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Analytical dynamics with constraint forces that do work in virtual displacements

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Abstract

Lagrangian mechanics is extended to cover situations in which constraint forces are permitted to do work on a system in virtual displacements. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Much of analytical dynamics is studied under the assumption that the constraint forces do no work on the system in any virtual displacement. Yet, it is clear that constraint forces such as friction *do* do work on the system in virtual displacements. The purpose of this paper is to show how such constraint forces may be taken into account in a systematic and convenient manner. A new physical principle which generalizes the principle of virtual work must be adduced.

Let the configuration of the system be described by the n generalized coordinates $q_1, q_2, \ldots, q_n$. Then the Lagrange equations of motion may be written as

$$M\ddot{q} = Q, \tag{1}$$

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where M is a positive definite symmetric matrix having dimensions n by n and depending on t and q. The generalized force vector Q is a function of t, q and $\dot{q}$, and is n-dimensional. The generalized acceleration vector $\ddot{q}$ is

$$\ddot{q} = (\ddot{q}_1, \ddot{q}_2, \ldots, \ddot{q}_n)^{\mathrm{T}}, \tag{2}$$

and, of course,

$$q = (q_1, q_2, \ldots, q_n)^{\mathrm{T}}.$$

The n-dimensional vector $\dot{q}$ is the generalized velocity vector.

Next assume that m constraints are specified of the form

$$\varphi_j(t, q, \dot{q}) = 0, \quad j = 1, 2, \ldots, m. \tag{3}$$

Then differentiation of both sides of these equations with respect to t yields the differentiated form of the constraint equation

$$A\ddot{q} = b, \tag{4}$$

where A, a function of t, q and $\dot{q}$, is m by n, and b, also a function of t, q, and $\dot{q}$, is an m-dimensional vector. Holonomic constraints are differentiated twice and nonholonomic constraints are differentiated once with respect to time to obtain Eq. (4). Together with initial conditions at some time these are equivalent to Eq. (3). We assume that the rank of A is $r \leqslant m$. The number of equality constraints m need not be less than n. We assume, though, that they are consistent.

The actual equation of motion then assumes the form

$$M\ddot{q} = Q + Q^c, \tag{5}$$

where Q^c is the generalized constraint force vector called into existence to maintain the constraints but do no work on the system in a virtual displacement. In Eqs. (4) and (5) there are $2n$ unknowns, the components of $\ddot{q}$ and Q^c, but there are only $n + r$ prescribed conditions. The needed $n - r$ additional linear relations are provided, conventionally, by the principle of virtual work, which requires that the constraint force Q^c do no work in a virtual displacement. This amounts to requiring that

$$v^{\mathrm{T}} Q^c = 0 \tag{6}$$

for all virtual displacements vectors v, of dimension n, for which

$$Av = 0.$$

The solution of the system of Eqs. (4)–(6) is [1]

$$\ddot{q} = a + M^{-1/2}(AM^{-1/2})^{+}(b - Aa), \tag{7}$$

where $a = M^{-1}Q$. The n-dimensional vector a is the generalized acceleration of the system that is free of constraints. Eq. (7) may be written as

$$M\ddot{q} = Q + M^{1/2}(AM^{-1/2})^{+}(b - Aa), \tag{8}$$

so that

$$Q^c = Q^{\mathrm{N}} = M^{-1/2}(AM^{1/2})^{+}(b - Aa), \tag{9}$$

is the generalized constraint force called into play to maintain the constraints while doing no work on the system in any virtual displacement.

2. Constraints forces that do work on the system

Now let us consider the incorporation of constraint forces which may do work on the system in a virtual displacement—friction is an example. We denote the n-dimensional vector of such forces as $c = c(t, q, \dot{q})$. In *any* displacement v such forces do the work $v^{\mathrm{T}}c$. Therefore, in a virtual displacement v, for which $Av = 0$, the actual constraint force, Q^c, will have to do the same amount of work, so that

$$v^{\mathrm{T}}Q^c = v^{\mathrm{T}}c, \tag{10}$$

which is a *new physical principle*. It generalizes Eq. (6), the principle of virtual work. Under the new principle the work done by the constraint forces in a virtual displacement may be positive or negative as well as zero.

