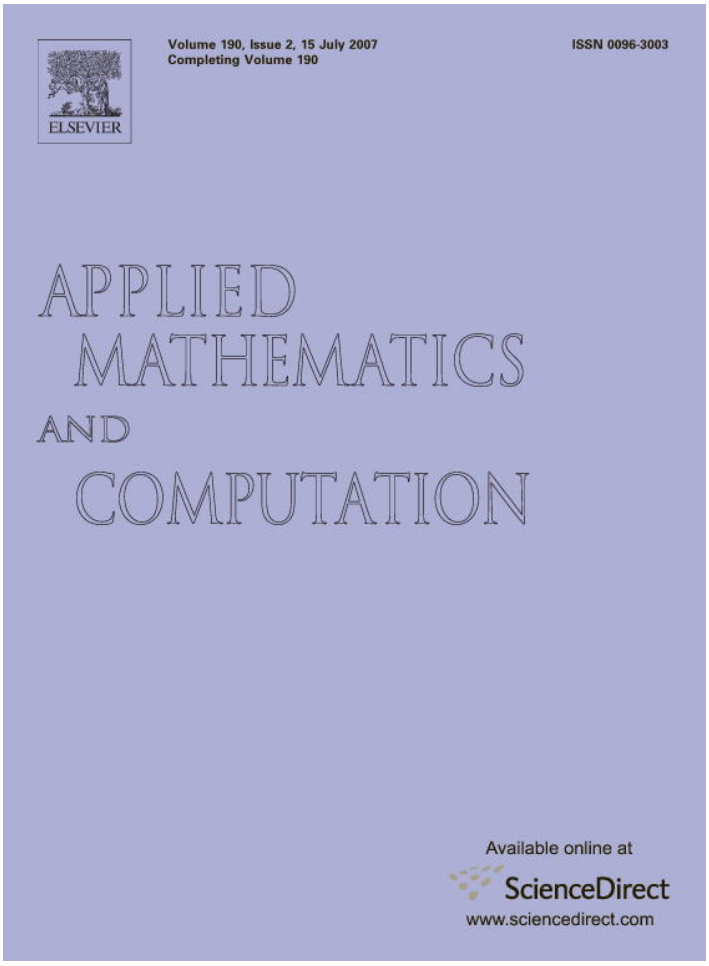
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Generalized LM-inverse of a matrix augmented by a column vector

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Abstract

This paper presents a direct proof of the recursive formulae for the generalized LM- inverse of a matrix augmented by a column vector. The recursive relations are proved by direct verification of the four conditions of the generalized LM-inverse. Several auxiliary results pertaining to generalized inverses are also provided.

Keywords: Generalized inverse; Moore–Penrose M-Inverse; Moore–Penrose LM-Inverse; Recursive formulae; Least squares problem

1. Introduction

Let us begin by considering a set of linear equations

$$Bx = b, \tag{1}$$

where B is an m by n matrix, b is an m-vector, and x is an n-vector.

The generalized LM-inverse of the matrix B is the matrix such that the solution

$$x = B^+_{LM} b \tag{2}$$

minimizes both

$$G = \left\|L^{1/2}(Bx - b)\right\|^2 = \|Bx - b\|^2_L \tag{3}$$

and

$$H = \left\|M^{1/2}x\right\|^2 = \|x\|^2_M, \tag{4}$$

where L is an m by m symmetric positive definite matrix and M is an n by n symmetric positive definite matrix.

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Below are the four conditions for the generalized LM-inverse [1].

$$\text{(i)}\quad BB^+_{LM}B = B, \tag{5}$$

$$\text{(ii)}\quad B^+_{LM}BB^+_{LM} = B^+_{LM}, \tag{6}$$

$$\text{(iii)}\quad LBB^+_{LM} \text{ is symmetric}, \tag{7}$$

$$\text{(iv)}\quad MB^+_{LM}B \text{ is symmetric}. \tag{8}$$

We note that the generalized LM-inverse is the more general kind of the Moore–Penrose inverse. The concept of Moore–Penrose (MP) inverses was first introduced by Moore [2] in 1920 and later independently by Penrose [3] in 1955. In 1960, Greville [4] gave the first formulae for recursively determining the Moore–Penrose inverse of a matrix. His algorithm provides an update of the MP inverse of a matrix whenever new information becomes available. As a result, the recursive formulae have found extensive use in many areas of applications. Among them are statistical inference [5], filtering theory, estimation theory [6], system identification [7], optimization and control, and most recently analytical dynamics [8,9]. In 1997 Udwadia and Kalaba [10] provided an alternative and simple constructive proof of Greville's formulae, and later [11,12] developed recursive relations for different types of generalized inverses of a matrix including the least-squares generalized inverse, the minimum-norm generalized inverse, and the Moore–Penrose (MP) inverse of a matrix.

