations do not appear to have been obtained hereto. The holonomic and/or nonholonomic constraints imposed can be nonlinear functions of the canonical momenta, the coordinates, and time, and they can be functionally dependent. These explicit equations of motion for constrained systems are obtained through the development of the connection between the Lagrangian concept of virtual displacements and Hamiltonian dynamics. A simple three-step approach for obtaining the explicit equations of motion of constrained Hamiltonian systems is presented. Four examples are provided illustrating the ease and simplicity with which these equations can be obtained by using the proposed three-step approach

Keywords Constrained Hamiltonian systems · Explicit equations of motion · Holonomic and nonholonomic constraints · Virtual displacements and Hamiltonian dynamics

F. E. Udwadia (✉)
Aerospace and Mechanical Engineering, Civil Engineering, Mathematics, Information and Operations Management, University of Southern California, 430K Olin Hall, Los Angeles, CA 90089-1453, USA
e-mail: feuusc@gmail.com

1 Introduction

The explicit equations of motion in the Lagrangian framework for mechanical systems that are subjected to holonomic and nonholonomic constraints have been available for some time now [1], whether or not they satisfy d'Alembert's principle [2], and whether or not the so-called mass matrix is strictly positive definite or semi-positive definite [3,4]. While these explicit equations may be considered as encompassing nearly all situations for the description of constrained mechanical systems, there may be some special situations where one might be interested in directly developing the equations of constrained motion for systems described by Hamilton's equations. There are many systems in which the Hamiltonian can be written down easily, and it is often useful to employ Hamilton's equations, both from an analytical and from a computational viewpoint, for systems that are known to preserve phase volume, and/or those that have symmetries, and/or those for which questions of integrability are considered important. One then needs an approach to directly obtain the constrained equations of motion when constraints are imposed on systems described in this manner. As will be shown, the development of the closed-form expressions for their constrained motion requires an understanding of the connection between the principle of virtual work and virtual displacements on the one hand, and the Hamiltonian formulation of mechanics on the other; it does not hinge on the Legendre transformation. These results help to deepen our understanding of these fundamental aspects of analytical dynamics, which to the best of the author's knowledge appear to have gone unnoticed so far.

It might be argued that for any given dynamical system, one can always transform the Hamiltonian (or Hamilton's equations for the unconstrained system) as well as the constraints imposed on it into an equivalent Lagrangian formulation using the Legendre transformation, then use the Lagrange formulation to obtain the constrained equations of motion as described, say, in Ref. [1], and finally, transform back these Lagrange equations into Hamiltonian form. While this is indeed possible, the effort required to work this out, even for a system with a small number of configuration coordinates and a few constraints, can be substantial. Besides, such a brute force approach gives no insights regarding the way in which different entities defined in terms of the Hamiltonian (and its derivatives) enter into the final constrained equation of motion of the system. An equation describing the constrained Hamiltonian system *directly* in terms of the canonical momenta and the coordinates—the so-called p's and the q's—that are used to describe the Hamiltonian of the unconstrained system as well as the constraints would be generally more preferable and expeditious to use when dealing with such problems. More importantly, the general equation developed herein for constrained Hamiltonian systems exhibits the underlying "structure" of constrained Hamiltonian dynamics. As in most of mathematical physics, the structure of the equation is significant providing new perspectives to our understanding of constrained Hamiltonian dynamics, hence the motivation for this work.

In this paper, it is assumed that the unconstrained system is described by a Hamiltonian, $H(p, q, t)$, where p is the canonical momentum n-vector (n by 1 vector) that is conjugate to the coordinate n-vector q, and t denotes time. The unconstrained equations of motion of the system are then

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{1}$$

where p and q are n by 1 column vectors (n-vectors).

This $2n$-dimensional unconstrained dynamical system described by Hamilton's equations is now further subjected to the constraints

$$\varphi(p, q, t) = 0 \tag{2}$$

where φ is an m-vector, and each row, $\varphi_i(p, q, t) = 0$, of this equation set constitutes a constraint on the system described by Eq. (1). It is assumed that each such constraint is at least a C^1 function of its arguments. Furthermore, each constraint can be any nonlinear function of its arguments. The set of constraints are permitted to be functionally dependent. The constraints described by Eq. (2) include holonomic and/or nonholonomic constraints.

Given the unconstrained equations of motion of a dynamical system by Eq. (1), our aim is to determine explicitly the constrained equations of motion of this dynamical system when it is subjected to the set of consistent constraints specified in Eq. (2). The equations obtained are new and have not been reported in the form obtained in the extant literature.

The organization of this paper is as follows. Section 2 deals with some preliminaries that relate the Lagrangians of mechanical systems and their Hamiltonians. Besides being useful in establishing notation, they will be used later on in the sequel. In this section, the pivotal connections between virtual displacements, virtual work, d'Alembert's principle, and the Hamiltonian formulation are obtained. In Sect. 3.1, the equations of motion describing a Hamiltonian system when subjected to the constraint given in Eq. (2) are developed. A three-step simple and systematic approach for obtaining these equations is developed in Sect. 3.2. Section 4 deals with four illustrative examples that demonstrate the efficacy and ease of application of the three-step approach presented in Sect. 3.2. The first two examples deal, respectively, with a holonomic constraint and a nonholonomic constraint on a particle. The third example deals with a system of particles subjected to constraints that are both holonomic and nonholonomic with the set being functionally dependent. The last example deals with a double pendulum wherein a nonholonomic constraint on the canonical momenta is imposed. Section 5 discusses the results, some areas of application, and provides some concluding remarks.