The vectors $\ddot{q}$ and Q^c are thus to be determined by the relations

$$M\ddot{q} = Q + Q^c, \tag{11}$$

$$A\ddot{q} = b, \tag{12}$$

and

$$v^{\mathrm{T}}Q^c = v^{\mathrm{T}}c \quad (v \text{ such that } Av = 0). \tag{13}$$

The task now is to solve these equations for $\ddot{q}$ and Q^c. (It can be shown that a solution exists and is unique.) Needed background concerning pseudoinverses is available in [1, Chapter 2].

3. Derivation of the new explicit equation of motion

We introduce

$$B = AM^{-1/2}, \tag{14}$$

$$\ddot{r} = M^{1/2}\ddot{q}, \tag{15}$$

so that Eq. (12) becomes

$$B\ddot{r} = b. \tag{16}$$

The general solution of this equation is

$$\ddot{r} = B^{+}b + (I - B^{+}B)w, \tag{17}$$

where B^{+} is the n by m pseudoinverse of the matrix B, and w is an arbitrary n-dimensional vector. We wish to determine this vector. The equation $Av = 0$ can be rewritten as

$$Bu = 0, \tag{18}$$

where $u = M^{1/2}v$. It follows, then, from Eqs. (13) and (11), that for all u such that $Bu = 0$,

$$u^{\mathrm{T}}M^{-1/2}[M\ddot{q} - Q] = u^{\mathrm{T}}M^{-1/2}c. \tag{19}$$

But this equation can be rewritten as

$$u^{\mathrm{T}}M^{-1/2}[M\ddot{q} - Q - c] = 0, \tag{20}$$

or

$$u^{\mathrm{T}}[\ddot{r} - \overline{Q} - \overline{c}] = 0, \tag{21}$$

where

$$M^{-1/2}Q = \overline{Q}, \tag{22}$$

and

$$M^{-1/2}c = \overline{c}. \tag{23}$$

The general solution of Eq. (18) has the form

$$u = (I - B^{+}B)z, \tag{24}$$

where z is an arbitrary n-dimensional vector. Eq. (21) becomes

$$z^{\mathrm{T}}(I - B^{+}B)\left[B^{+}b + (I - B^{+}B)w - \overline{Q} - \overline{c}\right] = 0, \tag{25}$$

where we have used Eq. (17). It follows that

$$z^{\mathrm{T}}(I - B^{+}B)\left[w - \overline{Q} - \overline{c}\right] = 0. \tag{26}$$

From this it follows that the vector

$$w - \overline{Q} - \overline{c} = B^{+}z_1, \tag{27}$$

where z_1 is again arbitrary. Upon substituting the value $w = \overline{Q} + \overline{c} + B^{+}z_1$ into Eq. (17) we find

$$\ddot{r} = B^{+}b + (I - B^{+}B)(\overline{Q} + \overline{c}). \tag{28}$$

From this it follows that

$$M^{1/2}\ddot{q} = B^{+}b + (I - B^{+}B)(M^{-1/2}Q + M^{-1/2}c),$$

$$\ddot{q} = M^{-1/2}B^{+}b + M^{-1}Q - M^{-1/2}B^{+}BM^{-1/2}Q + M^{-1/2}(I - B^{+}B)M^{-1/2}c. \tag{29}$$

Finally, we have the desired equation of motion

$$\ddot{q} = a + M^{-1/2}B^{+}(b - Aa) + M^{-1/2}(I - B^{+}B)M^{-1/2}c, \tag{30}$$

which may be rewritten in the form

$$M\ddot{q} = Q + M^{1/2}B^{+}(b - Aa) + M^{1/2}(I - B^{+}B)M^{-1/2}c. \tag{31}$$

The first two terms on the right side have been seen before in Eq. (8). They are the applied force Q and the constraint force Q^{N} that does no work in any virtual displacement but maintains the constraints, $A\ddot{q} = b$. The third term is occasioned by the specified constraint force c that does work in a virtual displacement. We may denote it by Q^{W}, so that Eq. (31) assumes the elegant form

$$M\ddot{q} = Q + Q^{\mathrm{N}} + Q^{\mathrm{W}}, \tag{32}$$

with

$$Q^{\mathrm{W}} = M^{1/2}(I - B^{+}B)M^{-1/2}c, \tag{33}$$

and

$$Q^{\mathrm{N}} = M^{1/2}B^{+}(b - Aa). \tag{33$'$}$$

Thus the constraint force Q^{c} has two components and may be written

$$Q^{c} = Q^{\mathrm{N}} + Q^{\mathrm{W}}, \tag{34}$$

where the forms of Q^{N} and Q^{W} are as shown above.