Recently, the recursive formulae for the generalized M-inverse [13,14] and for the generalized LM-inverse were obtained. Those for the generalized LM-inverse were proved constructively [15]. In this paper, we provide a much simpler and alternative proof for the recursive formulae of the generalized LM-inverse, B^+_{LM}, of any given matrix, B, partitioned as $B = [A \mid a]$, where A is an m by $n-1$ matrix and a is a column vector of m components. We show that the four conditions of the generalized LM-inverse of the recursive formulae are satisfied. Besides its inherent simplicity, our proof requires several subsidiary properties of the generalized LM-inverse of a matrix, many of which appear to be hereto unknown; they are presented in the Appendix. More general than the generalized M-inverse, the generalized LM-inverse finds use in an even wider range of application areas than the Moore–Penrose inverse – areas ranging from system theory, statistics, filtering, control theory, and optimization, to signal processing and mechanics.

2. Recursive formulae of the generalized *LM*-inverse of a matrix augmented by a column vector

2.1. Result

For any given matrix

$$B = [A \mid a] \tag{9}$$

its generalized LM-inverse formulae are given by

$$B^+_{LM} = [A \mid a]^+_{LM} = \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-}ad^+_L - pd^+_L \\ d^+_L \end{bmatrix}, \quad \text{when } d = (I - AA^+_{LM_-})a \neq 0, \tag{10}$$

$$= \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-}ah - ph \\ h \end{bmatrix}, \quad \text{when } d = (I - AA^+_{LM_-})a = 0, \tag{11}$$

where A is an m by $(n-1)$ matrix, a is a column vector of m components, $d^+_L = d^{\mathrm{T}}L/(d^{\mathrm{T}}Ld)$, $h = \frac{1}{\beta}q^{\mathrm{T}}MU$, $\beta = q^{\mathrm{T}}Mq$, $U = \begin{bmatrix} A^+_{LM_-} \\ 0_m \end{bmatrix}$, $q = \begin{bmatrix} v+p \\ -1 \end{bmatrix}$, $p = (I - A^+_{LM_-}A)M_-^{-1}\tilde{m}$, and $v = A^+_{M_-}a$. Note that L is a symmetric positive definite m by m matrix, and

$$M = \begin{bmatrix} M_- & \tilde{m} \\ \tilde{m}^{\mathrm{T}} & \bar{m} \end{bmatrix}, \tag{12}$$

where M is a symmetric positive definite n by n matrix, M_- is the symmetric positive definite $(n-1)$ by $(n-1)$, $\tilde{m}$ is the column vector of $(n-1)$ components, and $\bar{m}$ is the scalar.

It should be noted that the formulae are given in two separate cases; when $d \neq 0$ and when $d = 0$. When $d \neq 0$, the added column vector a is not a linear combination of the columns of A, and when $d = 0$, the added column vector a is a linear combination of the columns of A (see Appendix in Ref. [15] for a proof).

Proof. *Case* 1: (when $d \neq 0$)

Because the BB_M^+ and B_M^+B are repetitively used to verify all four properties of the generalized LM-inverse, we shall first evaluate BB_M^+ and B_M^+B. By Eqs. (9) and (10), we have

$$BB_{LM}^+ = [A \quad | \quad a]\begin{bmatrix} A_{LM\_}^+ - A_{LM\_}^+ad_L^+ - pd_L^+ \\ d_L^+ \end{bmatrix} = AA_{LM\_}^+ - AA_{LM\_}^+ad_L^+ - (Ap)d_L^+ + ad_L^+. \tag{13}$$

Since $Ap = 0$ (see Property 1 in Appendix) and $d = (I - AA_{LM\_}^+)a$, we get

$$BB_{LM}^+ = AA_{LM\_}^+ - AA_{LM\_}^+ad_L^+ + ad_L^+ = AA_{LM\_}^+ + (I - AA_{LM\_}^+)ad_L^+ = AA_{LM\_}^+ + dd_L^+. \tag{14}$$

Again by Eqs. (9) and (10), we obtain

$$B_{LM}^+B = \begin{bmatrix} A_{LM\_}^+ - A_{LM\_}^+ad_L^+ - pd_L^+ \\ d_L^+ \end{bmatrix}[A \quad | \quad a] \tag{15}$$

$$= \left[\begin{array}{c|c} A_{LM\_}^+A - A_{LM\_}^+a(d_L^+A) - p(d_L^+A) & A_{LM\_}^+a - A_{LM\_}^+a(d_L^+a) - p(d_L^+a) \\ d_L^+A & d_L^+a \end{array}\right]. \tag{16}$$

Using the relations $d_L^+A = 0$ and $d_L^+a = 1$ (see Properties 4 and 5 in Appendix), we have

$$B_{LM}^+B = \left[\begin{array}{c|c} A_{LM\_}^+A & A_{LM\_}^+a - A_{LM\_}^+a - p \\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} A_{LM\_}^+A & -p \\ 0 & 1 \end{array}\right]. \tag{17}$$

We now verify the four properties of the generalized LM-inverse.

