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An Alternate Proof for the Equation of Motion for Constrained Mechanical Systems

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ABSTRACT

The newly developed equation of motion for constrained mechanical systems are derived using D'Alembert's principle.

Consider an unconstrained nonlinear mechanical system consisting of n particles described in a rectangular inertial frame of reference by the equation

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t). \tag{1}$$

Here the $3n$-vector of position is denoted by the vector $\mathbf{x}$, and M is a constant, positive definite, diagonal $3n$ by $3n$ matrix. The dots refer to differentiation with respect to time. By unconstrained we mean that the number of degrees of freedom of the system equals $3n$. The "given force" $\mathbf{F}$ and the matrix M are assumed known.

Let the mechanical system, in addition, be constrained so that it satisfies the m consistent equations given by

$$\varphi_i(\mathbf{x}, \dot{\mathbf{x}}, t) = 0, \qquad i = 1, 2, \ldots, m, \tag{2}$$

which need not be functionally independent. On differentiating equations

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(2) we obtain the equation

$$A(\mathbf{x}, \dot{\mathbf{x}}, t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{3}$$

where the elements of the m by $3n$ matrix A as well as those of the m-vector $\mathbf{b}$ are known functions of $\mathbf{x}, \dot{\mathbf{x}}$, and t.

Given the set of initial conditions $\mathbf{x}(t_0)$ and $\dot{\mathbf{x}}(t_0)$ which satisfy the constraint equations (2), the equation of motion of the constrained mechanical system (described jointly by equations (1) and (3)) can now be explicitly expressed as [1]

$$M\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t) + M^{1/2}C^{+}(\mathbf{b} - A\mathbf{a}), \tag{4}$$

where the $3n$-vector $\mathbf{a} = M^{-1}F(\mathbf{x}, \dot{\mathbf{x}}, t)$ and the matrix $C = A(\mathbf{x}, \dot{\mathbf{x}}, t)M^{-1/2}$. The superscript "+" denotes the Moore–Penrose inverse of the matrix C. We note that the vector $\mathbf{a}$ is simply the acceleration corresponding to the unconstrained system as obtained by using equation (1).

This equation was obtained in [1] by appealing to Gauss' Principle. In this paper we present an alternate, and perhaps simpler, proof of this result by starting from D'Alembert's principle.

We shall find it convenient to recast equation (4) by premultiplying it by the matrix $M^{-1/2}$ to yield the equation

$$\ddot{\mathbf{x}}_s = \mathbf{a}_s + C^{+}(\mathbf{b} - C\mathbf{a}_s), \tag{5}$$

where the "scaled accelerations" $\ddot{\mathbf{x}}_s$ and $\mathbf{a}_s$ are defined, respectively, by the relations $\ddot{\mathbf{x}}_s = M^{1/2}\ddot{\mathbf{x}}$ and $\mathbf{a}_s = M^{1/2}\mathbf{a}$. It is this equation (5) which we now set out to prove. For ease of notation we will from here on drop the arguments of the vectors and the matrices.

By virtue of the presence of the constraints described by equation set (2), forces of constraint will brought into play and the equation of motion of the constrained system will therefore become

$$M\ddot{\mathbf{x}} = \mathbf{F} + \mathbf{F}^{c}, \tag{6}$$

where $\mathbf{F}^{c}$ is the force of constraint.

Multiplying (6) by $M^{-1/2}$ we get the equation

$$\ddot{\mathbf{x}}_s = \mathbf{a}_s + M^{-1/2}\mathbf{F}^{c}. \tag{7}$$

Similarly, noting that $C = AM^{-1/2}$, we can express equation (3) as

$$C\ddot{\mathbf{x}}_s = \mathbf{b}. \tag{8}$$

Substituting for $\ddot{\mathbf{x}}_s$ from (7) in (8) we get

$$C(M^{-1/2}\mathbf{F}^{c}) = \mathbf{b} - C\mathbf{a}_s, \tag{9}$$

which is a linear equation for the $3n$-vector $M^{-1/2}\mathbf{F}^{c}$. The general solution to this linear equation is

$$M^{-1/2}\mathbf{F}^{c} = C^{+}(\mathbf{b} - C\mathbf{a}_s) + (I - C^{+}C)\mathbf{h}, \tag{10}$$

where the $3n$-vector $\mathbf{h}$ is arbitrary.

