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A Unified Approach for the Recursive Determination of Generalized Inverses

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Abstract—Using a unified approach, simple derivations for the recursive determination of different types of generalized inverses of a matrix are presented. These include results for the generalized inverse, the least-squares generalized inverse, the minimum-norm generalized inverse, and the Moore-Penrose inverse of a matrix.

Keywords—Generalized inverses, Recursive determination, Unified approach, Dynamic programming.

1. INTRODUCTION

The recursive determination of a generalized inverse of a matrix finds extensive applications in the fields of statistical inference [1-3], filtering theory, estimation theory [4], and system identification [5]. More recently, generalized inverses have found renewed applicability in the field of analytical dynamics [6,7]. The reason for the extensive applicability of recursive relations is that they provide a systematic method to generate 'updates' whenever sequential addition of data or new information becomes available, and updated estimates which take into account this additional information are required.

The recursive scheme for the computation of the Moore-Penrose (MP) inverse [8,9] of a matrix was ingeniously obtained in a famous paper by Greville in 1960 [2]. Because of its extensive applicability, Greville's result is widely stated in almost every book that touches on the subject of generalized inverses of matrices. Yet, because of the complexity of his solution technique, Greville's proof is seldom, if ever, quoted or outlined, even in specialized texts which deal solely with generalized inverses of matrices. For example, in books like [4,10-12], Greville's result is stated, but no constructive proof is provided, most likely because of its perceived complexity.

In the same vein, Mitra and Bhimasankaram [13] provide several results for the recursive determination of generalized inverses of matrices; they state their results as several *Ansatze* and prove them by directly verifying their validity using a number of specialized results related to generalized inverses of matrices. Their results are equivalent to those presented here. However, they provide no constructive proofs for their results and their proofs for the various types of gen-

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eralized inverses have no underlying reasoning or thread running through them; only verifications of the various *Ansatze* are carried out.

In this paper, we present a simple constructive approach inspired in part by Bellman's optimality principle, to the recursive determination of various generalized inverses of a matrix. The approach relies on a unified underlying theme and shows clearly why and how the differences in the recursive forms of the various generalized inverses arise. Thus, our results encompass those of Greville [2], and our method of proof, being constructive, provides deeper insights into the nature of the recursive determination of generalized inverses.

For convenience, we introduce the following notation. Given a real matrix A, its MP-inverse G satisfies the following four conditions:

(1) $AGA = A$,
(2) $GAG = G$,
(3) AG is symmetric, and
(4) GA is symmetric.

We shall denote a matrix G which satisfies all four of these conditions by $A^{\{1,2,3,4\}}$. Similarly, a matrix which satisfies only the first and fourth condition above shall be denoted as $A^{\{1,4\}}$ and shall be referred to as the {1,4}-inverse of A, etc.

The most commonly used generalized inverses of a matrix are the MP-inverse (also denoted here as the {1,2,3,4}-inverse), the {1,3}-inverse, the {1,4}-inverse, and the {1}-inverse because these inverses are relevant to the solution x of the matrix equation $Ax = b$ or of the relation $Ax \approx b$. We shall begin by *defining* these generalized inverses (as in [12]) in terms of the relevant linear relations which they help solve. The MP-inverse provides the minimum-length solution $x = A^{\{1,2,3,4\}}b$ in the set of least-squares solutions of the possibly inconsistent equation $Ax \approx b$ for any b, the {1,3}-inverse provides a least-squares solution $A^{\{1,3\}}b$ to the possibly inconsistent equation $Ax \approx b$ for any b, the {1,4}-inverse provides a minimum-length solution $A^{\{1,4\}}b$ for any b for which the equation is consistent, and the {1}-inverse of A provides a solution $A^{\{1\}}b$ for any b for which the equation $Ax = b$ is consistent. This paper is concerned with these four commonly used generalized inverses defined above, which we shall denote, in general, by A^*. The solution x is then expressed, in general, as A^*b. Their generalized forms are given in [14].

Given a real m by k matrix A_k, one can partition it as $[A_{k-1}\ a]$ where A_{k-1} consists of the first $(k-1)$ columns of the matrix A_k and a is its last column. The column vector a comprises 'new' or additional information, while the matrix A_{k-1} comprises accumulated past data. The generalized inverse A_k^* of the updated matrix A_k is then sought in terms of the generalized inverse A_{k-1}^* of the matrix A_{k-1} which corresponds to past accumulated data, and the vector a containing new or additional information. The MP-inverse of a matrix A is unique. The other generalized inverses which we shall deal with here are not in general unique, and so, in what follows, by say $A_{k-1}^{\{1,4\}}$, we shall mean any one of the set of {1,4}-inverses of the matrix A_{k-1}.