2 Some preliminaries

2.1 Statement of the problem of constrained motion

(i) *Unconstrained Hamiltonian Systems*. Consider first the unconstrained Hamiltonian system given by Eq. (1). The canonical momentum n-vector p that is conjugate to the generalized coordinate n-vector q is defined as

$$p = \frac{\partial L}{\partial \dot{q}}. \tag{3}$$

in which $L(q, \dot{q}, t)$ denotes the Lagrangian of the system. It is assumed that the kinetic energy of the mechanical system can be expressed by a relation of the form

$$T = \frac{1}{2}\dot{q}^T M(q,t)\dot{q} + \tilde{m}(q,t)^T \dot{q} + \mu(q,t) \tag{4}$$

where $M(q,t)$ is a positive definite matrix, $\tilde{m}(q,t)$ is an n-vector, and $\mu(q,t)$ is a scalar. From relation (3), the canonical momentum for a conservative system can then be defined as

$$p = M\dot{q} + \tilde{m} \tag{5}$$

so that we can define

$$\dot{q}(p,q,t) = M^{-1}(p - \tilde{m}) \tag{6}$$

where we note that M^{-1} is positive definite. In the above two equations, and from here on, the arguments of the various quantities will be suppressed unless required for clarity.

We consider a conservative dynamical system and use relation (6) that expresses $\dot{q}$ in terms of p, q, and t, to define the Hamiltonian H of the system as

$$H(p,q,t) = p^T\dot{q}(p,q,t) - L(q, \dot{q}(p,q,t), t). \tag{7}$$

By taking arbitrary variations of q and $\dot{q}$ (or equivalently, arbitrary variations of q and p), while keeping t unvaried, one gets the *first* relation in Eq. (1) along with the relation $\frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}$ [5]. It is important to note that these two relations are obtained purely from the definition of H given by Eq. (7) and not the dynamics; they are relations that we define.

Using Lagrange's equation, one obtains the second relation in equation set (1) because

$$\dot{p} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} = -\frac{\partial H}{\partial q}. \tag{8}$$

Here, the second equality follows from Lagrange's equation, and the third from the above-mentioned definition of H. As seen, it is Eq. (8) that invokes the dynamics of the system; the first relation in Eq. (1) is simply a consequence of our definition of H.

To obtain a unique solution, the unconstrained system described by the two relations in Eq. (1) requires the initial condition n-vectors $p(t=0) = p_0$ and $q(t=0) = q_0$, where p_0 and q_0 can be arbitrarily prescribed. Eq. (1) along with these initial conditions then describes the initial value problem that defines the motion of unconstrained Hamiltonian system.

(ii) *Description of Constraints*. We now further subject the unconstrained Hamiltonian system described by Eq. (1) to the set of m constraints described in Eq. (2).

We note that if the unconstrained system is subjected to a holonomic constraint $\phi_i(q,t) = 0$, this constraint equation can be differentiated with respect to time t, to yield the corresponding component, $\varphi_i(p,q,t)$, of the m-vector φ given in Eq. (2) as

$$\frac{\partial \phi_i}{\partial q}\dot{q} + \frac{\partial \phi_i}{\partial t} = \frac{\partial \phi_i}{\partial q}M^{-1}(p - \tilde{m}) + \frac{\partial \phi_i}{\partial t} := \phi_i(p,q,t), \tag{9}$$

where $\dot{q}$ as defined in Eq. (6) is used in the first equality. Hence, Eq. (2) is general enough to include holonomic and/or nonholonomic constraints.

Upon differentiation of Eq. (2) with respect to time t, one obtains

$$\frac{\partial \varphi}{\partial p}\dot{p} = -\frac{\partial \varphi}{\partial q}\dot{q} - \frac{\partial \varphi}{\partial t} = -\frac{\partial \varphi}{\partial q}M^{-1}(p - \tilde{m}) - \frac{\partial \varphi}{\partial t}, \tag{10}$$

which can be written as

$$A_p(p,q,t)\dot{p} = b_p(p,q,t) \tag{11}$$

where

$$A_p = \frac{\partial \varphi}{\partial p} \text{ and } b_p = -\frac{\partial \varphi}{\partial q}M^{-1}(p - \tilde{m}) - \frac{\partial \varphi}{\partial t}. \tag{12}$$

The matrix A_p is an m by n matrix of rank k, and b_p is an m-vector.

We therefore find that $\dot{p}$, which is the rate of change in the canonical momentum of the constrained Hamiltonian system, must satisfy Eq. (11) at *every* instant of time.

Proper specification of the constrained dynamical system requires that the n-vectors of initial values, p_0 and q_0, can no longer be arbitrarily prescribed as before when the system was unconstrained. Assuming that these initial conditions are provided at time $t = 0$, the initial condition n-vectors p_0 and q_0 must now satisfy the m constraint equations given in Eq. (2), so that $\varphi(p_0, q_0, t = 0) = 0$. Given such an appropriate set of initial condition n-vectors that satisfy Eq. (2), Eq. (11) is then an alternate way of expressing Eq. (2).

(iii) *Dynamics of Constrained Hamiltonian Systems*. In the presence of the constraints (described by Eq. (2), or alternatively, by Eq. (11)) that the Hamiltonian system (described by Eq. (1)) is subjected to, the system's equations of motion are altered to

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} + Q^C(p, q, t) \tag{13}$$

where $Q^C(p, q, t)$ is the constraint force that arises because of the imposition of the constraints. Our aim is to obtain an explicit expression for Q^C. As pointed out before, the first equation in equation set (13) is purely a consequence of our definition of H, and it remains unchanged; the second equation that deals with the dynamical description of the system changes in order to accommodate the presence of the constraints.