According to D'Alembert's principle we assume that for any (nonzero) virtual "displacement" vector $\mathbf{v}$, the work done by the force of constraint $\mathbf{F}^{c}$ equals zero. But given the constraints $A\ddot{\mathbf{x}} = \mathbf{b}$, a virtual displacement vector $\mathbf{v}$ is defined as any nonzero (infinitesimal) vector which satisfies the equation

$$A\mathbf{v} = \mathbf{0}. \tag{11}$$

Hence by D'Alembert's principle we require that for all vectors $\mathbf{v}$ such that $A\mathbf{v} = 0$ we must have $\mathbf{v}^T\mathbf{F}^{c} = 0$. Relation (11) can be written in turn as $C\mathbf{u} = \mathbf{0}$ with the vector $\mathbf{u}$ defined as

$$\mathbf{v} = M^{-1/2}\mathbf{u}. \tag{12}$$

Hence D'Alembert's principle demands that for all (nonzero) vectors $\mathbf{u}$ such that $C\mathbf{u} = \mathbf{0}$ we must have $\mathbf{v}^T\mathbf{F}^{c} = \mathbf{u}^T M^{-1/2}\mathbf{F}^{c} = 0$. Using the expression for the $3n$-vector $M^{-1/2}\mathbf{F}^{c}$ obtained in equation (10), we get

$$\mathbf{u}^T(M^{-1/2}\mathbf{F}^{c}) = \mathbf{u}^T C^{+}(\mathbf{b} - C\mathbf{a}_s) + \mathbf{u}^T(I - C^{+}C)\mathbf{h}. \tag{13}$$

Also, $C\mathbf{u} = \mathbf{0}$ is equivalent to $\mathbf{u}^T C^{+} = \mathbf{0}$. D'Alembert's principle then amounts to requiring that for all (nonzero) vectors $\mathbf{u}$ for which $\mathbf{u}^T C^{+} = \mathbf{0}$, the right-hand side of (13) must be zero. But the right-hand side of (13) reduces to $\mathbf{u}^T\mathbf{h}$ when $\mathbf{u}$ is such that $\mathbf{u}^T C^{+} = \mathbf{0}$. Hence we must have $\mathbf{u}^T\mathbf{h} = 0$ for all vectors $\mathbf{u}$ which satisfy the relation $\mathbf{u}^T C^{+} = \mathbf{0}$.

This requires that the vector $\mathbf{h}$ belong to the column space of the matrix C^{+} and therefore that it can always be expressed as

$$\mathbf{h} = C^{+}\mathbf{w} \tag{14}$$

for a suitable m-vector $\mathbf{w}$. Using (14) in (10) yields

$$M^{-1/2}\mathbf{F}^{c} = C^{+}(\mathbf{b} - C\mathbf{a}_s) + \mathbf{u}^T(I - C^{+}C)C^{+}\mathbf{w} = C^{+}(\mathbf{b} - C\mathbf{a}_s), \tag{15}$$

where we have used the property $C^+CC^+ = C^+$ of the Moore–Penrose inverse in obtaining the last expression on the right-hand side of (15). Using this expression for the $3n$-vector $M^{-1/2}\mathbf{F}^c$ in the right-hand side of equation (7), we obtain equation (5), and our proof is complete.

REFERENCES

1 F. E. Udwadia and R. E. Kalaba, A new perspective on constrained motion, *Proc. Roy. Soc. London* 439:407–410 (1992).