2. MAIN RESULT

Let $A_k = [A_{k-1}\ a]$ be an m by k matrix whose last column is a. Let the m-vector $c = (I - A_{k-1}A_{k-1}^*)a$ and let the m-vector $d = (A_{k-1}^*)^\top A_{k-1}^* a/(1 + a^\top (A_{k-1}^*)^\top A_{k-1}^* a)$. Then,

$$A_k^* = \begin{bmatrix} A_{k-1}^* - A_{k-1}^* a u^\top \\ u^\top \end{bmatrix}, \qquad \text{for } c \neq 0 \tag{1a}$$

and

$$A_k^* = \begin{bmatrix} A_{k-1}^* - A_{k-1}^* a \nu^\top \\ \nu^\top \end{bmatrix}, \qquad \text{for } c = 0, \tag{1b}$$

when we have the following.

Part 1: $* = \{1,2,3,4\}$, $\quad u^\top = \dfrac{c^\top}{c^\top c}$, $\quad$ and $\nu = d$. (2)

Part 2: $* = \{1,3\}$, $\quad u^\top = \dfrac{c^\top}{c^\top c}$, $\quad$ and $\nu =$ any arbitrary m-vector q.(3)

Part 3: $* = \{1,4\}$, $\quad u^\top = \dfrac{c^\top\left(I - A_{k-1}A_{k-1}^*\right)}{c^\top c}$, $\quad$ and $\nu = d$. (4)

Part 4: $* = \{1\}$, $\quad u^\top = \dfrac{c^\top\left(I - A_{k-1}A_{k-1}^*\right)}{c^\top c}$, $\quad$ and $\nu =$ any arbitrary m-vector q.(5)

From equations (1a) and (1b), notice that the form of the inverse $A_{k-1}^*$is the same for $c = 0$ and for $c \neq 0$. We have used separate equations here only for convenience.

PROOF OF PART 1. We consider the solution of the least squares problem

$$A_k x = \begin{bmatrix} A_{k-1} & a \end{bmatrix} \begin{bmatrix} z \\ s \end{bmatrix} = A_{k-1}z + as \approx b, \tag{6}$$

where we have partitioned the vector x into the $(k-1)$-vector z and the scalar s. To determine the minimum-length-least-squares solution x of $A_k x \approx b$, we consider all those pairs (z, s) which minimize $J(z, s) = \|A_{k-1}z + as - b\|_2^2$, and from these pairs select the one whose length $z^\top z + s^2$ is a minimum.

We begin by setting $s = s_o$, where s_o is some fixed scalar. Thus, we have

$$J(z, s_o) = \|A_{k-1}z - (b - as_o)\|_2^2. \tag{7}$$

Minimizing (7) such that $\hat{z}^\top \hat{z}$ is also a minimum from among all $(k-1)$-vectors z, we obtain, from the definition of the MP-inverse,

$$\hat{z}(s_o) = A_{k-1}^{\{1,2,3,4\}}(b - as_o). \tag{8}$$

Thus, for a given value of the scalar s_o, the $(k-1)$-vector $\hat{z}$ is a function of s_o. Using equation (8) in equation (7), we can now find s_o such that

$$\begin{aligned} J(\hat{z}(s_o), s_o) &= \left\| A_{k-1}A_{k-1}^{\{1,2,3,4\}}(b - as_o) + as_o - b \right\|_2^2 \\ &= \left\| \left(I - A_{k-1}A_{k-1}^{\{1,2,3,4\}}\right) as_o - \left(I - A_{k-1}A_{k-1}^{\{1,2,3,4\}}\right) b \right\|_2^2 \end{aligned} \tag{9}$$

is a minimum. Depending on $c = (I - A_{k-1}A_{k-1}^{\{1,2,3,4\}})a$, we must now deal with two cases: the first when $c \neq 0$; the second when $c = 0$. The first case occurs when a does not lie in the column space of A_{k-1}; the second, when the vector a lies in the column space of the matrix A_{k-1}.

(i) For $c \neq 0$, the unique value of s_o which minimizes (9) is given by

$$\begin{aligned} \tilde{s}_o &= \frac{a^\top \left(I - A_{k-1}A_{k-1}^{\{1,2,3,4\}}\right)\left(I - A_{k-1}A_{k-1}^{\{1,2,3,4\}}\right) b}{c^\top c} \\ &= \frac{c^\top}{c^\top c} b = c^{\{1,2,3,4\}} b = u^\top b, \end{aligned} \tag{10}$$

where in the first equality, we have used the fact that the matrix $(I - A_{k-1}A_{k-1}^{\{1,2,3,4\}})$ is symmetric, and in the second equality, that it is idempotent. Having found the unique value $\tilde{s}_o$ which minimizes (9), we now obtain from (8), the minimum-length-least-squares solution of $A_k x \approx b$ as