2.2 Virtual displacements and Hamiltonian dynamics

Let us suppose that q and $\dot{q}$ are known at time t. The virtual displacement n-vector w at time t^+ (the instant immediately following time t) is defined as any nonzero n-vector from the actual configuration, q_a, at time t^+ to a "possible" configuration, q_{pos}, at time t^+ [6]. A "possible" configuration is defined as one that satisfies the constraints. Thus, we have

$$w(t+dt) = \left[q_{pos}(t) + dt\dot{q}_{pos}(t) + \frac{dt^2}{2}\ddot{q}_{pos}(t)\right] - \left[q_a(t) + dt\dot{q}_a(t) + \frac{dt^2}{2}\ddot{q}_a(t)\right] + O(dt^3) \tag{14}$$

Since we assume that q and $\dot{q}$ are known at time t, $q_{pos}(t) = q_a(t)$ and $\dot{q}_{pos}(t) = \dot{q}_a(t)$. Hence,

$$w(t+dt) = \frac{dt^2}{2}[\ddot{q}_{pos}(t) - \ddot{q}_a(t)] + O(dt^3). \tag{15}$$

Furthermore, both the actual and possible positions must satisfy the constraint Eq. (11), so that

$$A_p(q_{pos}(t), p_{pos}(t), t)\dot{p}_{pos} = b_p(q_{pos}(t), p_{pos}(t), t) \tag{16}$$

and

$$A_p(q_a(t), p_a(t), t)\dot{p}_a = b_p(q_a(t), p_a(t), t). \tag{17}$$

Since $q_{pos}(t) = q_a(t)$ and $\dot{q}_{pos}(t) = \dot{q}_a(t)$, we have (see Eq. (5)) $p_{pos}(t) = p_a(t)$, and therefore, subtracting Eq. (17) from Eq. (16), we get

$$A_p(p_a, q_a, t)[\dot{p}_{pos} - \dot{p}_a] = 0. \tag{18}$$

From Eq. (5), we get upon differentiating with respect to t,

$$\dot{p} = M(q, t)\ddot{q} + \dot{M}(q, t)\dot{q} + \dot{m}(q, t) \tag{19}$$

and noting again that $q_{pos}(t) = q_a(t)$ and $\dot{q}_{pos}(t) = \dot{q}_a(t)$, Eq. (18) after multiplication by $\frac{dt^2}{2}$ reduces to

$$A_p(p_a, q_a, t)M(q_a, t)[\ddot{q}_{pos} - \ddot{q}_a]\frac{dt^2}{2} = 0. \tag{20}$$

Using Eq. (15) and taking the limit as $dt \to 0$, Eq. (20) yields

$$A_p(p_a, q_a, t)M(q_a, t)w(t) = 0. \tag{21}$$

What we have thus found is that at any time t, a (nonzero) virtual displacement n-vector w must satisfy relation (21). In what follows, for greater clarity, we shall suppress the subscript "a" in the above equation.

2.3 D'Alembert's principle

In what follows we shall assume that d'Alembert's principle is valid. The principle states that at each instant of time t, for all virtual displacements, that is, for all nonzero vectors $w(t)$ that satisfy Eq. (21), we must have

$$w^T Q^C = 0. \tag{22}$$

We now have all the preliminaries to obtain the explicit constrained Hamilton's equations of motion.

3 Equations of motion

3.1 Explicit equations of motion of the constrained Hamiltonian system

We begin by noting that

$$\frac{\partial \dot{q}}{\partial p} = H_{pp} = M^{-1}(q, t) \tag{23}$$

where the positive definite matrix $H_{pp} = \frac{\partial^2 H}{\partial p^2}$. The first equality above comes from Eq. (1) and the second from Eq. (6). Thus, by Eq. (21), a virtual displacement is any nonzero n-vector $w(t)$ that satisfies the relation

$$A_p H_{pp}^{-1} w = 0. \tag{24}$$

Denoting $B_p := A_p H_{pp}^{-1/2}$ and $u := H_{pp}^{-1/2} w$, Eq. (24) becomes

$$B_p u = 0 \tag{25}$$

whose solution is [7,8]

$$u = (I - B_p^+ B_p)y \tag{26}$$

where y is an arbitrary n-vector, and X^+ denotes the Moore–Penrose inverse of the matrix X. Furthermore, since the constraints must be satisfied, the p's and q's must satisfy Eq. (2), or alternately Eq. (11), so that

$$A_p \dot{p} = B_p(H_{pp}^{1/2}\dot{p}) = b_p \tag{27}$$

whose solution is

$$H_{pp}^{1/2}\dot{p} = B_p^+ b_p + (I - B_p^+ B_p)\xi \tag{28}$$

where ξ is an arbitrary n-vector [9]. Our aim is now to explicitly find the second member on the right-hand side of Eq. (28) so that d'Alembert's principle is satisfied.

From relations (22) and (24), it is seen that d'Alembert's principle requires that at each instant of time t,

$$w^T Q^C = 0 \text{ for all nonzero } n\text{-vectors } w \text{ that satisfy } A_p H_{pp}^{-1} w = 0. \tag{29}$$

This can be restated as requiring that at each instant of time t, we must have

$$u^T H_{pp}^{1/2}\left[\dot{p} + \frac{\partial H}{\partial q}\right] = 0 \text{ for all nonzero } n\text{-vectors } u \text{ that satisfy } B_p u = 0, \tag{30}$$

where we have substituted for Q^C using Eq. (13). From Eq. (28), we find that

$$u^T H_{pp}^{1/2}\left[\dot{p} + \frac{\partial H}{\partial q}\right] = u^T\left[B_p^+ b_p + (I - B_p^+ B_p)\xi + H_{pp}^{1/2}\frac{\partial H}{\partial q}\right]. \tag{31}$$

Also, if u is such that $B_p u = 0$, then this would imply that $u^T B_p^+ = 0$ [6], and hence, d'Alembert's principle would require that at each instant of time,

$$u^T\left[\xi + H_{pp}^{1/2}\frac{\partial H}{\partial q}\right] = 0 \text{ for all } n\text{-vectors } u \text{ that satisfy } B_p u = 0. \tag{32}$$