Generalized LM-inverse condition 1: $BB_{LM}^+B = B$

Using Eqs. (9) and (14), we obtain

$$BB_{LM}^+B = (BB_{LM}^+)B = (AA_{LM\_}^+ + dd_L^+)[A \quad | \quad a] = [AA_{LM\_}^+A + d(d_L^+A) \quad | \quad AA_{LM\_}^+a + d(d_L^+a)]. \tag{18}$$

Because $AA_{LM\_}^+A = A$, $d_L^+A = 0$, $d_L^+a = 1$ (see Properties 4 and 5 in Appendix), and $d = (I - AA_{LM\_}^+)a$, we have

$$BB_{LM}^+B = [A \quad | \quad AA_{LM\_}^+a + (I - AA_{LM\_}^+)a] = [A \quad | \quad a] = B.$$

Generalized LM-inverse condition 2: $B_{LM}^+BB_{LM}^+ = B_{LM}^+$

Using Eqs. (11) and (14), we get

$$B_{LM}^+BB_{LM}^+ = B_{LM}^+(BB_{LM}^+) = \begin{bmatrix} A_{LM\_}^+ - A_{LM\_}^+ad_L^+ - pd_L^+ \\ d_L^+ \end{bmatrix}[AA_{LM\_}^+ + dd_L^+], \tag{19}$$

$$= \begin{bmatrix} A_{LM\_}^+AA_{LM\_}^+ + (A_{LM\_}^+d)d_L^+ - A_{LM\_}^+a(d_L^+A)A_{LM\_}^+ - A_{LM\_}^+ad_L^+dd_L^+ - p(d_L^+A)A_{LM\_}^+ - pd_L^+dd_L^+ \\ (d_L^+A)A_{LM\_}^+ + d_L^+dd_L^+ \end{bmatrix}. \tag{20}$$

Since $A_{LM\_}^+AA_{LM\_}^+ = A_{LM\_}^+$, $A_{LM\_}^+d = 0$, $d_L^+A = 0$ (see Properties 3 and 4 in Appendix), and $d_L^+dd_L^+ = d_L^+$, we obtain

$$B_{LM}^+BB_{LM}^+ = \begin{bmatrix} A_{LM\_}^+ - A_{LM\_}^+ad_L^+ - pd_L^+ \\ d_L^+ \end{bmatrix} = B_{LM}^+.$$

Generalized LM-inverse condition 3: $L(BB_{LM}^+)$ is symmetric

Since $LAA_{LM\_}^+$ and Ldd_L^+ are symmetric, using Eq. (14), we have

$$L(BB_{LM}^+) = L(AA_{LM\_}^+ + dd_L^+) = LAA_{LM\_}^+ + Ldd_L^+,$$

which is symmetric.

Generalized LM-inverse condition 4: $M(B^+_{LM}B)$ is symmetric

Using Eqs. (12) and (17), we obtain

$$M(B^+_{LM}B) = \begin{bmatrix} M_- & \tilde{m} \\ \tilde{m}^{\mathrm{T}} & \bar{m} \end{bmatrix} \begin{bmatrix} A^+_{LM_-}A & -p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} M_-A^+_{LM_-}A & -M_-p+\tilde{m} \\ \tilde{m}^{\mathrm{T}}A^+_{LM_-}A & -\tilde{m}^{\mathrm{T}}p+\bar{m} \end{bmatrix} = \begin{bmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{bmatrix}, \tag{21}$$

where $E_{1,1}$, $E_{1,2}$, $E_{2,1}$, and$E_{2,2}$ represent the elements (1,1), (1,2), (2,1), and (2,2) of the matrix $M(B^+_{LM}B)$, respectively. Note that $E_{1,1}$ is the $(n-1)$ by $(n-1)$ matrix, $E_{1,2}$ is the column vector of $(n-1)$ components, $E_{2,1}$ is the row vector of $(n-1)$ components, and $E_{2,2}$ is the scalar. We see that $E_{1,1} = M_-(A^+_{LM_-}A)$ is symmetric since $A^+_{LM_-}$ is the generalized LM_--inverse of A, while $E_{2,2}$ is a scalar, and is therefore symmetric. Thus, for $M(B^+_{LM}B)$ to be symmetric, we need to show that $E_{1,2}$ is the transpose of $E_{2,1}$.