$$x = A_k^{\{1,2,3,4\}} b = \begin{bmatrix} \hat{z}(\tilde{s}_o) \\ \tilde{s}_o \end{bmatrix} = \begin{bmatrix} A_{k-1}^{\{1,2,3,4\}} - A_{k-1}^{\{1,2,3,4\}} a u^\top \\ u^\top \end{bmatrix} b, \tag{11}$$

where the first equality follows by the definition of the MP-inverse. Hence,

$$A_k^{\{1,2,3,4\}} = \begin{bmatrix} A_{k-1}^{\{1,2,3,4\}} - A_{k-1}^{\{1,2,3,4\}} a u^\top \\ u^\top \end{bmatrix}, \qquad \text{for } c \neq 0. \tag{12}$$

(ii) For $c = 0$, we observe from equation (9) that $J(\hat{z}(s_o), s_o)$ is not a function of s_o. Thus, we only need to minimize $J_1(s_o) = \hat{z}(s_o)^\top \hat{z}(s_o) + s_o^2$ over all values of s_o, where $\hat{z}(s_o)$ is given in equation (8). For convenience, we can write this as

$$J_1(s_o) = s_o^2 + s_o^2 p_1^\top p_1 - 2 s_o p_1^\top p_2 + p_2^\top p_2, \tag{13}$$

where we have denoted $p_1 = A_{k-1}^{\{1,2,3,4\}} a$ and $p_2 = A_{k-1}^{\{1,2,3,4\}} b$. The scalar value $\tilde{s}_o$, which minimizes $J_1(s_o)$ is then given by

$$\tilde{s}_o = \frac{p_1^\top p_2}{\left(1 + p_1^\top p_1\right)} = \frac{a^\top \left(A_{k-1}^{\{1,2,3,4\}}\right)^\top A_{k-1}^{\{1,2,3,4\}}}{1 + a^\top \left(A_{k-1}^{\{1,2,3,4\}}\right)^\top A_{k-1}^{\{1,2,3,4\}} a} b = \nu^\top b. \tag{14}$$

Using equation (8) to obtain $\hat{z}(\tilde{s}_o)$, we get

$$x = A_k^{\{1,2,3,4\}} b = \begin{bmatrix} \hat{z}(\tilde{s}_o) \\ \tilde{s}_o \end{bmatrix} = \begin{bmatrix} A_{k-1}^{\{1,2,3,4\}} - A_{k-1}^{\{1,2,3,4\}} a \nu^\top \\ \nu^\top \end{bmatrix} \cdot b, \tag{15}$$

from which it follows, as before, that

$$A_k^{\{1,2,3.4\}} = \begin{bmatrix} A_{k-1}^{\{1,2,3,4\}} - A_{k-1}^{\{1,2,3,4\}} a \nu^\top \\ \nu^\top \end{bmatrix}, \qquad \text{for } c = 0. \tag{16}$$

This proves the first part of our result given in (2). ∎

PROOF OF PART 2. The {1,3}-inverse provides a least-squares solution $x = A_k^{\{1,3\}} b$ to the equation $A_k x \approx b$. As before, we partition the vector x into a $(k-1)$-vector z and a scalar s. For a fixed s_o, we minimize

$$J(z, s_o) = \|A_{k-1} z - (b - a s_o)\|_2^2 \tag{17}$$

to yield

$$\hat{z}(s_o) = A_{k-1}^{\{1,3\}} (b - a s_o), \tag{18}$$

for some {1,3}-inverse of the m by $(k-1)$ matrix A_{k-1}. We next minimize

$$\begin{aligned} J(\hat{z}(s_o), s_o) &= \left\| A_{k-1} A_{k-1}^{\{1,3\}} (b - a s_o) + a s_o - b \right\|_2^2 \\ &= \left\| \left(I - A_{k-1} A_{k-1}^{\{1,3,\}}\right) a s_o - \left(I - A_{k-1} A_{k-1}^{\{1,3,\}}\right) b \right\|_2^2, \end{aligned} \tag{19}$$

with respect to s_o and again need to consider two cases: when $c \neq 0$ and where $c = 0$.

(i) For $c \neq 0$, we obtain the unique value of s_o which minimizes (19) to be

$$\tilde{s}_o = \frac{a^\top \left(I - A_{k-1} A_{k-1}^{\{1,3\}}\right)\left(I - A_{k-1} A_{k-1}^{\{1,3\}}\right)}{c^\top c} b = u^\top b, \tag{20}$$

where we have again used the fact that the matrix $(I - A_{k-1} A_{k-1}^{\{1,3\}})$ is symmetric and idempotent. Following the same sort of steps as before, we then obtain the required result for this case.