But the vectors u that satisfy the relation $B_p u = 0$ are given in Eq. (26), where y is an arbitrary n-vector. Hence, we must have

$$y^T (I - B_p^+ B_p)\left[\xi + H_{pp}^{1/2}\frac{\partial H}{\partial q}\right] = 0 \text{ for all } n\text{-vectors } y.$$

The last equality then requires that

$$(I - B_p^+ B_p)\xi = -(I - B_p^+ B_p)H_{pp}^{1/2}\frac{\partial H}{\partial q}. \tag{33}$$

Using Eq. (33) in the second member on the right of Eq. (28), we obtain the equation of motion of the constrained Hamiltonian system as

$$\dot{p} = -\frac{\partial H}{\partial q} + H_{pp}^{-1/2} B_p^+\left(b_p + B_p H_{pp}^{1/2}\frac{\partial H}{\partial q}\right). \tag{34}$$

Noting that $B_p = A_p H_{pp}^{-1/2}$, the equations of motion of the Hamiltonian system (1) when it is further subjected to the constraint m-vector $\varphi(p, q, t) = 0$ given in Eq. (2) are explicitly given by

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q} + H_{pp}^{-1/2}(A_p H_{pp}^{-1/2})^+\left(b_p + A_p\frac{\partial H}{\partial q}\right). \tag{35}$$

Another alternative and useful form of the second relation in Eq. (35) can be obtained by using the identity $X^+ = X^T(XX^T)^+$ which is valid for any matrix X [8]. This yields

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q} + H_{pp}^{-1} A_p^T (A_p H_{pp}^{-1} A_p^T)^+\left(b_p + A_p\frac{\partial H}{\partial q}\right). \tag{36}$$

Remark 1 When the Hamiltonian system is subjected to a single constraint, then A_p is a 1 by n matrix, and then $A_p H_{pp}^{-1} A_p^T > 0$ becomes a scalar. In such situations, Eq. (36) is easier to use since the generalized inverse of a nonzero scalar is simply its reciprocal (see the example in Sect. 4.4). □

Remark 2 When the constraints are all independent so that the matrix A has full rank m, the equation for $\dot{p}$ in Eq. (36) simplifies because $(A_p H_{pp}^{-1} A_p^T)^+ = (A_p H_{pp}^{-1} A_p^T)^{-1}$, and one can then simply use the regular inverse of the matrix on the right-hand side. □

As stated before, the first equation in the equation set (35) (or the set (36)) is simply the outcome of our definition of the Hamiltonian. The second equation in the equation set (35) [or the set (36)] deals with the dynamics of constrained Hamiltonian systems and exhibits the deep structure of constrained motion. Let us then consider this second equation in greater detail.

The first member on the right-hand side of this (second) equation, $-\frac{\partial H}{\partial q}$, is the $\dot{p}$ corresponding to the unconstrained Hamiltonian system (see the second relation in Eq. (1)). The second member on the right-hand side (in this second equation) can be written as $H_{pp}^{-1/2}(A_p H_{pp}^{-1/2})^+[b_p - A_p(-\frac{\partial H}{\partial q})]$. We note that $\dot{p}$ for the constrained system *must* satisfy the constraint $b_p - A_p\dot{p} = 0$ given in Eq. (11). Were we to substitute $\dot{p}$ $(= -\frac{\partial H}{\partial q})$ obtained from the unconstrained Hamiltonian system into this constraint relation, it would, in general, not be satisfied. In fact, we would then get the error, $e(p, q, t) := [b_p - A_p(-\frac{\partial H}{\partial q})]$, which is simply *the extent to which the* $\dot{p}$ *of the unconstrained Hamiltonian system does not satisfy the constraint given by* Eq. (11). Thus, the second equation of the set (35) informs us that the canonical momentum of the unconstrained system that is expressed by $-\frac{\partial H}{\partial q}$, is *altered* by the presence of the constraint given by Eq. (2) through an additional additive term given by the second member on the right-hand side, namely $H_{pp}^{-1/2}(A_p H_{pp}^{-1/2})^+e$. This additional term explicitly shows the effect of the presence of constraints on Hamiltonian systems. We observe that the term is directly proportional to the error e. The matrix of proportionality is $H_{pp}^{-1/2}(A_p H_{pp}^{-1/2})^+$, and each of the elements of this matrix are, in general, nonlinear functions of p, q, and t.

3.2 A three-step approach for obtaining the explicit equations of motion of the Hamiltonian systems that are constrained

Consider an unconstrained mechanical system described by Hamilton's equations as in Eq. (1). This unconstrained system is further subjected to a set of m consistent (holonomic and/or nonholonomic) constraints as in Eq. (2). Our discussion above points to a simple, systematic three-step approach for obtaining the equations of motion of such a system in the presence of these m constraints.

(i) First, Hamilton's equations for the unconstrained system are obtained (or written down).
(ii) Second, the matrix A_p and the column vector b_p that specify the constraints are determined (see Eq. (11)). H_{pp} is found using the first equality in Eq. (23).
(iii) Finally, the constrained equation of motion of the system is directly computed using either Eq. (35) or Eq. (36).

4 Illustrative examples

In this section, four examples of the use of the explicit equations obtained in Sect. 3.1 are provided. The examples are purposely chosen to be relatively simple because they are meant to be purely illustrative. The three-step approach stated in Sect. 3.2 is used, and the three steps are clearly identified in each example.

4.1 Spherical pendulum

(i) *Unconstrained Hamiltonian system*

Following the three-step approach described above, in the first step, we obtain Hamilton's equations for the unconstrained system consisting of a particle of mass m subjected to a downward force of gravity.