Equation (31) is the explicit equation of motion for mechanical systems that have constraint forces that may do work on the system in a virtual displacement. Equation (30) is the form that is used for numerical integration.

4. An example

Consider a bead of unit mass that moves along a vertical circle of unit radius under the influence of gravity. We introduce the rectangular coordinates x_1 and x_2. Then

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} 0 \\ -g \end{pmatrix}.$$

The constraint equation $x_1^2 + x_2^2 = 1$ leads, through two differentiations with respect to t, to the linear constraint on the acceleration components $x_1\ddot{x}_1 + x_2\ddot{x}_2 = -(\dot{x}_1^2 + \dot{x}_2^2)$. From this we see that $A = (x_1 \quad x_2)$, a 1 by 2 matrix and $b = -(\dot{x}_1^2 + \dot{x}_2^2)$, a scalar. If no friction is present, the equation of motion is simply

$$\ddot{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\cdot\cdot} = a + A^+(b - Aa). \tag{35}$$

Since $A = (x_1 \quad x_2)$, its pseudoinverse, A^+, is

$$A^+ = \frac{1}{x_1^2 + x_2^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{36}$$

The equation of motion is thus

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\cdot\cdot} = \begin{pmatrix} 0 \\ -g \end{pmatrix} + \frac{1}{x_1^2 + x_2^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\left[gx_2 - (\dot{x}_1^2 + \dot{x}_2^2)\right] = \begin{pmatrix} 0 \\ -g \end{pmatrix} + Q^{\mathrm{N}}, \tag{37}$$

since

$$-Aa = -(x_1 \quad x_2)\begin{pmatrix} 0 \\ -g \end{pmatrix} = +gx_2.$$

Let us assume that the force of friction, the usual Coulomb friction force, c, is specified as being "proportional to the normal thrust". This means that

$$c = -\mu\frac{1}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2}}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}|Q^{\mathrm{N}}|, \tag{38}$$

where, according to Eq. (37), the normal thrust, Q^{N}, is given by

$$Q^{\mathrm{N}} = \frac{1}{x_1^2 + x_2^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\left[gx_2 - (\dot{x}_1^2 + \dot{x}_2^2)\right]. \tag{39}$$

The constant μ is the coefficient of friction. If follows from Eq. (30) that the equation of motion, including friction, is

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{\cdot\cdot} = \begin{pmatrix} 0 \\ -g \end{pmatrix} + \frac{1}{x_1^2 + x_2^2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\left[gx_2 - (\dot{x}_1^2 + \dot{x}_2^2)\right] \\ + \left\{I - \frac{1}{x_1^2 + x_2^2}\begin{pmatrix} x_1x_1 & x_1x_2 \\ x_2x_1 & x_2x_2 \end{pmatrix}\right\}c,\end{aligned} \tag{40}$$

where c is specified in Eq. (38).

In view of the fact that one differentiation of the constraint $x_1^2 + x_2^2 = 1$ gives $x_1\dot{x}_1 + x_2\dot{x}_2 = 0$, we see that the last term on the right in Eq. (40) simplifies to $\{I - O\}c = c$, in this special case. In general, Eq. (33) shows that if the vector

$M^{-1/2}c$ lies in the null space of the matrix B, then $Q^{\mathrm{W}} = c$; if $M^{-1/2}c$ lies in the range space of B^{T}, then $Q^{\mathrm{W}} = 0$.

5. Discussion

A simple and comprehensive method for incorporating into Lagrangian mechanics the effects of constraint forces that may do work on a system in a virtual displacement has been given. First, a generalized form of the principle of virtual work was given in Section 2. Then the new equations of motion were given in Section 3, and an example was adumbrated in Section 4. Since modern computing environments, such as MATLAB, contain commands for obtaining pseudoinverses of matrices, numerical experiments are now under way, and results will be reported subsequently.

Reference

[1] F. Udwadia, R. Kalaba, in: Analytical Dynamics, Cambridge University Press, New York, 1996.