Using $p = (I - A^+_{LM_-}A)M_-^{-1}\tilde{m}$, the element $E_{1,2}$ can be written as

$$-M_-p + \tilde{m} = -M_-(I - A^+_{LM_-}A)M_-^{-1}\tilde{m} + \tilde{m} = M_-(A^+_{LM_-}A)M_-^{-1}\tilde{m} = (A^+_{LM_-}A)^{\mathrm{T}}\tilde{m}, \tag{22}$$

which is the transpose of the element $E_{2,1}$. Hence, $M(B^+_{LM}B)$ is symmetric.

We have shown that all four generalized LM-inverse conditions are satisfied. Hence, the formula (10) is verified. □

Case 2: (when $d = 0$)

We begin again by evaluating BB^+_{LM} and $B^+_{LM}B$ quantities that we will need further along. Using Eqs. (9) and (11), we have

$$BB^+_{LM} = [A \mid a]\begin{bmatrix} A^+_{LM_-} - A^+_{LM_-}ah - ph \\ h \end{bmatrix} = [AA^+_{LM_-} - (AA^+_{LM_-}a)h - (Ap)h + ah]. \tag{23}$$

Since $AA^+_{LM_-}a = a$ and $Ap = 0$ (see Properties 1 and 2 in Appendix), we get

$$BB^+_{LM} = [AA^+_{LM_-} - ah + ah] = AA^+_{LM_-}. \tag{24}$$

Using Eqs. (9) and (11), we have

$$B^+_{LM}B = \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-}ah - ph \\ h \end{bmatrix}[A \mid a] = \begin{bmatrix} A^+_{LM_-}A - A^+_{LM_-}ahA - phA & A^+_{LM_-}a - A^+_{LM_-}aha - pha \\ hA & ha \end{bmatrix}. \tag{25}$$

We next verify the four properties of the Generalized LM-inverse.

Generalized LM-inverse condition 1: $BB^+_{LM}B = B$

Using the fact that $AA^+_{LM_-}A = A$ and $AA^+_{LM_-}a = a$ (see Property 2 in Appendix), by Eqs. (9) and (24) we obtain

$$BB^+_{LM}B = (BB^+_{LM})B = AA^+_{LM_-}[A \mid a] = [AA^+_{LM_-}A \mid AA^+_{LM_-}a] = [A \mid a] = B.$$

Generalized LM-inverse condition 2: $B^+_{LM}BB^+_{LM} = B^+_{LM}$

By Eqs. (11) and (24), we have

$$B^+_{LM}BB^+_{LM} = B^+_{LM}(BB^+_{LM}) = \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-}ah - ph \\ h \end{bmatrix}[AA^+_{LM_-}] = \begin{bmatrix} A^+_{LM_-}AA^+_{LM_-} - A^+_{LM_-}a(hAA^+_{LM_-}) - p(hAA^+_{LM_-}) \\ hAA^+_{LM_-} \end{bmatrix}. \tag{26}$$

Since $A^+_{LM_-}AA^+_{LM_-} = A^+_{LM_-}$and $hAA^+_{LM_-} = h$ (see Property 9 in Appendix), we have

$$B^+_{LM}BB^+_{LM} = \begin{bmatrix} A^+_{LM_-} - A^+_{LM_-}ah - ph \\ h \end{bmatrix} = B^+_{LM}.$$

Generalized LM-inverse condition 3: $L(BB^+_{LM})$ is symmetric

Because $LAA^+_{LM\_}$ is symmetric, by Eq. (24) we have

$$(BB^+_{LM})^{\mathrm{T}} = (AA^+_{LM\_})^{\mathrm{T}} = LAA^+_{LM\_}L^{-1} = LBB^+_{LM}L^{-1}.$$

Generalized LM-inverse condition 4: $M(B^+_{LM}B)$ is symmetric.