Using an inertial coordinate frame whose z-axis points vertically downwards, and denoting the position of the particle by the three-vector $q = [x, y, z]^T$, the Hamiltonian of the particle is given by

$$H = \frac{1}{2m}p^T p - mgz \tag{37}$$

where $p = [p_x, p_y, p_z]^T$ and p_x, p_y, and p_z denote the momenta in the x-, y-, and z-directions, respectively. Hamilton's equations for the unconstrained system are then

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial q} = [0, 0, mg]^T. \tag{38}$$

(ii) *Description of constraints*

In order to model the spherical pendulum, the particle is subjected to the holonomic constraint specifying that its distance from the origin O of the coordinate system is a constant L. This constraint can be expressed as

$$x^2 + y^2 + z^2 - L^2 = q^T q - L^2 = 0. \tag{39}$$

Differentiating Eq. (38) with respect to time, we obtain the constraint equation

$$\varphi(p, q) := q^T p = 0. \tag{40}$$

A second differentiation with respect to time yields the constraint equation

$$A_p\dot{p} = b_p \tag{41}$$

where $A_p = q^T$ and $b_p = -\frac{1}{m}p^T p$.

The matrix $H_{pp}^{-1} = mI_3$.

(iii) *Dynamics of constrained Hamiltonian system*

Substituting in Eq. (36) now yields the equation of motion of the spherical pendulum as

$$\dot{q} = \frac{p}{m}, \text{ and}$$
$$\dot{p} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} - \frac{m}{L^2}\left(\frac{1}{m^2}p^T p + gz\right)q. \tag{42}$$

4.2 Particle with nonholonomic constraint

(i) *Unconstrained Hamiltonian System*

Consider a free particle with coordinates $q = [x, y, z]^T$ in an inertial frame of reference. Its Hamiltonian is

$$H = \frac{1}{2m}p^T p \tag{43}$$

where $p = [p_x, p_y, p_z]^T$ is its momentum.

(ii) *Description of Constraints*

Let the particle be constrained so that

$$\varphi(p, q) := z^2 p_x - p_y = 0. \tag{44}$$

Differentiation of this equation with respect to time yields

$$[z^2, -1, 0]\dot{p} = -\frac{2zp_x p_z}{m} \tag{45}$$

so that $A_p = [z^2, -1, 0]$, and $b_p = -2zp_x p_z/m$. From Eq. (43), $H_{pp}^{-1} = mI_3$.

(iii) *Dynamics of Constrained Hamiltonian System*

Using Eq. (36), the equations of motion of the nonholonomically constrained particle are obtained as

$$\dot{q} = \frac{p}{m}, \text{ and}$$
$$\dot{p} = -\frac{2zp_x p_z}{m(1+z^4)}\begin{bmatrix} z^2 \\ -1 \\ 0 \end{bmatrix}. \tag{46}$$

4.3 System of two particles on an inclined plane subjected to functionally dependent holonomic and nonholonomic constraints

Consider two-point particles with masses m_1 and m_2 that move on a frictionless plane inclined at an angle α to the horizontal. The particles are connected by a massless (rigid) rod of length L, and each mass is constrained so that its velocity lies in the plane and is always perpendicular to the line going through them. See Fig. 1. This model is motivated by the problem of describing the motion of two thin circular discs (wheels) whose centers are connected by a light axle of length L. The explicit equations of motion for this system will be obtained.

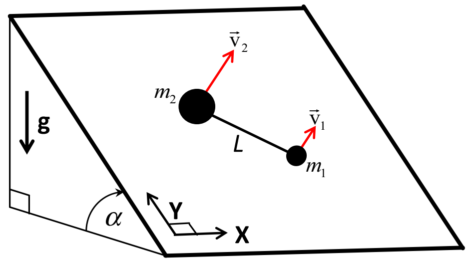

Fig. 1 Two-point particles of masses m_1 and m_2 moving on a frictionless inclined plane. The particles are always to be at a constant distance L from one another, and their velocities $\vec{v}_1$ and $\vec{v}_2$ are always required to be perpendicular to the line joining them

(i) *Unconstrained Hamiltonian System*

The unconstrained Hamiltonian system will comprise of the two-point masses on the inclined plane, each subjected to the force of gravity. For convenience, an inertial frame whose X- and Y-axes lie in the plane of motion and whose Z-axis is normal to the plane of motion is used (see Fig. 1). The Hamiltonian of the unconstrained two-dimensional motion of the system in the XY plane is given by

$$H = \sum_{i=1}^{2}\left[\frac{p_{x_i}^2}{2m_i} + \frac{p_{y_i}^2}{2m_i}\right] + g(m_1 y_1 + m_2 y_2)\sin\alpha \tag{47}$$

where p_{x_i} and p_{y_i} are the momenta in the X- and Y-directions, respectively, of mass m_i. Hamilton's equations for this unconstrained system are then

$$\dot{x} := \begin{bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = \left[\frac{p_{x_1}}{m_1}, \frac{p_{y_1}}{m_1}, \frac{p_{x_2}}{m_2}, \frac{p_{y_2}}{m_2}\right]^T,$$
$$\dot{p} := \begin{bmatrix} \dot{p}_{x_1} \\ \dot{p}_{y_1} \\ \dot{p}_{x_2} \\ \dot{p}_{y_2} \end{bmatrix} = -g\sin\alpha\begin{bmatrix} 0 \\ m_1 \\ 0 \\ m_2 \end{bmatrix}. \tag{48}$$

(ii) *Description of Constraints*

The constraints on the system are described by the following three equations:

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = L^2 \tag{49}$$

$$\frac{(x_1 - x_2)}{m_1} p_{x_1} + \frac{(y_1 - y_2)}{m_1} p_{y_1} = 0 \tag{50}$$

$$\frac{(x_1 - x_2)}{m_2} p_{x_2} + \frac{(y_1 - y_2)}{m_2} p_{y_2} = 0. \tag{51}$$

The first of these equations constrains the distance between the two masses to equal L, while the remaining two ensure that the velocity of each mass is always perpendicular to the line going through the two of them.