(ii) For $c = 0$, equation (19) shows that $J(\hat{z}(s_o) s_o)$ is not a function of s_o; hence, the choice of s_o is arbitrary. If we let $s_o = q^\top b$, where q is any arbitrary m-vector, we obtain statement (3) of our result for this case. The matrix $A_k^{\{1,3\}}$ so obtained is not unique. Moreover, once a {1,3}-inverse is obtained through the use of equations (1a), (1b), and (3), other {1,3}-inverses may be generated by adding to this {1,3}-inverse any matrix R such that $A_k R = 0$. ∎

PROOF OF PART 3. The {1,4}-inverse provides a minimum length solution $x = A_k^{\{1,4\}} b$ to the consistent equation $A_k x = b$. Again we partition the vector x into a $(k-1)$-vector z and a scalar s. For a fixed s_o, we solve the consistent equation

$$A_{k-1} z = b - a s_o, \tag{21}$$

yielding the minimum length solution

$$\hat{z}(s_o) = A_{k-1}^{\{1,4\}} (b - a s_o) \tag{22}$$

for some {1,4}-inverse of A_{k-1}. We now use equation (22) in equation (21) so that

$$\left(I - A_{k-1} A_{k-1}^{\{1,4\}}\right) a s_o = \left(I - A_{k-1} A_{k-1}^{\{1,4\}}\right) b, \tag{23}$$

obtaining the two cases $c = (I - A_{k-1} A_{k-1}^{\{1,4\}}) a \neq 0$, and $c = (I - A_{k-1} A_{k-1}^{\{1,4\}}) a = 0$ as before.

(i) For $c \neq 0$, the solution of the consistent equation (23) yields

$$s_o = \frac{c^\top \left(I - A_{k-1} A_{k-1}^{\{1,4\}}\right)}{c^\top c} b = u^\top b, \tag{24}$$

from which the result for this case follows on using equation (22).

(ii) For $c = 0$, both the left- and the right-hand sides of equation (23) are zero. We then need to find s_o so as to minimize $J_1(s_o) = \hat{z}(s_o)^\top \hat{z}(s_o) + s_o^2$, where $\hat{z}(s_o)$ is given by equation (22). Setting $p_1 = A_{k-1}^{\{1,4\}} a$ and $p_2 = A_{k-1}^{\{1,4\}} b$, and following the reasoning in Part 1, Case (ii), we obtain the result given in (4).

We note in passing that the vector b must lie in the range space of the matrix A_k. Using the {1,4}-inverse obtained from equation (1a), (1b), and (4), others can be obtained by adding to this {1,4}-inverse any matrix L which satisfies the relation $LA_k = 0$. (See [14].) ∎

PROOF OF PART 4. The {1}-inverse provides a solution $x = A_k^{\{1\}} b$ to the consistent equation $A_k x = b$. Partitioning the vector x as before, for a fixed $s = s_o$, we obtain equation (21), whose solution now is given by

$$\hat{z}(s_o) = A_{k-1}^{\{1\}} (b - a s_o), \tag{25}$$

for some {1}-inverse of A_{k-1}.

Using this result in (21), we obtain

$$\left(I - A_{k-1} A_{k-1}^{\{1\}}\right) a s_o = \left(I - A_{k-1} A_{k-1}^{\{1\}}\right) b. \tag{26}$$

When $c = (I - A_{k-1} A_{k-1}^{\{1\}}) a \neq 0$, the unique solution of (26) is

$$s_o = \frac{c^\top \left(I - A_{k-1} A_{k-1}^{\{1\}}\right)}{c^\top c} b = u^\top b. \tag{27}$$

From this, result (5) follows for this case. When $c = 0$, as in Part 3, both the left- and right-hand sides of the consistent equation (26) are zero; hence, s_o can be arbitrary. We can then choose s_o to equal $q^\top b$ where q is any arbitrary m-vector yielding the result provided for this case in (5).

Again as in Parts 2 and 3 above, the {1}-inverse obtained from equations (1a), (1b), and (5) can be used to obtain other {1}-inverses by adding to this {1}-inverse any matrix $(L + R)$ where L is such that $LA_k = 0$ and R is such that $A_k R = 0$. (See [14].) ∎

3. CONCLUSIONS

In this paper, we present a unified approach for obtaining recursive relations for several of the commonly used generalized inverses of an m by k matrix A. Part 1 of our main result which deals with the MP-inverse was obtained by Greville [2], but in a more complex manner; our proof of this part is substantially simpler. The unifying theme used in this paper is brought about by defining generalized inverses in terms of the solution(s) $x = A^*b$ of the matrix equation $Ax = b$ or of the relation $Ax \approx b$, and then using a procedure akin to dynamic programming. This results in similar lines of reasoning for obtaining recursive relations for the various types of generalized inverses, while providing insight into why and how the differences among them arise.

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