Using Eqs. (12), (25), and $v = A^+_{LM\_}a$, we obtain

$$\begin{aligned} M(B^+_{LM}B) &= \left[\begin{array}{c|c} M_- & \tilde{m} \\ \tilde{m}^{\mathrm{T}} & \bar{m} \end{array}\right]\left[\begin{array}{c|c} A^+_{LM\_}A - vhA - phA & v - vha - pha \\ hA & ha \end{array}\right], \\ &= \left[\begin{array}{c|c} M_-(A^+_{LM\_}A - vhA - phA) + \tilde{m}(hA) & M_-(v - vha - pha) + \tilde{m}(ha) \\ \tilde{m}^{\mathrm{T}}(A^+_{LM\_}A - vhA - phA) + \bar{m}(hA) & \tilde{m}^{\mathrm{T}}(v - vha - pha) + \bar{m}(ha) \end{array}\right] = \begin{bmatrix} E_{1,1} & E_{1,2} \\ E_{2,1} & E_{2,2} \end{bmatrix}, \end{aligned} \tag{27}$$

where $E_{1,1}$, $E_{1,2}$, $E_{2,1}$, and $E_{2,2}$ represent the elements (1,1), (1,2), (2,1), and (2,2) of the matrix $M(B^+_{LM}B)$, respectively. Note that $E_{1,1}$ is the $(n-1)$ by $(n-1)$ square matrix, $E_{1,2}$ is the column vector of $(n-1)$ components, $E_{2,1}$ is the row vector of $(n-1)$ components, and $E_{2,2}$ is the scalar. For $M(B^+_{LM}B)$ to be symmetric, we need to show that $E_{1,1}$ is symmetric and $E_{1,2}$ is the transpose of $E_{2,1}$.

Let us first show that $E_{1,1}$ is symmetric. We can rewrite $E_{1,1}$ as

$$E_{1,1} = M_-A^+_{LM\_}A - (M_-v + M_-p - \tilde{m})(hA). \tag{28}$$

Since $M_-v + M_-p - \tilde{m} = \beta(hA)^{\mathrm{T}}$ (see Property 12 in Appendix), we get

$$E_{1,1} = M_-A^+_{LM\_}A - \beta(hA)^{\mathrm{T}}(hA). \tag{29}$$

Because $M_-A^+_{LM\_}A$ and $\beta(hA)^{\mathrm{T}}(hA)$ are symmetric, $E_{1,1}$ is symmetric.

Next, we will show that $E_{1,2}$ is the transpose of $E_{2,1}$. Let us rewrite $E_{1,2}$ as

$$E_{1,2} = M_-v - (M_-v + M_-p - \tilde{m})(ha). \tag{30}$$

Using $ha = \frac{1}{\beta}(v^{\mathrm{T}}M_-v - \tilde{m}^{\mathrm{T}}v)$ and $M_-p - \tilde{m} = -(A^+_{LM\_}A)^{\mathrm{T}}\tilde{m}$ (see Properties 10 and 11 in Appendix) in Eq. (30), we obtain

$$E_{1,2} = M_-v - \left[M_-v - (A^+_{LM\_}A)^{\mathrm{T}}\tilde{m}\right]\left[v^{\mathrm{T}}M_-v - \tilde{m}^{\mathrm{T}}v\right]\frac{1}{\beta}. \tag{31}$$

On the other hand, $E_{2,1}$ can be written as

$$E_{2,1} = \tilde{m}^{\mathrm{T}}\left(A^+_{LM\_}A\right) - \left(\tilde{m}^{\mathrm{T}}v + \tilde{m}^{\mathrm{T}}p - \bar{m}\right)(hA). \tag{32}$$

Using $hA = \frac{1}{\beta}(v^{\mathrm{T}}M_- - \tilde{m}^{\mathrm{T}}A^+_{LM\_}A)$ and $\tilde{m}^{\mathrm{T}}v + \tilde{m}^{\mathrm{T}}p - \bar{m} = v^{\mathrm{T}}M_-v - v^{\mathrm{T}}\tilde{m} - \beta$ (see Properties 8 and 14 in Appendix) in Eq. (32), we get

$$E_{2,1} = \tilde{m}^{\mathrm{T}}(A^+_{LM\_}A) - \left[v^{\mathrm{T}}M_-v - v^{\mathrm{T}}\tilde{m} - \beta\right]\left[v^{\mathrm{T}}M_- - \tilde{m}^{\mathrm{T}}A^+_{LM\_}A\right]\frac{1}{\beta}, \tag{33}$$

which can be simplified to

$$E_{2,1} = v^{\mathrm{T}}M_- - \left[v^{\mathrm{T}}M_-v - v^{\mathrm{T}}\tilde{m}\right]\left[v^{\mathrm{T}}M_- - \tilde{m}^{\mathrm{T}}A^+_{LM\_}A\right]\frac{1}{\beta}. \tag{34}$$

It can be seen from Eqs. (31) and (34) that $E_{1,2}$ is the transpose of $E_{2,1}$. Since we have already shown that $E_{1,1}$ is symmetric and since $E_{2,2}$ is scalar which is symmetric, the symmetry of $M(B^+_{LM}B)$ is verified.