Differentiating Eq. (49) twice with respect to time and the other two once with respect to time, we obtain the relations

$$(x_1 - x_2)\left(\frac{\dot{p}_{x_1}}{m_1} - \frac{\dot{p}_{x_2}}{m_2}\right) + (y_1 - y_2)\left(\frac{\dot{p}_{y_1}}{m_1} - \frac{\dot{p}_{y_2}}{m_2}\right) = -\left(\frac{p_{x_1}}{m_1} - \frac{p_{x_2}}{m_2}\right)^2 - \left(\frac{p_{y_1}}{m_1} - \frac{p_{y_2}}{m_2}\right)^2 \tag{52}$$

$$\frac{(x_1 - x_2)}{m_1} \dot{p}_{x_1} + \frac{(y_1 - y_2)}{m_1} \dot{p}_{y_1} = \frac{(p_{x_1} p_{x_2} + p_{y_1} p_{y_2})}{m_1 m_2} - \frac{p_{x_1}^2 + p_{y_1}^2}{m_1^2} \tag{53}$$

$$\frac{(x_1 - x_2)}{m_2} \dot{p}_{x_2} + \frac{(y_1 - y_2)}{m_2} \dot{p}_{y_2} = \frac{p_{x_2}^2 + p_{y_2}^2}{m_2^2} - \frac{(p_{x_1} p_{x_2} + p_{y_1} p_{y_2})}{m_1 m_2}, \tag{54}$$

so that

$$A_p = \begin{bmatrix} a_1 & b_1 & -a_2 & -b_2 \\ a_1 & b_1 & 0 & 0 \\ 0 & 0 & a_2 & b_2 \end{bmatrix} \text{ and }$$

$$b_p = \begin{bmatrix} \frac{2p_{12}}{m_1 m_2} - \frac{p_1^2}{m_1^2} - \frac{p_2^2}{m_2^2} \\ \frac{p_{12}}{m_1 m_2} - \frac{p_1^2}{m_1^2} \\ \frac{p_2^2}{m_2^2} - \frac{p_{12}}{m_1 m_2} \end{bmatrix} \tag{55}$$

where

$$a_i = (x_1 - x_2)/m_i, b_i = (y_1 - y_2)/m_i, \tag{56}$$

$$p_i^2 = p_{x_i}^2 + p_{y_i}^2, i = 1, 2, \text{ and} \tag{57}$$

$$p_{12} = p_{x_1} p_{x_2} + p_{y_1} p_{y_2}. \tag{58}$$

One observes that the three constraints given in Eqs. (52)–(54) are not independent since the three rows of the matrix A_p are indeed not independent. We might have inferred this from a physical point of view since the requirement at each instant of time that the velocities of the two masses always be perpendicular to the line joining them necessitates that the distance between the two masses remain a constant, L—the same distance that the two masses started out with at time $t = 0$. In what follows, we continue to use the three functionally dependent constraints to demonstrate that Eq. (36) is valid even when the constraints are functionally dependent. While in this example this dependence could have been inferred from the start from an understanding of the physics, in complex mechanical systems where there may be many constraints, such dependencies are often difficult to decipher, especially when several nonholonomic constraints are involved. It is important to notice that were the Lagrange multiplier approach used to obtain the constrained equations of motion, it would run into difficulties when the constraints are functionally dependent.

Noting relation (48), the 4-by-4 block diagonal matrix $H_{pp} = Diag\left[\frac{I_2}{m_1}, \frac{I_2}{m_2}\right]$, where I_2 is the 2-by-2 identity matrix.

(iii) *Dynamics of constrained Hamiltonian system*

Using Eq. (36), we assemble the various elements on the right-hand side of its second member:

$$(A_p H_{pp}^{-1} A_p^T)^+ = \frac{1}{9L^2} \begin{bmatrix} m_1 + m_2 & 2m_1 - m_2 & m_1 - 2m_2 \\ 2m_1 - m_2 & 4m_1 + m_2 & 2(m_1 + m_2) \\ m_1 - 2m_2 & 2(m_1 + m_2) & m_1 + 4m_2 \end{bmatrix}, \tag{59}$$

$$H_{pp}^{-1} A_p^T (A_p H_{pp}^{-1} A_p^T)^+ = \frac{1}{3L^2} \begin{bmatrix} m_1(x_1 - x_2) & 2m_1(x_1 - x_2) & m_1(x_1 - x_2) \\ m_1(y_1 - y_2) & 2m_1(y_1 - y_2) & m_1(y_1 - y_2) \\ m_2(x_2 - x_1) & m_2(x_1 - x_2) & 2m_2(x_1 - x_2) \\ m_2(y_2 - y_1) & m_2(y_1 - y_2) & 2m_2(y_1 - y_2) \end{bmatrix} \tag{60}$$

and

$$b_p + A_p \frac{\partial H}{\partial q} = \begin{bmatrix} \frac{2p_{12}}{m_1 m_2} - \frac{p_1^2}{m_1^2} - \frac{p_2^2}{m_2^2} \\ \frac{p_{12}}{m_1 m_2} - \frac{p_1^2}{m_1^2} + (y_1 - y_2) g \sin\alpha \\ \frac{p_2^2}{m_2^2} - \frac{p_{12}}{m_1 m_2} + (y_1 - y_2) g \sin\alpha \end{bmatrix}. \tag{61}$$

The equation of motion of the constrained system is then given by the relations

$$\dot{x} = \left[\frac{p_{x_1}}{m_1}, \frac{p_{y_1}}{m_1}, \frac{p_{x_2}}{m_2}, \frac{p_{y_2}}{m_2}\right]^T, \tag{62}$$

and

$$\dot{p} = -g\sin\alpha \begin{bmatrix} 0 \\ m_1 \\ 0 \\ m_2 \end{bmatrix} + \frac{1}{3L^2}\begin{bmatrix} m_1(x_1-x_2) & 2m_1(x_1-x_2) & m_1(x_1-x_2) \\ m_1(y_1-y_2) & 2m_1(y_1-y_2) & m_1(y_1-y_2) \\ m_2(x_2-x_1) & m_2(x_1-x_2) & 2m_2(x_1-x_2) \\ m_2(y_2-y_1) & m_2(y_1-y_2) & 2m_2(y_1-y_2) \end{bmatrix} \times \begin{bmatrix} \frac{2p_{12}}{m_1m_2} - \frac{p_1^2}{m_1^2} - \frac{p_2^2}{m_2^2} \\ \frac{p_{12}}{m_1m_2} - \frac{p_1^2}{m_1^2} + (y_1-y_2)g\sin\alpha \\ \frac{p_2^2}{m_2^2} - \frac{p_{12}}{m_1m_2} + (y_1-y_2)g\sin\alpha \end{bmatrix}. \quad (63)$$

Simplifying Eq. (63), the equations of motion of the constrained Hamiltonian system become

$$\dot{x} := \left[\frac{p_{x_1}}{m_1}, \frac{p_{y_1}}{m_1}, \frac{p_{x_2}}{m_2}, \frac{p_{y_2}}{m_2}\right]^T, \text{ and}$$

$$\dot{p} = -g\sin\alpha\,[0, m_1, 0, m_2]^T + \frac{1}{m_1m_2L^2}\left[(x_1-x_2)f_1, (y_1-y_2)f_1, (x_1-x_2)f_2, (y_1-y_2)f_2\right]^T \quad (64)$$

where

$$f_1 = [m_1p_{12} - m_2p_1^2 + m_1^2m_2g(y_1-y_2)\sin\alpha] \quad (65)$$

and

$$f_2 = [m_1p_2^2 - m_2p_{12} + m_1m_2^2g(y_1-y_2)\sin\alpha]. \quad (66)$$

The second member on the right-hand sides of Eqs. (63) and (64) explicitly gives the contribution to $\dot{p}$ caused by the presence of the constraints given in Eqs. (49)–(51).

While not obvious from these equations, it can be shown with a little algebra that for a given set of initial conditions, the motion of the system is independent of the values of the masses m_1 and m_2, perhaps a somewhat nonintuitive result.

4.4 Constrained double pendulum

Consider the double pendulum shown in Fig. 2. The arms of the pendulum are of lengths L_1 and L_2. The point masses at the end of each arm have values m_1 and m_2, and the two arms of the pendulum are assumed to be massless. The upper mass is m_1. The system is Hamiltonian. The upper arm of length L_1 makes an angle θ_1 with the vertical, and the lower arm makes an angle θ_2 as shown in Fig. 2. The aim is to determine the explicit equation of motion of the system in the presence of the nonholonomic constraint $p_1(t) = \alpha p_2(t)\sin(\omega t)$ where $p_1(t)$ and $p_2(t)$ are the canonical momenta conjugate to the coordinates θ_1 and θ_2. The constants $\omega \neq 0$ and $\alpha \neq 0$ are scalars.

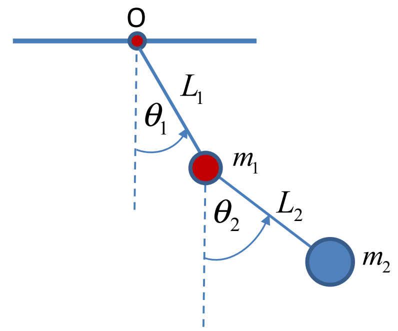

Fig. 2 A double pendulum with point masses m_1 and m_2. The datum for measuring the potential energy is the horizontal going through the point of support, O

(i) *Unconstrained Hamiltonian System*

From the kinetic energy, T, and the potential energy, V, of the unconstrained double pendulum system (see Fig. 2), its Hamiltonian (energy) can be obtained as

$$H = T + V = \frac{1}{2}\frac{(m_1+m_2)l_1^2p_2^2 + m_2l_2^2p_1^2 - 2m_2l_1l_2p_1p_2\cos\theta_{12}}{m_2l_1^2l_2^2\Delta} - m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2) \quad (67)$$

where $\Delta = m_1 + m_2\sin^2(\theta_{12})$, and $\theta_{12} = \theta_1 - \theta_2$. The canonical momenta are given by

$$p := \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} (m_1+m_2)l_1^2 & m_2l_1l_2\cos\theta_{12} \\ m_2l_1l_2\cos\theta_{12} & m_2l_2^2 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}. \quad (68)$$

We note that the matrix on the right-hand side of Eq. (68) is H_{pp}^{-1} (see Eqs. (5) and (23)).

Hamilton's equations of motion describing the unconstrained system are then

$$\dot{\theta} := \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \frac{\partial H}{\partial p} = \begin{bmatrix} \frac{l_2p_1 - l_1p_2\cos(\theta_{12})}{l_1^2l_2\Delta} \\ \frac{(m_1+m_2)l_1p_2 - m_2l_2p_1\cos(\theta_{12})}{m_2l_1l_2^2\Delta} \end{bmatrix} \quad (69)$$

$$\dot{p} := \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = -\frac{\partial H}{\partial \theta} = \begin{bmatrix} \Sigma - (m_1+m_2)l_1g\sin\theta_1 \\ -\Sigma - m_2l_2g\sin\theta_2 \end{bmatrix} \quad (70)$$

where

$$\Sigma(p,\theta) = \frac{m_2 l_1 l_2 T \sin(2\theta_{12}) - p_1 p_2 \sin(\theta_{12})}{l_1 l_2 \Delta} \quad (71)$$

and the kinetic energy T is given by the first member on the right-hand side in Eq. (67).