We have shown that all four generalized *LM*-inverse conditions are satisfied. Hence, the formula (11) is verified. □

3. Conclusions

The recursive formulae for obtaining the generalized LM-inverse of any general matrix augmented by a column vector were first given in Ref. 15. There they were derived in a constructive manner. We herein provide an alternative proof of their formulae by directly verifying that the four conditions of the generalized LM-inverse are satisfied, thereby confirming the validity of the formulae, and providing several new auxiliary results related to these generalized inverses.

Appendix A

This section provides some properties that are used for verifying the recursive formulae for determining of the generalized LM-inverse of a matrix.

Property 1. $Ap = 0$.

Proof. Since $p = (I - A^+_{LM_-}A)M_-^{-1}\tilde{m}$, we have

$$Ap = A(I - A^+_{LM_-}A)M_-^{-1}\tilde{m} = 0. \qquad \square$$

Property 2. $a = AA^+_{LM_-}a$ (*when* $d = 0$).

Proof. Because $d = (I - AA^+_{LM_-})a = 0$, we get

$$a = AA^+_{LM_-}a. \qquad \square$$

Property 3. $A^+_{LM_-}d = 0$.

Proof. Since $d = (I - AA^+_{LM_-})a$, we obtain

$$A^+_{LM_-}d = A^+_{LM_-}(I - AA^+_{LM_-})a = 0. \qquad \square$$

Property 4. $d^+_L A = 0$.

Proof. Since $d = (I - AA^+_{LM_-})a$ and $d^+_L = (d^{\mathrm{T}}Ld)^{-1}d^{\mathrm{T}}L$, we have

$$\begin{aligned} d^+_L A &= (d^{\mathrm{T}}Ld)^{-1}d^{\mathrm{T}}LA = (d^{\mathrm{T}}Ld)^{-1}\left[(I - AA^+_{LM_-})a\right]^{\mathrm{T}}LA = (d^{\mathrm{T}}Ld)^{-1}a^{\mathrm{T}}(I - AA^+_{LM_-})^{\mathrm{T}}LA \\ &= (d^{\mathrm{T}}Ld)^{-1}a^{\mathrm{T}}L(I - AA^+_{LM_-})L^{-1}LA = 0. \qquad \square \end{aligned}$$

Property 5. $d^+_L a = 1$.

Proof. Using $d^+_L = (d^{\mathrm{T}}Ld)^{-1}d^{\mathrm{T}}L$ and $d = (I - AA^+_{LM_-})a$, we have

$$\begin{aligned} d^+_L a &= \frac{d^{\mathrm{T}}La}{d^{\mathrm{T}}Ld} = \frac{[(I - AA^+_{LM_-})a]^{\mathrm{T}}La}{[(I - AA^+_{LM_-})a]^{\mathrm{T}}L[(I - AA^+_{LM_-})a]} = \frac{a^{\mathrm{T}}L(I - AA^+_{LM_-})L^{-1}La}{a^{\mathrm{T}}L(I - AA^+_{LM_-})L^{-1}L(I - AA^+_{LM_-})a} \\ &= \frac{a^{\mathrm{T}}L(I - AA^+_{LM_-})a}{a^{\mathrm{T}}L(I - AA^+_{LM_-})a} = 1. \qquad \square \end{aligned}$$

Property 6. $h = \frac{1}{\beta}(v^{\mathrm{T}}M_- - \tilde{m}^{\mathrm{T}})A^+_{LM_-}$.

Proof. Since $h = \frac{1}{\beta}q^{\mathrm{T}}MU$, where $\beta = q^{\mathrm{T}}Mq$, $q = \begin{bmatrix} v + p \\ -1 \end{bmatrix}$, $v = A^+_{LM_-}a$, $p = (I - A^+_{LM_-}A)M_-^{-1}\tilde{m}$, $M = \left[\begin{array}{c|c} M_- & \tilde{m} \\ \tilde{m}^{\mathrm{T}} & \bar{m} \end{array}\right]$, and $U = \begin{bmatrix} A^+_{LM_-} \\ 0_m \end{bmatrix}$, we have

$$h = \frac{1}{\beta} q^{\mathrm{T}} M U = \frac{1}{\beta} \begin{bmatrix} v + p \\ -1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} M_- & \tilde{m} \\ \tilde{m}^{\mathrm{T}} & \tilde{m} \end{bmatrix} \begin{bmatrix} A^+_{LM_-} \\ 0_m \end{bmatrix} = \frac{1}{\beta} \begin{bmatrix} v^{\mathrm{T}} + p^{\mathrm{T}} \mid -1 \end{bmatrix} \begin{bmatrix} M_- A^+_{LM_-} \\ \tilde{m}^{\mathrm{T}} A^+_{LM_-} \end{bmatrix},$$
$$= \frac{1}{\beta} \left[v^{\mathrm{T}} M_- A^+_{LM_-} + p^{\mathrm{T}} M_- A^+_{LM_-} - \tilde{m}^{\mathrm{T}} A^+_{LM_-} \right].$$