(ii) *Description of constraints*

Differentiating the constraint, $p_1(t) = \alpha p_2(t) \sin(\omega t)$, with respect to time, we obtain the relation

$$\dot{p}_1 - \alpha \dot{p}_2 \sin(\omega t) = \omega\alpha \cos(\omega t) p_2 \quad (72)$$

so that $A_p = [1 - \alpha \sin(\omega t)]$ and $b_p = \omega\alpha \cos(\omega t) p_2$.

(iii) *Dynamics of constrained Hamiltonian system*

The second equation in Eq. (36) that describes the constrained system can be written simply as

$$\begin{aligned}\dot{p} &= -\frac{\partial H}{\partial \theta} + \frac{\left(b_p + A_p \frac{\partial H}{\partial \theta}\right)}{A_p H_{pp}^{-1} A_p^T} H_{pp}^{-1} A_p^T \\ &= -\frac{\partial H}{\partial \theta} + \frac{\gamma_2}{\gamma_1} H_{pp}^{-1} A_p^T. \end{aligned} \quad (73)$$

This is because A_p is a row vector, and hence, the matrix $\gamma_1 := (A_p H_{pp}^{-1} A_p^T)$ is a scalar whose generalized inverse is just its reciprocal (see Remark 1). Also, $\gamma_2 := b_p + A_p \frac{\partial H}{\partial \theta}$ is a scalar that can be readily computed. Furthermore, the matrix $H_{pp}^{-1} A_p^T$ is simply the matrix shown in Eq. (68) multiplied by the column vector A_p^T.

Using Eq. (70), the constrained motion of the double pendulum is then described by the equations

$$\begin{aligned}\dot{\theta} &= \begin{bmatrix} \frac{l_2 p_1 - l_1 p_2 \cos(\theta_{12})}{l_1^2 l_2 \Delta} \\ \frac{(m_1+m_2) l_1 p_2 - m_2 l_2 p_1 \cos(\theta_{12})}{m_2 l_1 l_2^2 \Delta} \end{bmatrix} \\ \dot{p} &:= \begin{bmatrix} \Sigma - (m_1+m_2) l_1 g \sin\theta_1 \\ -\Sigma - m_2 l_2 g \sin\theta_2 \end{bmatrix} \\ &\quad + \frac{\gamma_2}{\gamma_1} \begin{bmatrix} (m_1+m_2) l_1^2 - \alpha m_2 l_1 l_2 \cos\theta_{12} \sin(\omega t) \\ m_2 l_1 l_2 \cos\theta_{12} - \alpha m_2 l_2^2 \sin(\omega t) \end{bmatrix} \end{aligned} \quad (74)$$

where

$$\begin{aligned}\gamma_1 &= (m_1+m_2) l_1^2 - 2\alpha m_2 l_1 l_2 \cos(\theta_{12}) \sin(\omega t) \\ &\quad + \alpha^2 m_2 l_2^2 \sin^2(\omega t) \end{aligned} \quad (75)$$

and

$$\begin{aligned}\gamma_2 &= \alpha\omega p_2 \cos(\omega t) - [1 + \alpha \sin(\omega t)]\Sigma + (m_1+m_2) \\ &\quad l_1 g \sin(\theta_1) - \alpha \sin(\omega t) m_2 l_2 g \sin(\theta_2). \end{aligned} \quad (76)$$

5 Conclusions

This paper gives the explicit equations of motion for general Hamiltonian systems subjected to holonomic and/or nonholonomic constraints of the form $\varphi_i(p,q,t) = 0, i = 1, 2, \ldots, m$. These equations do not seem to be available as of now in the current literature. The constraints can be nonlinear in their arguments and functionally dependent. Starting with an unconstrained Hamiltonian system, the central connection between virtual displacements and the Hamiltonian formulation is first established. From this, the explicit equations of motion for the constrained system are then developed. A simple and systematic three-step approach is presented for getting these equations.

For purposes of illustration, four model examples employing these new equations are provided. The examples cover holonomic and/or nonholonomic constraints. The ease with which the equations of the constrained system can be obtained is demonstrated, and the three-step approach is clearly identified in each example. Functionally dependent constraints as well as time-dependent constraints are included.

It is important to point out that the entire approach presented herein is innocent of the notion of Lagrange multipliers, a notion that is usually employed to find the equations of motion of constrained mechanical systems [5,10]. The present approach has the advantage that it gives explicit equations of motion of constrained systems, (i) even when the constraints are nonlinear functions of the arguments and hence are not restricted to so-called Pfaffian forms, and (ii) even when they are functionally dependent. The latter situation commonly arises in modeling complex mechanical systems that may require several nonholonomic constraints where functional independence can become difficult to establish.

The explicit equations describing constrained motion that are obtained in this paper for systems whose unconstrained dynamics are described by Hamiltonian's equations are the analogs of those obtained earlier for systems whose unconstrained motion is described by Lagrange's equations [1] and by Poincare's equations [11,12].

The usefulness of the explicit equations of motion for constrained mechanical systems goes beyond the field of analytical dynamics and has significant implications in areas like nonlinear control. This is because the placement of control requirements on nonlinear, nonautonomous dynamical systems can very often be regarded as the imposition of a set of constraints on them. Use of these explicit equations for constrained motion then provides a simple means of finding, in closed form, the (generalized) control forces required to exactly satisfy the control requirements placed on nonlinear nonautonomous dynamical systems. In addition, it can be shown that these control forces can be made to simultaneously minimize a suitable norm of the control cost at each instant of time (see, for example, Refs. [13,14]).

Compliance with ethical standards

Conflicts of interest The author declares that he has no conflict of interest.

Research involving Human Participants and/or Animals The research carried out in this article does not involve any human participants or animals.

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