Because $p^{\mathrm{T}} M_- A^+_{LM_-} = [\tilde{m}^{\mathrm{T}} M_-^{-1} (I - A^+_{LM_-} A)^{\mathrm{T}}] M_- A^+_{LM_-} = \tilde{m}^{\mathrm{T}} M_-^{-1} M_- (I - A^+_{LM_-} A) M_-^{-1} M_- A^+_{LM_-} = 0$, we obtain

$$h = \frac{1}{\beta} (v^{\mathrm{T}} M_- A^+_{LM_-} - \tilde{m}^{\mathrm{T}} A^+_{LM_-}) = \frac{1}{\beta} (v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}}) A^+_{LM_-}. \qquad \square$$

Property 7. $v^{\mathrm{T}} M_- A^+_{LM_-} A = v^{\mathrm{T}} M_-$.

Proof. Since $v = A^+_{LM_-} a$ and $M_- A^+_{LM_-} A = (A^+_{LM_-} A)^{\mathrm{T}} M_-$, we have

$$v^{\mathrm{T}} M_- A^+_{LM_-} A = (A^+_{LM_-} a)^{\mathrm{T}} (A^+_{LM_-} A)^{\mathrm{T}} M_- = (A^+_{LM_-} A A^+_{LM_-} a)^{\mathrm{T}} M_- = (A^+_{LM_-} a)^{\mathrm{T}} M_- = v^{\mathrm{T}} M_-. \qquad \square$$

Property 8. $hA = \frac{1}{\beta} (v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}} A^+_{LM_-} A)$.

Proof. Since $h = \frac{1}{\beta} (v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}}) A^+_{LM_-}$ and $v^{\mathrm{T}} M_- A^+_{LM_-} A = v^{\mathrm{T}} M_-$ (see Properties 6 and 7 above), we have

$$hA = \frac{1}{\beta} (v^{\mathrm{T}} M_- A^+_{LM_-} A - \tilde{m}^{\mathrm{T}} A^+_{LM_-} A) = \frac{1}{\beta} (v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}} A^+_{LM_-} A). \qquad \square$$

Property 9. $hAA^+_{LM_-} = h$.

Proof. Since $h = \frac{1}{\beta} (v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}}) A^+_{LM_-}$ (see Property 6 above) and $A^+_{LM_-} A A^+_{LM_-} = A^+_{LM_-}$, we get

$$hAA^+_{LM_-} = \frac{1}{\beta} (v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}}) A^+_{LM_-} A A^+_{LM_-} = \frac{1}{\beta} (v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}}) A^+_{LM_-} = h. \qquad \square$$

Property 10. $ha = \frac{1}{\beta} (v^{\mathrm{T}} M_- v - \tilde{m}^{\mathrm{T}} v)$.

Proof. Because $A^+_{LM_-} a = v$ and $h = \frac{1}{\beta} (v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}}) A^+_{LM_-}$ (see Property 6 above), we have

$$ha = \frac{1}{\beta} (v^{\mathrm{T}} M_- A^+_{LM_-} a - \tilde{m}^{\mathrm{T}} A^+_{LM_-} a) = \frac{1}{\beta} (v^{\mathrm{T}} M_- v - \tilde{m}^{\mathrm{T}} v). \qquad \square$$

Property 11. $M_- p - \tilde{m} = -(A^+_{LM_-} A)^{\mathrm{T}} \tilde{m}$.

Proof. Since $p = (I - A^+_{LM_-} A) M_-^{-1} \tilde{m}$, we get

$$M_- p - \tilde{m} = M_- [(I - A^+_{LM_-} A) M_-^{-1} \tilde{m}] - \tilde{m} = -M_- A^+_{LM_-} A M_-^{-1} \tilde{m} = -(A^+_{LM_-} A)^{\mathrm{T}} \tilde{m}. \qquad \square$$

Property 12. $M_- v + M_- p - \tilde{m} = [M_- v - (A^+_{LM_-} A)^{\mathrm{T}} \tilde{m}]^{\mathrm{T}} = \beta (hA)^{\mathrm{T}}$.

Proof. Since $M_- p - \tilde{m} = -(A^+_{LM_-} A)^{\mathrm{T}} \tilde{m}$ and $v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}} A^+_{LM_-} A = \beta h A$ (see Properties 8 and 11 above), we get

$$M_- v + M_- p - \tilde{m} = M_- v - (A^+_{LM_-} A)^{\mathrm{T}} \tilde{m} = [v^{\mathrm{T}} M_- - \tilde{m}^{\mathrm{T}} A^+_{LM_-} A]^{\mathrm{T}} = \beta (hA)^{\mathrm{T}}. \qquad \square$$

Property 13. $p^{\mathrm{T}} M_- = \tilde{m}^{\mathrm{T}} (I - A^+_{LM_-} A)$.

Proof. Because $p = (I - A^+_{LM_-} A) M^{-1} \tilde{m}$ and $v = A^+_{LM_-} a$, we have

$$p^{\mathrm{T}} M_- = [(I - A^+_{LM_-} A) M_-^{-1} \tilde{m}]^{\mathrm{T}} M_- = \tilde{m}^{\mathrm{T}} M_-^{-1} [M_- (I - A^+_{LM_-} A) M_-^{-1}] M_- = \tilde{m}^{\mathrm{T}} (I - A^+_{LM_-} A). \qquad \square$$

Property 14. $\tilde{m}^{\mathrm{T}}v + \tilde{m}^{\mathrm{T}}p - \bar{m} = v^{\mathrm{T}}M_{-}v - v^{\mathrm{T}}\tilde{m} - \beta.$

Proof. Since $q = \begin{bmatrix} v+p \\ -1 \end{bmatrix}$, and $M = \left[\begin{array}{c|c} M_{-} & \tilde{m} \\ \tilde{m}^{\mathrm{T}} & \bar{m} \end{array}\right]$, we have

$$\beta = q^{\mathrm{T}}Mq = \begin{bmatrix} v+p \\ -1 \end{bmatrix}^{\mathrm{T}} \left[\begin{array}{c|c} M_{-} & \tilde{m} \\ \tilde{m}^{\mathrm{T}} & \bar{m} \end{array}\right] \begin{bmatrix} v+p \\ -1 \end{bmatrix} = v^{\mathrm{T}}M_{-}v + 2p^{\mathrm{T}}M_{-}v - 2\tilde{m}^{\mathrm{T}}v + p^{\mathrm{T}}M_{-}p - 2\tilde{m}^{\mathrm{T}}p + \bar{m},$$

where we have used $p^{\mathrm{T}}Mv = v^{\mathrm{T}}Mp$, $\tilde{m}^{\mathrm{T}}v = v^{\mathrm{T}}\tilde{m}$, and $\tilde{m}^{\mathrm{T}}p = p^{\mathrm{T}}\tilde{m}$ since they are scalars.

Using $p^{\mathrm{T}}M_{-} = \tilde{m}^{\mathrm{T}}(I - A^{+}_{LM\_}A)$ (see Property 13 above), we have $p^{\mathrm{T}}M_{-}v = \tilde{m}^{\mathrm{T}}(I - A^{+}_{LM\_}A)A^{+}_{LM\_}a = 0$ and $p^{\mathrm{T}}M_{-}p = [\tilde{m}^{\mathrm{T}}(I - A^{+}_{LM\_}A)][(I - A^{+}_{LM\_}A)M_{-}^{-1}\tilde{m}] = \tilde{m}^{\mathrm{T}}[(I - A^{+}_{LM\_}A)M_{-}^{-1}\tilde{m}] = \tilde{m}^{\mathrm{T}}p.$

Thus, we obtain $\beta = v^{\mathrm{T}}M_{-}v - 2\tilde{m}^{\mathrm{T}}v - \tilde{m}^{\mathrm{T}}p + \bar{m}$, which gives

$$\bar{m} = \beta - v^{\mathrm{T}}M_{-}v + 2\tilde{m}^{\mathrm{T}}v + \tilde{m}^{\mathrm{T}}p.$$

Using the relation above and again the fact that $\tilde{m}^{\mathrm{T}}v = v^{\mathrm{T}}\tilde{m}$, we have

$$\tilde{m}^{\mathrm{T}}v + \tilde{m}^{\mathrm{T}}p - \bar{m} = \tilde{m}^{\mathrm{T}}v + \tilde{m}^{\mathrm{T}}p - (\beta - v^{\mathrm{T}}M_{-}v + 2\tilde{m}^{\mathrm{T}}v + \tilde{m}^{\mathrm{T}}p) = v^{\mathrm{T}}M_{-}v - v^{\mathrm{T}}\tilde{m} - \beta. \quad \square$$

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